

SHOCK WAVE AND  
HIGH PRESSURE PHENOMENA

R. P. Drake

# High-Energy-Density Physics

Fundamentals,  
Inertial Fusion,  
and Experimental  
Astrophysics

 Springer

Drake



High-Energy-Density Physics

The raw numbers of high-energy-density physics are amazing: shock waves at hundreds of km/s (approaching a million km per hour), temperatures of millions of degrees, and pressures that exceed 100 million atmospheres. This book introduces the reader to the fundamental tools and discoveries of high-energy-density physics. It surveys the production of high-energy-density conditions, the fundamental plasma and hydrodynamic models that can describe them and the problem of scaling from the laboratory to the cosmos. Connections to astrophysics are discussed throughout. The book is intended to support coursework in high-energy-density physics, to meet the needs of new researchers in this field, and also to serve as a useful reference on the Fundamentals. Specifically the book has been designed to enable academics in physics, astrophysics, applied physics and engineering departments to provide in a single-course introduction to fluid mechanics and radiative transfer, with dramatic applications in the field of high-energy-density systems.

Shock Wave  
and High Pressure  
Phenomena

R. P. Drake

High-Energy-Density  
Physics

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## Shock Wave and High Pressure Phenomena

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R. Paul Drake

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# High-Energy-Density Physics

Fundamentals, Inertial Fusion,  
and Experimental Astrophysics

With 172 Figures

 Springer

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Dedicated to Kent Estabrook

## Preface

This book has two goals. One goal is to provide a means for those new to high-energy-density physics to gain a broad foundation from one text. The second goal is to provide a useful working reference for those in the field.

This book has at least four possible applications in an academic context. It can be used for training in high-energy-density physics, in support of the growing number of university and laboratory research groups working in this area. It also can be used by schools with an emphasis on ultrafast lasers, to provide some introduction to issues present in all laser–target experiments with high-power lasers, and with thorough coverage of the material in Chap. 11 on relativistic systems. In addition, it could be used by physics, applied physics, or engineering departments to provide in a single course an introduction to the basics of fluid mechanics and radiative transfer, with dramatic applications. Finally, it could be used by astrophysics departments for a similar purpose, with the parallel benefit of training the students in the similarities and differences between laboratory and astrophysical systems.

The notation in this text is deliberately sparse and when possible a given symbol has only one meaning. A definition of the symbols used is given in Appendix A. In various cases, additional subscripts are added to distinguish among cases of the same quantity, as for example in the use of  $\rho_1$  and  $\rho_2$  to distinguish the mass density in two different regions. With the goals of minimizing the total number of symbols and of using them uniquely, the text avoids various common usages. An example is the use of  $\mu$  for the coefficient of viscosity, which is avoided, with the viscosity expressed always as the product or  $\rho\nu$ , where  $\nu$  is the kinematic coefficient of viscosity.

Much of the homework throughout this text is only feasible using a computational mathematics program. The author prefers *Mathematica*, which has been an essential tool in the preparation of this text, but there are now and will be several such programs available. This departure from traditional norms reflects the emergence of such programs as effective tools. They should be part of the standard toolkit of all future scientists. This dramatically changes the meaning of “simple” solutions to problems. For example, an eighth-order polynomial equation is not necessarily difficult to deal with. Appendix B includes examples of *Mathematica* code for two of the issues discussed in the book, to help the reader get started.

A word on the use of units is in order. The metric system in a broad sense is the common language of science. But the world in general and high-energy-density systems, in particular, are not conveniently analyzed within any single standard subsystem of these units. Each of the SI system, the Gaussian cgs system, and other systems is the most convenient for certain problems, as are a few other specific units such as the electron Volt. This is why these systems exist. It is an *essential* tool for a practicing scientist to be able to readily convert between systems of units “on the fly”. This is true because the existing literature is presented for the most part in convenient units, which working scientists use because they are convenient. But comparisons of one system to another are very important as checks on one’s reasoning, and this often leads to the need to convert units. Thus, the author is an adamant opponent of the SI purists who would commit nothing to print that is not in SI units and an adamant advocate for defining one’s units in all work one does. When feasible, the equations in this book are written in unit-independent form. When this is not possible, as for example with the Lorentz force, the units are specified and are usually in the Gaussian cgs system, which is the most convenient for most plasma applications. The units are also specified when practical equations are given. At least this was the author’s intent. Please let him know where he failed. Finally, the appendix on units in Jackson’s *Classical Electrodynamics* is an excellent reference on this subject.

Bibliographic references are sparse in most chapters of this text. Most of the references are to published books that address a certain topic in more detail than is feasible here. The journal literature is cited only when there are as yet no relevant books, and such citations often fail to reflect the scope of work in the journal literature. This was deliberate for several reasons. One of my goals has been to write a book that will prove useful for many years. The archival literature changes rapidly and the present era is one of very rapid advance in high-energy-density physics and in astrophysics. As a result, any references to the current literature will rapidly become dated. In addition, the era of immediate bibliographic database searches is here to stay, so future readers will readily be able to find up-to-date references in the archival literature of their time.

A second goal has been to present the material here with a common voice, because in the opinion of the author this is pedagogically most effective. A book that ties itself too closely to published literature can become disjointed. A third goal has been to show that this material is “simple”, in the sense that a physicist would use. A rich panoply of phenomena evolves straightforwardly from what are at root a few and simple starting equations. In the spirit of Richard Feynman, one can understand a great deal without needing more than clear thinking. (Though one must add that a computational mathematics program helps a lot for some nonlinear problems.) The greatest



departure from this goal has been in Chaps. 3 and 8, where to avoid very protracted discussions we have been forced to ask the reader to accept some details without much explanation.

Throughout this text, there are a number of figures showing the results of computer simulations, in order to display hydrodynamic and radiation hydrodynamic phenomena. Unless otherwise noted, these simulations were done using the HYADES computer code authored by Jon Larsen and available at this writing from Cascade Sciences Inc. A number of similar tools exist; they prove very useful for calculations to evaluate possible experiments and to identify the most important physical mechanisms in specific physical systems of interest.

Writing acknowledgments is rather daunting, given the many individuals who contribute to a project such as this. To those overlooked: remind me and I will at least buy you dinner. I must thank my family and my current research group for tolerating the time required for such a project. Dmitri Ryutov has been a source of inspiration, instruction, and encouragement, in addition to a vital collaborator, for a number of years, and also reviewed two chapters. Alexander Velikovich reviewed two chapters, made time for several delightful conversations, and significantly broadened my understanding of several issues. Harry Robey provided valuable insight into hydrodynamic instabilities and found an important error. Robert Kauffman and David Montgomery provided specific useful figures. Enam Chowdhury provided useful input and graciously allowed me to use some of his work. Michael DesJarlais, Warren Mori, Mordecai Rosen, Mark Hermann, James Knauer, Riccardo Betti, and Bedros Afeyan found time to comment on or discuss some of the material. Farhat Beg and William Kruer taught from the draft text. Ralph Schneider was a source of enduring encouragement.

The students in the lectures at Michigan in 2003 and 2005 and the 28 attendees of the summer school in 2004, though too numerous to list, helped identify errors and provided opportunities to improve the text. My own current graduate students Amy Reighard, Carolyn Kuranz, Eric Harding, and Tony Visco suffered through working with the draft, even while providing continuing motivation. Korbie Dannenberg, in addition to having done some of the work reflected in examples herein, kept my group moving forward when I was off writing. Jan Beltran provided a wide range of administrative assistance with the summer school and with the book, all of which I greatly appreciate. Of course, the responsibility for the errors in the text rests solely with me.

Beyond this specific group, I have enjoyed collaborations with a large community of scientists, engineers, and technicians during the past 20-plus years. A few of the key individuals not mentioned above are Dave Arnett, Jim Asay, Hector Baldis, Steve Batha, Bruno Bauer, Serge Bouquet, Jim Carroll, John DeGroot, Kent Estabrook, Adam Frank, Gail Glendinning, Martin Goldman, Tudor Johnston, Jave Kane, Paul Keiter, Alexei Khokhlov, Marcus

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I love to work in coffee shops and was fortunate that my local favorite, Espresso Royale, opened a branch on Plymouth Road near my home early during this project. I did a lot of writing, editing, and deriving at their tables. To Sarah and all the staff who have worked there, thanks for the hospitality.

Finally, this book would not exist without two people. E. Michael Campbell talked me into entering this field when it had troubled times, supported doing the science needed to make inertial fusion succeed, and helped me move on when the time came for that. Bruce A. Remington talked me into jumping into the astrophysical applications of high-energy-density tools when this was a new idea and has continued to be a valuable collaborator since that time. I thank them both.

Ann Arbor, Michigan  
December 2005

*R. Paul Drake*

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# 1 Introduction to High-Energy-Density Physics

This book concerns itself with the physics of systems having high-energy-density. By *high-energy-density* systems, we refer to those having a pressure above one million atmospheres. The units of this pressure can be designated by 1 Mbar,  $10^{11}$  Pascals,  $10^{11}$  J/m<sup>3</sup>,  $10^{12}$  dynes/cm<sup>2</sup>, or  $10^{12}$  ergs/cm<sup>3</sup>. We will tend to express the pressure in Mbars, as this is the most common of these units found in the relevant literature. This characterization was developed in a report of the National Academy of Sciences in the U.S., entitled *Frontiers in High-Energy-Density Physics: the X-games of Contemporary Science*. It reflects several observations. For example, one learned in school that solids and liquids are “incompressible”, but this is not strictly true. If one applies a pressure exceeding  $\sim 1$  Mbar to ordinary solid matter, it compresses. Another way to make this point is to say that the internal energy of a hydrogen molecule is  $\sim 1$  Mbar.

Thinking further on this point, one might realize that once the energy that holds a collection of particles together, whether as applied pressure or as binding energy, becomes of the order of the internal energy of molecules and atoms; these will behave more as ions and electrons than as neutral particles. Indeed, the fact that metals have conduction electrons might be viewed as the beginning of this type of phenomenon. At higher pressures (or lower densities, so that 1 Mbar corresponds to higher temperatures) the material will definitely be an ionized medium. An ionized medium is a plasma, but traditional plasma theory applies to ionized gasses. We will see in Chap. 2 that very little of the regime of high-energy-density physics can be described validly by traditional plasma theory. Thus, another way to characterize high-energy-density systems is as plasma that is too dense for traditional plasma theory. We explore the implications of this in Chap. 3. We discuss how to produce these dense, high-energy-density plasmas in Chap. 8. One might like to take this up sooner, but in fact the concepts developed in Chaps. 3–7 are essential to presenting a comprehensive and comprehensible discussion.

Chapter 8 discusses how to create these conditions. For example, one might launch a shock wave that converts ordinary matter into high-energy-density matter. Such shock waves have velocities above 10 km/s. At constant pressure, shock velocities increase as density decreases, so that shock waves above 100 km/s ( $>360,000$  km per hour) are common in high-energy-density



physics. Alternatively, one might produce an intense beam of photons, electrons, or ions that can penetrate the matter and directly heat it.

High-energy-density physics encompasses more than the regime of dense plasma, in the sense just described. It also includes conditions in which pressures  $>1$  Mbar result from very high temperature at very low density. For example, air at a density of  $1 \text{ mg/cm}^3$ , of the order of atmospheric density, reaches a pressure of  $\sim 1$  Mbar at a temperature above 10 keV. (Throughout this text we express temperature in the energy unit of an electron Volt, so that the Boltzmann constant is  $1.6 \times 10^{-12}$  ergs/eV or  $1.6 \times 10^{-19}$  J/eV.) A temperature of 10 keV is roughly 100 million degree Kelvin, so that temperatures of millions of degrees or more are common in high-energy-density physics. As the density decreases further, conditions in which the pressure remains above one Mbar soon become relativistic, and thus also outside the realm of traditional plasma theory. Overall, what high-energy-density systems have in common with traditional plasmas and with condensed-matter systems is that collective effects are an essential aspect of the behavior. The difference from traditional plasma physics is that the particles are more correlated, relativistic, or both. The difference from traditional condensed-matter physics is that ionization and Coulomb interactions are essential.

The present text is the first book to be written as a textbook in high-energy-density physics. (We place it in the context of prior work later in this chapter.) This reflects the fact that high-energy-density physics is in some sense a new field. One can see that the regimes just discussed offer some challenges beyond established areas of physics, but one might wonder both in what sense this is new and why. The material discussed here, as in condensed-matter physics and other areas, is entirely built on the foundations of classical and modern physics as established from the mid 19th to the mid 20th century. In addition, much of the material discussed herein is discussed in more depth in one of a dozen or so more-advanced books. The fundamental sense in which this is a new area is that there are new tools and that new tools beget new areas of science. It is now practical for scientists in an academic or laboratory setting to perform experiments to study the fundamental behavior of high-energy-density systems over a significant range of parameters. This creates a need for the treatment of this material as an integrated subject, moving from fundamentals to their applications, for the presentation of the material in a common voice suitable for graduate courses and as a first working reference, and for a discussion that spans the range of conditions now (or soon to be) available for study. Hence the emergence of high-energy-density physics as a distinct field and hence this text.

## 1.1 Some Historical Remarks

Let us consider the new tools that brought this about. One can identify several precursors. The development of particle accelerators in the 1930s

began the effort to focus large numbers of particles to small areas. The advent of nuclear weapons in the 1940s produced high-energy-density conditions, but not in a way that permitted systematic study. In the 1950s the ill-fated Z-pinch was developed, in pursuit of controlled thermonuclear fusion for power production. The Z-pinch was intended at first to gradually compress and heat matter through the attraction of parallel channels of current. In the late 1950s and through the 1960s the laser was invented and developed until relatively intense, pulsed lasers became possible. All of this set the technological stage for what followed. We summarize this here, with the intent of giving the flow of key technical developments, as opposed to a thorough historical review.

By the mid 1960s it became sensible to ask whether lasers might be used to produce controlled thermonuclear fusion. This would be accomplished by creating momentary collections of burning fusion fuel, held together by only their inertia. *Atzeni and Meyer-ter-Vehn*, in their book, review the foundations for this specific development, some of which were in classified research programs, and credit a number of key contributors not mentioned here. The effort to address this question in the USSR was led by Basov. In 1972, *Nuckolls, Lowell, and Wood* published the key paper arguing that this approach to fusion might be feasible. Programs to pursue what became known as *inertial confinement fusion* were begun in the U.S., the Soviet Union, Europe, and Japan. A key figure during the development of the necessary lasers was John Emmett, who led the program at Lawrence Livermore that first produced lasers delivering  $>1$  kJ in 1 ns. By the end of the 1970s, there were lasers in several countries that could deliver a number of kJ to volumes of a cubic mm or less in pulses of order 1 ns in duration. One Mbar is  $100 \text{ J/mm}^3$ , so these systems produced high-energy-density conditions. During the same period, devices that could drive currents above 1 MA, known as pulsed-power devices, were also developed, motivated as well by their potential application to inertial confinement fusion. In the U.S. this was done at Sandia National Laboratories. The initial intent was that these devices could create inertial fusion using particle beams, but in the end they contributed to the revolution in Z-pinches described below. Other lasers were also developed as high-energy sources, including  $\text{CO}_2$  lasers at the Los Alamos Laboratory and Iodine lasers in the Soviet Union. These lasers did not work out for fusion, but in some sense they encouraged the development of KrF lasers, pursued further in the U.S. at the Naval Research Laboratory and at this writing a candidate for use in producing electricity powered by inertial fusion.

These tools could create high-energy-density systems, but not in a way that permitted systematic study. During the 1970s a few research projects and programs began to do systematic fundamental science at high energy density, notably in Europe and at the Naval Research Laboratory. This is perhaps too harsh, as one can find a sequence of refereed journal papers tracing progress in the science from all of the participants. But all such efforts were hampered

by a lack of experimental technique and diagnostic hardware. They were also hampered by a tendency to focus on the goal of fusion to the exclusion of its fundamental underpinnings. As an extreme example, the head of the project to build the Nova laser, completed in the mid-1980s, once told the author that the only diagnostic needed by Nova for the success of inertial fusion was a single neutron detector. This proved to be far from the truth. Experiments and diagnostics in a number of areas, in addition to substantial improvements to the facility itself, were required before Nova could achieve the compression of DT fusion fuel to 1,000 times liquid density (a remarkable accomplishment).

Meanwhile, a revolution in experimental technique was occurring in the U.S. and around the world. Low-energy lasers had been in use as probe beams for some time, and this continued. Now the collection of high-energy laser beams began to be used for several independent purposes. Some beams could strike a target to produce a desired system, other beams might be used to drive some process or event in that system, and still others might be used as diagnostics, often by producing x-rays whose transmission or scattering could be measured. This required that the beams be independently timed and controlled, which is easy to say but imposes considerable additional cost and complexity. In addition, the 1980s saw the realization and demonstration of affordable instrumentation that could obtain data on a sub-ns timescale, including snapshots and time histories as images or as spectra. It may be unfair to single out this decade, as these developments occurred both before and after as well, but in my view it was during the 1980s that doing high-energy-density physics with lasers became practical. This was in part driven by an increased focus on studying the elements of the physics that were required for inertial fusion. While many researchers around the world contributed to this developing focus, the one individual who had the biggest impact in the biggest program was a young group leader named Michael Campbell, at the Lawrence Livermore National Laboratory.

The 1980s also saw the invention of *chirped-pulse amplification* by Gerard Mourou, described in Chap. 8. This made it practical to drive the irradiance of lasers above  $10^{18}$  W/cm<sup>2</sup>, and to begin to produce relativistic effects. Such systems have short pulses, typically below 1 ps, and so are known as *ultrafast lasers*. They have contributed the tools that enable exploration of the low-density, relativistic regime of high-energy-density physics.

The available experimental tools for high-energy-density physics expanded again in the 1990s with the development of the wire-array Z pinch. A modification known as the fast Z pinch had been under exploration for fusion since the late 1970s. A fast Z pinch avoids the magnetohydrodynamic instabilities that disabled the Z pinches of the 1950s, by using the pinch to briefly accelerate material inward, after which the stagnation of the imploding material converts kinetic energy to internal energy. There is hope that this might provide an alternative approach to fusion. Whether or not this works out, such

pinches are large and efficient radiation sources. When they distribute the current across hundreds of metallic wires, they can produce energies of MJs in volumes of cubic centimeters. This development, also discussed further in Chap. 8, provided yet another environment for the pursuit of high-energy-density physics, and there was a veritable explosion in such activity using pinches at around the turn of the century.

Meanwhile, particle accelerators continued to develop, driven primarily by the needs of particle physics. By the 1990s, these accelerators could also produce collections of numerous relativistic particles in the high-energy-density regime. Both ion beams and electron beams can produce high-energy-density conditions. As one example, the Stanford Linear Accelerator has been used to produce bunches of order  $10^{10}$  electrons at an energy of 50 GeV, in a 5 ps pulse. These bunches form a  $3 \mu\text{m}$  spot, and so have an irradiance of  $10^{20}$  W/cm<sup>2</sup>. They deliver 150 J per pulse so a target, and they arrive at a rate that can exceed 100 Hz. These electrons are themselves a high-energy-density medium and can be caused to interact with materials of choice.

All of the above developments produced an environment within which it became possible to pursue questions in high-energy-density physics for their own sake. Researchers around the world can now address the properties of matter, the development of dynamic structure and of instabilities, the properties of radiation, the effect of radiation on the dynamic behavior, and relativistic phenomena in this regime. These fundamentals are what we take up in the next six chapters and Chap. 11. Researchers can then use this knowledge to invent novel approaches to inertial fusion (Chap. 9), to learn things needed for astrophysics (Chap. 10), and to develop technologies ranging from improved lithographic systems to novel medical therapies. Before turning to these tasks, the following provides some further overview of the regimes of high-energy-density physics and of its applications to fusion and to astrophysics.

## 1.2 Regimes of High-Energy-Density Physics

The report of the U.S. National Academy of Sciences mentioned above includes a figure that summarizes the parameter regime of high-energy-density physics. We have adapted this here as Fig. 1.1. This figure merits an extensive discussion, which will point the way to much of our work throughout this text. The horizontal axes show density (mass density below and number density above, for hydrogen). The vertical axes show temperature, in degree Kelvin to the left and in eV to the right. All the axes are logarithmic, so that this figure spans 23 orders of magnitude in density and more than 9 orders of magnitude in temperature. It shows a number of boundaries and curves. These boundaries and curves assume that the matter and radiation are approximately in equilibrium. We will work our way through these and see what they tell us.

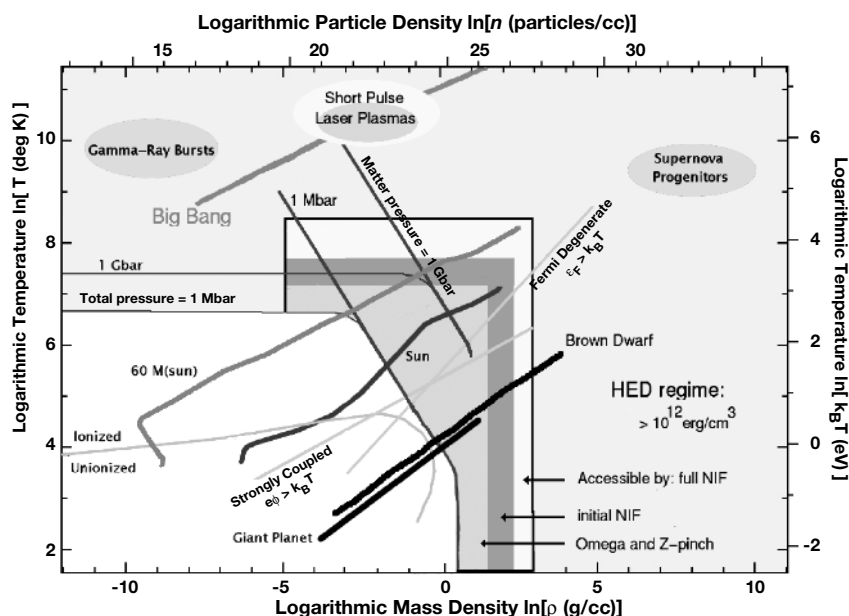


Fig. 1.1. Regimes of high-energy-density physics. Adapted from the NRC Report: *High Energy Density Physics: The X-Games of Contemporary Science*

We begin along the left edge of this figure. Working up in temperature, the first curve encountered shows the boundary between ionized and unionized matter. Once the temperature reaches about an eV, one begins to see some ionization. The required temperature increases as density increases, reflecting the increasing density of electrons that can recombine with the ions. Then, at densities approaching  $1 \text{ g/cm}^3$  (g/cc in the axis label), the curve turns over. At high-enough density, matter is ionized at any temperature, for reasons discussed in Chap. 3. This phenomenon is known as *pressure ionization*.

Continuing up the left axis, one sees horizontal lines labeled “Total pressure = 1 Mbar” and “1 Gbar”. These lines correspond to the temperature at which an equilibrium radiation field produces these pressures. Above the lines the plasma is *radiation-dominated*, if the system is in equilibrium. A plasma at low-enough density becomes radiation-dominated at a temperature of about 1 keV. We introduce radiation-dominated plasmas in Chap. 3 and discuss them in more depth in Chs. 6 and 7. Following these horizontal lines to the right, one sees that they tilt downward. This occurs as the thermal pressure of the matter becomes equal to and then dominant over the radiation pressure. Thus, for example, the thermal pressure at a density of  $1 \text{ g/cm}^3$  and a temperature of 1 eV is roughly 1 Mbar. At a density just above  $1 \text{ g/cm}^3$ , the curve showing a pressure of 1 Mbar becomes vertical. This is the density where the Fermi pressure of the electrons exceeds 1 Mbar. We

discuss the properties of the electrons as fermions in Chap. 3, and they are important for inertial fusion as discussed in Chap. 9. The 1 Gbar curve, if continued, would become vertical at a higher density.

Two lines cross the middle of the graph. One of these is labeled “Strongly Coupled  $e\phi > k_B T$ ” and the other is labeled “Fermi Degenerate  $\epsilon_F > k_B T$ ”. To the right of the first of these lines, the energy of Coulomb interactions exceeds the thermal energy. Such plasmas are often labeled *strongly coupled plasmas*. We discuss strongly coupled plasmas in Chap. 3, and develop an approach to describing their behavior in the regime that can be studied in the early 21st century. As we shall see, the boundary above which traditional plasma theory is valid lies above and to the left of this line. In the region to the right of the second of these lines, the Fermi energy exceeds the thermal energy, and one has a plasma that is *Fermi degenerate*. This is obviously connected with the vertical boundary in pressure discussed previously. We discuss Fermi-degenerate plasmas in Chap. 3.

The figure also shows some laboratory and astrophysical benchmarks. In the core of the plot, one can see the regions known to be accessible to selected, early-21st-century facilities in the U.S. These include the Omega laser at the University of Rochester, the largest Z pinches at the Sandia National Laboratories, and NIF (the National Ignition Facility) at Lawrence Livermore National Laboratory. This core regime, corresponding to ion densities from about  $10^{19} \text{ cm}^{-3}$  to  $10^{24} \text{ cm}^{-3}$  and to temperatures from 1 eV to 1 keV, will occupy much of our attention. Above this region is shown the plasmas that can be produced by the ultrafast lasers mentioned in Sect. 1.1. These plasmas are not in equilibrium, but do produce MeV temperatures in two senses. They produce beams of particles with exponential distributions having MeV energy scales, and they produce electrons that oscillate with MeV kinetic energies.

The astrophysical benchmarks in Fig. 1.1 include gamma-ray bursts (upper left), supernova progenitors (upper right), and several dark curves showing the evolution of astrophysical systems in time or space. At the top of the plot, the dark curve shows the evolution of the Big Bang, with later times moving down and to the left. Near the center of the plot, four curves trace the structure of various stars and planets from their core (at high temperature and density, to the upper right) toward their surface (to the lower left). One can see that the cores of these objects are in the high-energy-density regime, and that brown dwarfs and giant planets contain strongly coupled, Fermi-degenerate plasmas. Work at high-energy-density facilities clearly has the potential to produce knowledge relevant to the interiors of these astrophysical objects.

However, Fig. 1.1 also provides an incomplete picture of high-energy-density physics, because it assumes equilibrium. What is missing is dynamical processes. Phenomena such as shock waves, radiation waves, material ablation, radiative cooling, and hydrodynamic instabilities are not included. Much of Chaps. 4, 5, and 7 are concerned with these dynamical processes.

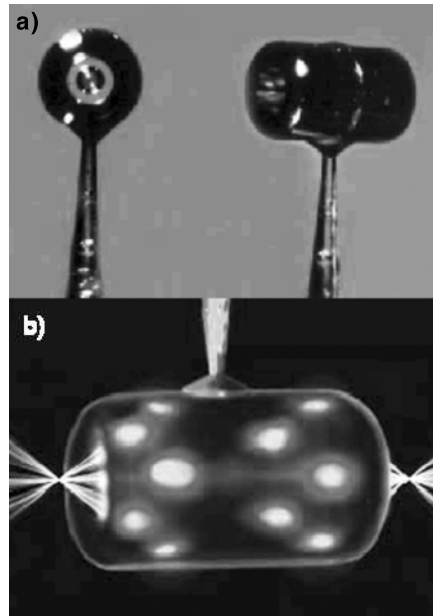
Dynamical processes are also essential to the production of high-energy-density conditions (Chap. 8), to the achievement of inertial fusion (Chap. 9), and to the simulation of astrophysical phenomena (Chap. 10). There is no simple plot that summarizes all this, but we can provide a summary introduction to inertial fusion and to experimental astrophysics before we proceed to our detailed task.

### 1.3 An Introduction to Inertial Confinement Fusion

We mentioned in Sect. 1.1 that inertial confinement fusion or ICF is the application that has driven much of the development of high-energy-density technology and science. Chapter 9 discusses ICF, beginning with fundamentals. ICF can produce a net energy gain because light elements release energy when they are combined to form heavier elements. This requires a high temperature, to overcome the Coulomb repulsion of the nuclei, but the energy released can be used to sustain the high temperature. This is very much like ordinary combustion, and so fusion fuel is said to ignite and to burn under the proper conditions. The applications of ICF will expand as the energy gain of ICF systems increases. Here energy gain is the ratio of the electricity used to produce an ICF event to the energy of the neutrons (or x-rays) it produces. At modest gains, even of order 1, ICF will produce large amounts of neutrons and/or x-rays that can be used for further areas of research. At larger gains (of order 10), the neutrons from ICF events might be used to breed fuel for electric power plants powered by nuclear fission. At large enough gains (of order 100), ICF events might be used directly as the energy source in electric power plants.

An ICF power plant would operate on a rhythmic cycle that has been compared to the cycle of an internal combustion engine. (The author is unsure whom to credit for this analogy.) Let us work our way through this cycle, discussing its elements. The cycle of an ICF power plant would begin with injection of an ICF *target* into a reactor chamber, in analogy to the injection of fuel into the cylinder of an engine. The target is a structure designed to produce energy by fusion when energy is delivered to it in a specific way. The target will include a *capsule* of fusion fuel (probably DT fuel – deuterium and tritium – and very likely spherical in shape) covered by a material known as an *ablator*. The target may also include other structures necessary for the operation of a specific fusion design. Fig. 1.2a shows an example. Here the spherical object is a fusion capsule while the cylindrical structure surrounding it is an object known as a *hohlraum* (see Chap. 8 and the discussion below). In Fig. 1.2a the target is mounted on a glass stalk; in a power plant it would be dropped into position, very likely while spinning to help maintain its orientation.

The phase in internal combustion that follows injection is compression of the fuel, generally by the motion of a piston. This provides energy to the fuel



**Fig. 1.2.** Inertial fusion targets. (a) A hohlraum target containing a fusion capsule. (b) Image of a thin-walled hohlraums showing where the laser beams strike. Credit: Lawrence Livermore National Laboratory

as it compresses the fuel. This phase exists in ICF, though it is somewhat more involved. The first phase is to deliver energy to the target with the required spatial distribution and uniformity. The device that provides the energy is called a *driver* in the jargon of ICF. It might be a laser beam or a particle beam or conceivably an intense photon source. The target might be driven by direct irradiation (a condition known as *direct drive*) or indirectly (known as *indirect drive*). Indirect drive might be accomplished, for example, through the conversion of the energy from the driver into some other form of energy such as thermal x-ray photons. The hohlraum of Fig. 1.2a converts laser energy into x-rays, producing an x-ray environment with a temperature of order 2 million degrees (200 eV). Fig. 1.2b, obtained with a thin hohlraum that allows one to see where the laser beams strike the interior walls, shows an example of such irradiation.

This delivery of energy to the ablator causes it to ablate away (hence the name). This, however, is not a passive process. The delivery of energy to the ablator produces temperatures of millions of degrees and pressures of order 100 Million atmospheres. High-velocity, ablated material is propelled away from the hot, high-pressure material at the surface of the ablator. In reaction, the remaining material and the fuel are accelerated inward. This process is identical to rocket propulsion, so that an ICF capsule is sometimes



described as “spherical rocket”. Evidently, the inward motion of the capsule will compress the fuel, thus completing the compression phase of the cycle. The fuel is compressed somewhat more than hydrocarbon fuels are, reaching a final density of 1,000 to several thousand times the density of liquid DT.

The next phase of the cycle in both internal combustion and ICF is ignition. One can have spontaneous ignition, as in diesel engines, or sparkplug ignition, as in gasoline engines. Both these approaches are possible in principle for ICF. On the one hand, one can design the target so that the fuel at the center of the imploding capsule ignites when the fuel stagnates against this central material. This is known as ignition from a central hot spot. On the other hand, one can compress the fuel and then use an external energy source to ignite it. This approach is known as *fast ignition*. At this writing, it is not clear which of these will prove at first most productive, or in the long run most practical. The ignition and subsequent burning of the fusion fuel creates a large quantity of energy, completing this phase of the cycle.

The next phase of the cycle is energy extraction. In internal combustion, this occurs as the expanding hot gas does work on the piston. It is more complicated in ICF, as neither neutrons nor x-rays, nor even high-energy particles, are able to push effectively on solid matter. They penetrate rather than push. Instead one must extract their energy in some other way. When ICF uses DT fuel, most of the energy emerges as neutrons, and the only known way to extract their energy is to use them to heat a large volume of matter. This hot matter can in turn heat water to drive a steam cycle, in which energy is extracted by driving large turbines with steam. This may or may not prove economical for electric power production. Steam cycles are not particularly efficient, so one may hope that in the long run one can use fusion fuel that produces only charged-particle output. This is more demanding but offers the potential of directly extracting the energy, with high efficiency. In the absence of new physics, the compact fusion plants that drive many spaceships in science-fiction would only be possible using these advanced fuels.

Returning our focus to high-energy-density physics, many elements of the ICF process depend upon such physics for their success. These include the production of the energy that drives the target, the delivery of energy to the ablator, the implosion process itself, the final properties of the fuel, and any attempt at fast ignition. ICF depends upon the properties of high-energy-density matter (Chap. 3), on the production of shock waves and related effects (Chap. 4), on limitation of hydrodynamic instabilities (Chap. 5), on the transport of radiation within the target (Chap. 6), and on the impact of radiation on material motions (Chap. 7). Furthermore, the basic approaches to ICF reflect the various options for producing high-energy-density conditions (Chap. 8). This makes it fairly clear why the science of high-energy-density physics grew out of ICF and its facilities, and why knowledge of this physics is essential if one is to deeply understand ICF.

## 1.4 An Introduction to Experimental Astrophysics

We made the point above that new sciences arise from new tools. A second and more specific example of this is the emergence of high-energy-density experimental astrophysics. One might say that the human brain as a tool gave rise to astronomy. Adam Frank has observed that the spectrometer can be argued to have given rise to astrophysics. In the same sense, the ability to do high-energy-density physics in the laboratory has given rise to this branch of experimental astrophysics. *Remington, Drake, and Ryutov* provide a review of work in this area through 2004. Some summary remarks follow here. Various examples are given throughout the text. In Chap. 10 we will take up the specific problem of doing experiments that are sensibly scaled from the astrophysical system to the laboratory.

The potential for contributions to astrophysics from high-energy-density physics is clear from Fig. 1.1. That figure leads one to focus on the possibility of measuring the equation of state, which for example might relate the pressure, density, and temperature of materials of astrophysical interest. At this writing, for example, it is not yet clear just when and how dense hydrogen enters its metallic state. This is particularly important for gas giant planets such as Jupiter. Figure 1.3 shows a pie-shaped slice to illustrate a segment of the spherical cross section of this planet. The equation of state of hydrogen determines whether Jupiter must have a rock core. The nature of the transition to the metallic state constrains how the dynamo in Jupiter produces its magnetic fields. This connection has been recognized for some time but equation of state experiments are difficult, as they require very high precision. By the 1980s one could begin to attempt measurements using the tools capable of such work at that time. These were principally devices called gas guns and rail guns that can accelerate slabs of material to high velocity, producing high pressure when they collide with other slabs of material. Until recently, such guns could not produce pressures as large as 1 Mbar. The emergence of pulsed-power devices that could drive 10 MA currents, and of lasers capable of creating very large pressures led at around the turn of the century to work at pressures of several Mbar.

The equation of state is one example of a property of matter in its equilibrium state that has implications for astrophysics. Another example is the x-ray opacity. Much of the energy transported within stars and other hot, dense objects is carried by radiation. The absorption of this radiation turns out to control some of the properties of these objects. Yet this radiation absorption is often dominated by the electronic transitions in various ions, especially those up to and including ions of iron. The measurement of the opacity of materials of astrophysical interest, and the comparison of these measurements with newly available computer codes that could calculate the opacities accurately, began in the early 1990s.

Measurements of equation of state and opacity carry out the research suggested naturally by Fig. 1.1, determining important equilibrium properties

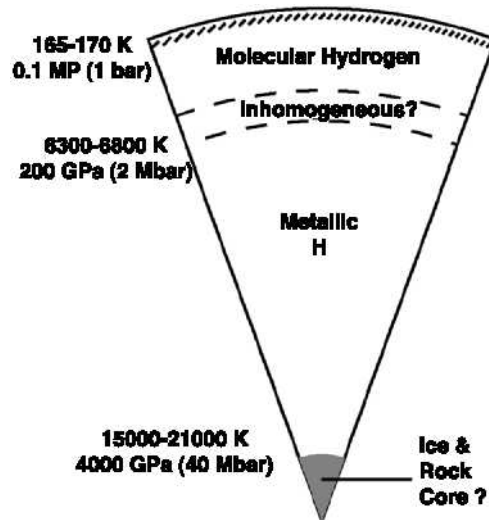
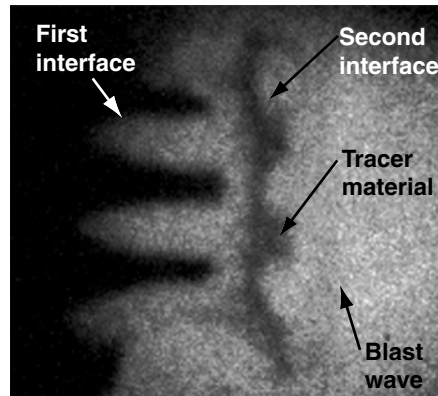


Fig. 1.3. Schematic interior of Jupiter

of materials. An explosion of work in laboratory astrophysics followed the realization, independently by Hideaki Takabe in Japan and Bruce Remington in the U.S., that one could also use high-energy-density tools to explore the large-scale dynamics that matters for astrophysics. To that point, our Earth-bound knowledge of astrophysical dynamics depended entirely on computer simulations that could be tested against one another but not against any benchmark data. Some of the resulting work is discussed further in Chap. 10. Here we provide a partial overview.

It is most straightforward to produce a well-scaled experiment in systems that are purely hydrodynamic, so that viscous dissipation, heat conduction, and radiation are negligible. This may seem like too much simplification to be worthwhile, but that is not the case. It turns out that many astrophysical systems behave mostly or entirely as hydrodynamic systems, including supernovae, some supernova remnants, blast waves, and some astrophysical jets. Some of these systems also very likely produce turbulence that is fundamentally hydrodynamic. In addition, the complex three-dimensional instabilities within these systems and interactions among them are beyond the capabilities of turn-of-the-century simulations to reproduce. This creates a role for experiments. Early work, around the turn of the century, focused on the unstable phenomena that occur during supernova explosions. Figure 1.4 shows an example. As a result of the passage of a blast wave like that in supernova, the structure at a first interface affected the evolution of a second interface. Other early experiments in this area also explored the simulation of processes



**Fig. 1.4.** Structure produced when a blast wave crosses two interfaces. The second interface is visible only where a diagnostic tracer is present behind it. There is little contrast across the blast wave itself, making it difficult to see

in supernova remnants, the dynamic behavior of hydrodynamic astrophysical jets, and the crushing of clouds by shock waves.

Once the transport of energy by radiation becomes important in the evolution of the matter, one has entered the regime of radiation hydrodynamics (Chap. 7). Important astrophysical phenomena in this regime include radiation waves, radiative shock waves, and radiative jets, not to mention the interiors of stars. Here, however, one nearly always loses the ability to do a precisely scaled experiment. Instead, one can hope to scale the essential dimensionless parameters so that the dynamics of interest to astrophysics are present in the laboratory. One example is the radiative jet, in which there are three key parameters: *(i)* are the internal Mach number (the ratio of the jet velocity to the sound speed of the material in the jet), *(ii)* the ratio of density in the jet to the ambient density around it, and *(iii)* the ratio of the distance along the jet required for significant cooling to occur to the radius of the jet. An experiment can in principle scale these three numbers. Such an experiment and an astrophysical jet would be expected to show qualitatively similar behavior, even if some detailed processes, such as the nature of the radiative cooling, were different. Similarly, for radiative shock waves there are three key parameters. The first two are the transmission of radiation by the regions ahead of and behind the density jump produced by the shock wave. The third parameter is the ratio of the radiation energy flux produced in the shocked matter just behind the shock to the material energy flux coming to the shock (in a frame of reference in which the shock is at rest). These parameters can also in principle be scaled from an experiment to an astrophysical case.

An additional area in which astrophysical processes can be simulated in high-energy-density experiments is the production of *photoionized plasmas*.

Most Earth-bound plasmas are produced by electron-impact ionization, including those in fluorescent light bulbs, in magnetized laboratory plasmas, and in laser-irradiated materials. This is also true of many astrophysical environments, such as the solar corona. However, there are a number of astrophysical environments in which the dominant source of plasma is photoionization. (This is also true of portions of the ionosphere above the Earth.) Indeed, since we depend primarily upon spectroscopy for the study of astrophysical objects, the data we get from very energetic environments such as neutron stars and the space near black holes come from photoionized plasmas. Without well-grounded knowledge of the properties of such plasmas, we cannot hope to interpret the data to see, for example, whether Einstein's theories of gravity accurately describe black holes. Beginning in the mid 1990s, experiments began to produce photoionized plasmas in well scaled environments and to measure their properties.

Finally, astrophysical environments often produce relativistic effects. These are many. A few examples include the production of electron-positron pairs in intense radiation environments, the alteration of atomic structure in very strong magnetic fields, the propagation of relativistic jets through magnetic fields, and the evolution of gamma-ray bursts, which seem likely to involve relativistic radiation hydrodynamics. At this writing, the use of ultrafast lasers to explore such phenomena has yet to begin, but seems likely to begin soon.

## 1.5 Some Connections to Prior Work

As indicated above, high-energy-density physics is a new field in some sense, having evolved out of some historical precursors. It might have emerged sooner in the absence of classification of fusion research to various degrees. However, it might not have, because the ability to do systematic experiments remains relatively recent. Even so, there are a number of historical books in related fields and notably in astrophysics that provide a deeper discussion of some of the material covered here. This section provides an overview of some of this work, oriented to the order of the following chapters. There are two areas for which no more detailed books are available. These are experimental astrophysics (Chap. 10) and relativistic high-energy-density physics (Chap. 11). These are both relatively new areas of research at this writing. Some review papers exist, no doubt soon to be superceded by others. One will have to wait for book-length treatments.

The properties of matter at high-energy-density (Chap. 3) could be addressed more thoroughly by a detailed study of the statistical mechanics of matter in the high-energy-density regime. Such a work does not exist but various references approach aspects of this problem. Many books in statistical mechanics address various fundamentals; one good example is Volume 5 of the series by *Landau and Lifshitz*. Even so, the specific issues that arise in

ionized media and otherwise at high energy density are not addressed there. The book on equations of state by *Eliezer, Ghatak, and Hora* considers some of these issues, as does the book on plasma spectroscopy by *Salzman*. They are also dealt with, on a fairly *ad hoc* basis, in the books to be highlighted later by *Zeldovich and Raizer* and *Mihalas and Mihalas*. *Griem* addresses several specific issues in his book on plasma spectroscopy, which is alas not so easy to read.

The behavior of shock waves and rarefactions (Chap. 4) is addressed to various degrees in a number of texts, notably including those by *Sedov* and by *Whitham*. Principal among such works, for problems of interest to high-energy-density physics, is the book by *Zeldovich and Raizer*. This book quite rightly has been venerated for many years. If there is a single text that might be said to be a precursor to the field of high-energy-density physics, it is this. It has also long been the best general introduction to high-energy-density physics, despite being too detailed in many respects and incomplete in others. The author hopes that the present text might take over this function. Even so, *Zeldovich and Raizer* will remain the next book of choice for details of one-dimensional hydrodynamics and approaches to the key physics based on similarity solutions.

In the area of hydrodynamic instabilities (Chap. 5), the work that stands clearly at the forefront in the analysis of hydrodynamic instabilities is that of *Chandrasekhar*. It is thorough, it is fundamental, and it works a number of problems of central interest to high-energy-density systems. Unfortunately, having been written in the 1950s this text does not discuss some aspects of compressible hydrodynamic instabilities that are also of substantial interest. For these, to go beyond what little is done in the present text one must head to the archival literature. In the specific area of incompressible hydrodynamic turbulence, the outstanding introductory text is by *Tennekes and Lumley* and the thorough and definitive tome is by *Hinze*.

For issues relevant to high-energy-density physics, radiative transfer (Chap. 6) and radiation hydrodynamics (Chap. 7) have never been treated in independent books, so we will discuss them together. The field of radiative transfer has many applications, as this kind of process is essential to the behavior of planetary atmospheres, stellar interiors, energetic astrophysical events, and high-energy-density laboratory systems. Chandrasekhar for example also contributed a book on radiative transfer. However, the book that is without question most relevant to radiative transfer and to radiation hydrodynamics in high-energy-density physics is that by *Mihalas and Mihalas*. The material on these areas in *Zeldovich and Raizer* is also relevant and is very insightful. Unfortunately, *Mihalas and Mihalas* is well known to be difficult to read. This may be in part because of the excellent connections it makes with the literature that existed when it was written. The author hopes that the present text, with its more pedagogical focus and simplified presentation, will provide a foundation that enables a better appreciation of

their thorough discussions. Because of its emphasis on the essential physics, the present text has steered clear of the issue that occupies most of the time of most of the people working in radiative transfer and radiation hydrodynamics, which is the discovery and implementation of computer algorithms that can produce practical approximate solutions of radiative transfer problems, either in isolation or in the context of radiation hydrodynamics. There is some discussion of these issues in *Mihalas and Mihalas*. In addition, *Castor* has published the definitive text in this area in 2004.

We now turn to experiments and applications. The production of high-energy-density systems touches on a number of areas of research. Some of these have many more-detailed books while others have none. There are many books, for example, on lasers, no one of which seems uniquely relevant. Much less is available on what happens when the laser strikes the target, with the principal reference being the text on laser-plasma interactions by *Kruer*. The book on Z pinches by *Lieberman et al.* has some material that is relevant to high energy density, and much that is not.

It is no surprise that more is available in the area of inertial fusion, as this has been the driving application. Two books in this area deserve specific mention. *Atzeni and Meyer-ter-Vehn* published in 2004 an extensive book whose focus is inertial fusion. This book takes up most of the topics covered here, and a number of topics not covered here, all in the context of how they impact inertial fusion as an application and a goal. It is heavy on formulae, including parametric fits to many complex relationships, and comparatively light on discussion and explanations. As a result, it will be a very useful tool for experts. An earlier book by *Lindl*, published soon after the declassification of fusion using hohlraums, has more of an engineering orientation. It addresses primarily the specific issues involved in producing fusion by this method, about which it includes much useful detail. It also includes a number of useful formulae based on studies by computer simulation. In a number of cases in Chap. 8 and Chap. 9, we compare results from a fundamental and simplified analysis with those reported by Lindl.

## 1.6 Variables and Notation

One difficulty with work that involves contributions from various disciplines is conflict of variable definitions. Certain symbols find very common use to mean different things. For example,  $\gamma$  is used as the ratio of specific heats throughout fluid dynamics but is also used extensively as the symbol for a growth rate in instability theory and as the Lorentz factor in relativity. The goal here has been to use a notation that was consistent throughout the text while keeping close to standard usage where feasible. For this reason, we have made extensive use of subscripts. Appendix A includes a list of variables used in the text, and of the symbols used for a number of common constants. This list is intended to include all variables that are used in more

than one section and many of the dimensional variables. It may not include some variables that are used solely within the context of a single derivation or a single section of the text, especially when these are nondimensional. As discussed at more length in the preface, the intent in the equations has been to either ensure that an equation has evident and consistent units, so that any system of units can be used, or to give the units used explicitly. In nearly all cases, we write equations involving a temperature as equations in energy units, representing the contribution of the temperature as  $k_B T$ , where  $k_B$  is the Boltzmann constant and  $T$  is the temperature.

In writing variables and equations, we have used italic text for scalar quantities and boldface text for vector quantities. For tensors of the second rank, we have used either underlined boldface symbols, such as  $\underline{\mathbf{P}}$ , or dyadic notation, such as  $\mathbf{uw}$ . (The element in the  $i$ th column and  $j$ th row of a tensor written in dyadic notation are  $(\mathbf{uw})_{ij} = u_i w_j$ ). In writing differential equations, we have explicitly written the derivatives as fractions rather than using a more compact notation.



## 2 Descriptions of Fluids and Plasmas

Not long ago, one could say that 99% of the known universe is plasma. The recent discovery of dark matter and dark energy imply that this may no longer be true, but it will remain the case that 99% of the readily observed universe is plasma. The interstellar medium, stars, and more exotic compact objects are all composed of or surrounded by ionized matter. Without understanding something about plasmas, one cannot hope to understand the universe.

It is equally true that knowledge of plasmas is essential to high-energy-density physics. To reach pressures above a megabar at densities of a few times solid density or smaller requires temperatures large enough to ionize the matter. Thus, in most high-energy-density systems the matter is in the plasma state. In various contexts and regimes, this plasma may behave like a simple fluid, like a traditional plasma, or like a plasma beyond the scope of traditional plasma theories. In addition, plasma behavior is essential to the use of lasers to produce high-energy-density systems (Chap. 8). When the behavior of high-energy-density systems departs from that of simple hydrodynamic fluids, either plasma effects or radiation effects are typically responsible. In addition, the models of use in describing plasmas are supersets of those used to describe simpler fluids. We discuss various approaches to describing plasmas and fluids here. Radiation and radiation effects are described in Chaps. 6 and 11.

If the reader has studied plasma physics, then parts of this chapter will be review. If the reader has studied electrodynamics or fluid dynamics, but not plasma physics as such, then other parts of this chapter will be review. In addition, our focus here is not at all to span fluid dynamics and plasma physics in a few pages, but rather to introduce the models we will need in the pages that follow. Moreover, we assume here that all motions are non-relativistic. Relativistic motions are considered in Chaps. 6 and 11.

### 2.1 The Euler Equations for a Polytopic Gas

In books on plasma physics, it is common to begin with collections of individual particles, to determine how to describe their behavior statistically using the Boltzmann equation, and then to average their behavior in ways that produce simpler models of plasma dynamics. Here we take the reverse path,

beginning with the very simple averaged equations that are useful in many high-energy-density contexts, and working our way toward more-complex descriptions that are more powerful but also less-often necessary. In this spirit, we begin with the Euler equations for a polytropic gas:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \mathbf{u} = 0, \quad (2.1)$$

$$\rho \left( \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) = -\nabla p, \text{ and} \quad (2.2)$$

$$\frac{\partial p}{\partial t} + \mathbf{u} \cdot \nabla p = -\gamma p \nabla \cdot \mathbf{u}, \quad (2.3)$$

where  $\mathbf{u}$ ,  $\rho$ , and  $p$  are the velocity, density, and pressure, respectively. Here (2.1) is the continuity equation, describing conservation of mass, (2.2) is the equation of motion, describing the change in momentum density, and (2.3) is the energy equation. Equation (2.3) assumes that the fluid is polytropic so  $p \propto \rho^\gamma$  where  $\gamma$  is the *adiabatic index* (the ratio of specific heats). *Landau and Lifshitz* report that “polytropic processes” is a historic term for processes in which pressure is proportional to some inverse power of volume. This is why a fluid or gas with  $p \propto \rho^\gamma$  is described as a polytropic gas, and why  $\gamma$  is often called the *polytropic index* or *polytrope*. For a fully ionized nonrelativistic gas (at a high enough temperature and a low enough density)  $\gamma$  is equal to 5/3; for a gas where radiation pressure is dominant,  $\gamma$  is equal to 4/3; for a diatomic molecular gas,  $\gamma$  is equal to 7/5. We discuss more general versions of these equations in Sect. 2.3. In some of the later discussion, especially of shock waves, it will prove useful to have the energy equation in conservative form, discussed next. In this form the energy equation is

$$\frac{\partial}{\partial t} \left( \rho \epsilon + \frac{\rho u^2}{2} \right) = -\nabla \cdot \left[ \rho \mathbf{u} \left( \epsilon + \frac{u^2}{2} \right) + p \mathbf{u} \right]. \quad (2.4)$$

Other useful forms of the energy equation are discussed later in this section.

Although it is not evident at first glance, all three of these equations are at root continuity equations, in which the total amount of something is changed only by sources (sinks are negative sources). They all can be written in “conservative form”, in which the change in the density,  $\rho_Q$ , of some quantity  $Q$  is determined by the flux of that quantity,  $\mathbf{I}_Q$ , having units of  $Q$  per unit area per unit time, and the net volumetric sources of that quantity,  $S_Q$ . The conservative form is then

$$\frac{\partial}{\partial t} \rho_Q + \nabla \cdot \mathbf{I}_Q = S_Q. \quad (2.5)$$

If one integrates such an equation over some volume with a surface,  $\sigma$ , and applies the Gauss’ law, one obtains

$$\frac{\partial}{\partial t}Q + \oint_{\sigma} \Gamma_Q \cdot d\mathbf{A} = \text{Net source in volume}, \quad (2.6)$$

in which the second term is the net flow of  $Q$  into or out of the volume and  $\oint_{\sigma}$  represents the integral over the closed surface  $\sigma$ .

### Homework 2.1

One approach to deriving the Euler equations is to identify the density, flux, and sources of mass, momentum, and energy and then to use (2.5). Do this for a polytopic gas and then simplify the results to obtain (2.1) through (2.3).

Equation (2.1) is often referred to as the continuity equation. It constrains the dynamics of a fixed amount of matter. One can recognize the second term on the left as the divergence of a flux. As a result, if one integrates this equation over a finite volume, the change of mass within the volume will equal the flow of mass into or out of the volume. If there were mass sources or mass sinks, these would appear on the right-hand side of (2.1). This equation, though simple, is a key factor in the complex behavior of hydrodynamic systems, because in many cases the variation of both  $\rho$  and  $\mathbf{v}$  is important. This makes (2.1) an essentially nonlinear partial differential equation, not readily solved by any analytic technique.

Equation (2.2) is the momentum equation, or more accurately is an equation derived from the calculation of the rate of change of momentum density. This specific equation applies when electric and magnetic fields, viscous momentum transfer, and radiative forcing are all negligible. The only remaining momentum source is the pressure gradient, which causes compression or decompression of the plasma. Its effect is represented by the term on the right-hand side. The second term on the left describes the convection of momentum.

An important simple application of (2.1) and (2.2), for which an analytic solution is straightforward, is that they describe small-amplitude acoustic waves. We consider this here as an example of the technique called “linearization”, because linearization will be important in numerous contexts later in the book. Linearization is possible when the variation of every variable in a problem can be described as a small deviation from a constant average value (which might be zero). The essence of linearization is the realization that terms that are linear in the small deviations are vastly larger than terms that are nonlinear in these quantities. Here we use the subscript 0 for the average values and the subscript 1 for the small deviations, and we also assume that  $p$  can be described as a function of  $\rho$ . Then with  $\mathbf{u}_0 = 0$ , (2.1) becomes

$$\frac{\partial \rho_1}{\partial t} + \rho_0 \nabla \cdot \mathbf{u}_1 + \mathbf{u}_1 \cdot \nabla \rho_1 + \rho_1 \nabla \cdot \mathbf{u}_1 = 0 \quad (2.7)$$

and (2.2) becomes

$$\rho_0 \left( \frac{\partial \mathbf{u}_1}{\partial t} + \mathbf{u}_1 \cdot \nabla \mathbf{u}_1 \right) + \rho_1 \left( \frac{\partial \mathbf{u}_1}{\partial t} + \mathbf{u}_1 \cdot \nabla \mathbf{u}_1 \right) = -\frac{\partial p}{\partial \rho} \nabla \rho_1. \quad (2.8)$$

Then the nonlinear terms, with products involving  $\rho_1$  and/or  $\mathbf{u}_1$ , can be discarded as small. To show this formally one should rework these equations so that every small quantity is expressed as a ratio that is actually small. This is left as a homework problem, as is the derivation from these two equations of the acoustic wave equation

$$\frac{\partial^2}{\partial t^2} \rho - \frac{\partial p}{\partial \rho} \nabla^2 \rho = 0. \quad (2.9)$$

Here the square of the sound speed,  $c_s$ , is  $c_s^2 = \partial p / \partial \rho$ , which equals  $\gamma p / \rho$  for a polytropic gas. The definition of the sound speed squared is often expressed as the partial derivative of pressure with respect to density at constant entropy. In more detail, this partial derivative is taken according to the properties of the system under study. If the fluctuations are adiabatic, then it is taken at constant entropy. If rapid heat transport keeps the temperature constant, then it is taken at constant temperature, and so on.

### Homework 2.2

Linearize the Euler equations to derive (2.7) and (2.8). Find appropriate divisors to make these equations nondimensional and discuss which terms are smaller than others. Then derive (2.9).

It is also useful preparation for later analysis to discuss the solutions of (2.9). A formal solution involves a sum over terms that include  $\exp[i(\pm \mathbf{k} \cdot \mathbf{x} \pm \omega t)]$ , in which the angular frequency of the oscillation is  $\omega$  and its wavevector is  $\mathbf{k}$  (whose magnitude  $k$  is related to the wavelength  $\lambda$  by  $k = 2\pi/\lambda$ ). Throughout this book  $\mathbf{x}$  and  $t$  are used as variables for position and time, with boldface indicating vector quantities. A formal solution also must include a consideration of how real physical quantities are to be related to the complex mathematics. (For a discussion of this last point see for example Chap. 6 of the electrodynamics text by *Jackson*). If, however, one either assumes that the variation in  $\rho$  is proportional to any one of the exponential forms, such as  $\exp[i(\mathbf{k} \cdot \mathbf{x} - \omega t)]$ , or if one takes the Fourier transform of (2.9), one finds the dispersion relation for acoustic waves:

$$\omega^2 - c_s^2 k^2 = 0, \quad (2.10)$$

from which one sees that  $c_s$  is both the phase velocity ( $\omega/k$ ) and the group velocity ( $\partial \omega / \partial k$ ) of these waves. Note that the fluctuating amplitude  $\rho_1$  cancels out of the dispersion relation. This quantity is thus not constrained by (2.1) and (2.2), until it becomes large enough that linearization becomes invalid.

**Homework 2.3**

Take the actual, mathematical Fourier transform of (2.9) to find (2.10).

**Homework 2.4**

Substitute, for the density in (2.9), the actual, mathematical Fourier transform of the spectral density  $\tilde{\rho}(\mathbf{k}, \omega)$ . Show how the result is related to (2.10).

Indeed, one aspect of the behavior of acoustic waves of larger amplitude is worth mentioning. This is *acoustic wave steepening*, encountered more often in theory than in practice. The crest of a large acoustic wave is at a higher density than the trough, and the sound speed is correspondingly larger. This has the effect that any given wave crest tends to overtake the trough ahead of it. This in turn causes a steepening of the “front” of the wave. In a system determined by the Euler equations, ordinary acoustic waves steepen until they become a series of abrupt increases in the plasma pressure and density connected by smooth decreases. In any actual plasma, finite viscosity will limit the steepening of the wave fronts.

Equation (2.3) explicitly describes the variation of the plasma pressure but in fact is the simplified equation obtained by calculating the rate of change of the energy density in the plasma. For any polytropic medium, the total energy density is proportional to the pressure and is given by  $p/(\gamma - 1)$ . In (2.3) the left-hand side describes the temporal and convective variation of the plasma pressure. In an incompressible fluid, for which  $\nabla \cdot \mathbf{u}$  is zero, this is the entire story of energy conservation. In a compressible fluid, in contrast, the work done during compression or decompression, during which  $\nabla \cdot \mathbf{u}$  is nonzero, is part of the flow of energy. An important additional point here is that (2.2) and (2.3) describe the pressure as a scalar quantity. In general the pressure is a tensor,  $\underline{P}$ , and what we write as  $\nabla p$  would in general be the vector given by  $\nabla \cdot \underline{P}$ . In most circumstances in high-energy-density physics, the concept of an isotropic scalar pressure applies very well. We allow for the tensorial nature of pressure when including viscous effects in Sect. 2.3. Other cases where the pressure is not scalar include work with solids, which can sustain shear stresses and other asymmetric internal forces, and with well-magnetized plasmas, in which case the behavior along the magnetic field is fundamentally different than behavior perpendicular to it.

Equation (2.3) also has an important relation to the entropy of the plasma. To be specific, we consider a polytropic gas, although the conclusion is more general than this. The specific entropy (the entropy per unit mass) of a polytropic gas,  $s$ , can be expressed (*Sedov*, p. 261) as

$$s = c_V \ln \left( \frac{p}{\rho^\gamma} \right) - c_V \ln \left( \frac{p_o}{\rho_o^\gamma} \right) + s_o, \quad (2.11)$$

in which  $s_o$  is the value of  $s$  in a reference state, for which  $p = p_o$  and  $\rho = \rho_o$ , and  $c_V$  is the specific heat at constant volume. If one evaluates the total

derivative of  $s$ ,  $Ds/Dt = \partial s/\partial t + \mathbf{u} \cdot \nabla s$ , one finds

$$\frac{1}{c_V} \frac{Ds}{Dt} = \frac{1}{p} \frac{\partial p}{\partial t} + \frac{1}{p} \mathbf{u} \cdot \nabla p + \gamma \nabla \cdot \mathbf{u}. \quad (2.12)$$

Equation (2.3) then implies that  $Ds/Dt = 0$ , so one concludes that specific entropy is conserved across regions of time and space where (2.3) is continuously valid. Note that entropy is not conserved across transitions with a discontinuous change in the fluid parameters, such as shock waves. In addition, heat transport or other dissipative processes lead to a change of entropy. In simulations of hydrodynamic systems that include dissipative processes, one can evaluate their importance by examining the change in entropy of the fluid elements. The ways in which these equations apply to shock waves are discussed in Chap. 4. (Note that the total derivative, discussed at more length in texts on fluid dynamics, is the rate of change of the some quantity within some specific parcel of fluid. It includes the inherent time dependence of the quantity and also the rate of change resulting from fluid motion in the presence of a spatial derivative.)

The energy equation can be expressed in other useful ways, which will matter for later chapters. If one expands (2.4) and collects all the terms involving derivatives of density, one finds that these terms sum to zero by the continuity equation. By taking the dot product of  $\mathbf{u}$  with (2.2), one obtains an equation for the mechanical energy in the plasma. Subtracting this from the energy equation, and substituting for  $\nabla \cdot \mathbf{u}$  from the continuity equation, one obtains what is sometimes called the gas-energy equation,

$$\left( \frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla \right) \epsilon - \frac{p}{\rho^2} \left( \frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla \right) \rho = 0. \quad (2.13)$$

In the presence of energy sources or heat transport, the right-hand side of this equation would not be zero. For an ideal gas, with  $\epsilon = p/[\rho(\gamma - 1)]$ , this equation reduces to a particularly useful form:

$$\left( \frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla \right) p - c_s^2 \left( \frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla \right) \rho = 0. \quad (2.14)$$

### Homework 2.5

Derive (2.14) from (2.1), (2.2), and (2.4).

## 2.2 The Maxwell Equations

Many simplified fluid equations have some electrodynamic component. To understand these, we will need the Maxwell equations of electrodynamics. They are written here for reference and to allow some discussion of Gaussian

cgs and other units. We write these equations assuming that the media are not inherently magnetized or electrically polarized, so that we can account explicitly for all charges and currents, and can take the polarization and magnetization fields to be zero. This is common in plasma physics but not in other areas where polarization electric field and material magnetization are useful concepts. We have Gauss' law,

$$\nabla \cdot \mathbf{E} = 4\pi k_1 \rho_c, \quad (2.15)$$

the absence of magnetic monopoles,

$$\nabla \cdot \mathbf{B} = 0, \quad (2.16)$$

Faraday's law

$$\nabla \times \mathbf{E} = -k_3 \frac{\partial \mathbf{B}}{\partial t}, \quad (2.17)$$

and Maxwell's generalization of Ampere's Law

$$\nabla \times \mathbf{B} = \frac{k_2}{k_1 k_3} \frac{\partial \mathbf{E}}{\partial t} + 4\pi \frac{k_2}{k_3} \mathbf{J}. \quad (2.18)$$

Here  $\rho_c$  is the charge density and  $\mathbf{J}$  is the current density. Here we have followed Jackson in expressing these equations in unit-independent form. In most applications in this text, the equations will be written in Gaussian cgs units, which turn out to produce convenient expressions for plasma phenomena. In such equations,  $B$  is in Gauss and other quantities are in cgs units. The constants are  $k_1 = 1$ ,  $k_2 = 1/c^2$ , and  $k_3 = 1/c$ . In these cases one tends not to be interested in the electric quantities – few researchers actually use statvolts/cm as a unit of electric field. The cgs unit of energy is the erg, which may be of use or may be converted to eV or keV ( $1 \text{ eV} = 1.6 \times 10^{-12} \text{ ergs} = 1.6 \times 10^{-19} \text{ J}$ ).

Researchers who need to calculate magnetic fields typically work with the SI version of Ampere law:

$$\frac{1}{\mu_o} \nabla \times \mathbf{B} = \epsilon_o \frac{\partial \mathbf{E}}{\partial t} + \mathbf{J}, \quad (2.19)$$

in which  $\mu_o = 4\pi \times 10^{-7} \text{ H/m}$ ,  $\epsilon_o = 10^7/(4\pi c^2) \text{ F/m}$  (Farads/m) with  $c$  in m/s,  $B$  is in Tesla,  $E$  is in V/m, and  $J$  is in  $\text{A/m}^2$  ( $\text{Amps/m}^2$ ). More generally, for SI units  $k_1 = 1/(4\pi\epsilon_o) \text{ m/F} = 10^{-7} c^2$ ,  $k_2 = \mu_o/(4\pi) = 10^{-7} \text{ H/m}$ , and  $k_3 = 1$ .

One also needs the Lorentz force, which acts on any charge  $q$  with velocity  $\mathbf{v}$  and is

$$\mathbf{F}_L = q \left( \mathbf{E} + \frac{\mathbf{v} \times \mathbf{B}}{c} \right)_{(\text{cgs})} = q (\mathbf{E} + \mathbf{v} \times \mathbf{B})_{(\text{SI})}, \quad (2.20)$$

in which the units are designated by the text in parentheses.

We will also find it convenient to work with the vector potential,  $\mathbf{A}$ , so that

$$\mathbf{B} = \nabla \times \mathbf{A} \quad (2.21)$$

and

$$\mathbf{E} = -\nabla\Phi - \frac{1}{c} \frac{\partial \mathbf{A}}{\partial t}, \quad (2.22)$$

in which the scalar potential is  $\Phi$ , using the Coulomb gauge, so  $\nabla \cdot \mathbf{A} = 0$ .

We will also need at times to work with the energy density and energy flux of electromagnetic field. In cgs units the energy densities  $W_E$  and  $W_B$  of the electric and magnetic fields, respectively, are

$$W_E = \frac{E^2}{8\pi} \quad (2.23)$$

and

$$W_B = \frac{B^2}{8\pi}. \quad (2.24)$$

Note that when one averages over many cycles of a fluctuating field to obtain an averaged energy density, these quantities are divided by 2. This is also the case for the Poynting flux,

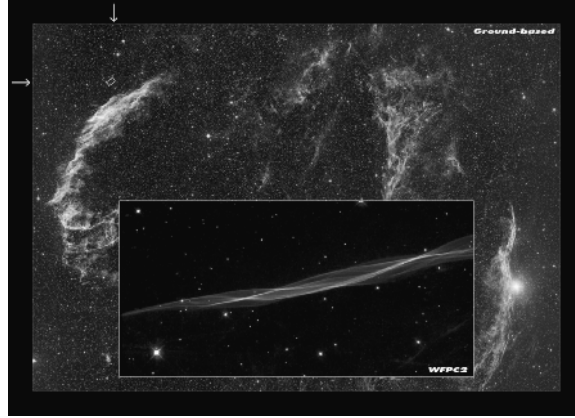
$$\mathbf{S} = \frac{v_g}{4\pi} \mathbf{E} \times \mathbf{B}, \quad (2.25)$$

in which the group velocity of the wave is  $v_g$ .

### 2.3 More General and Complete Single-Fluid Equations

Figure 2.1 shows an image of the Cygnus loop. This object, 6 times the size of the moon when viewed from the Earth, is the result of a supernova that occurred about 15,000 years ago. It features very thin, crenellated layers of matter. Spectroscopic imaging of the emission from different elements shows where they are produced. Analysis of these emissions has found that the various features are produced by shock waves and has revealed some of their properties. The Cygnus loop is of note here because it cannot be described using only the Euler equations. In this case, the phenomenon that is missing and that matters is radiative heat transport. There are many cases in which one or more phenomena, beyond the interplay of pressure and momentum, are essential to the behavior of a system of interest. Even so, one very often can ignore the fact that real plasmas include some combination of ions and electrons. A great deal of the behavior of plasmas, especially including high-energy-density ones, can be described by treating the plasma as a single fluid that can be charged, carry currents, and interact with radiation. In this section we discuss such single-fluid equations and a few specific limits of interest.





**Fig. 2.1.** The Cygnus loop supernova remnant. The background image is an optical image from a ground-based telescope. The inset, from the WFPC2 instrument on the Hubble Space Telescope, shows the very thin layer of emission (by hydrogen at 656.3 nm) produced by the upward-moving shock wave in the small box aligned with the arrows. The ground-based image of the Cygnus loop (shown in the background) measures  $3^\circ \times 2^\circ$  and was taken by CalTech with the Oschin Schmidt Telescope and scanned as part of the Digitized Sky Survey. Hubble Image credit: European Space Agency. [http://origins.jpl.nasa.gov/library/story/101100-a\\_ld.html](http://origins.jpl.nasa.gov/library/story/101100-a_ld.html)

### 2.3.1 General Single-Fluid Equations

We discussed, with reference to (2.4), the general structure of transport equations. This structure still applies here. The more complex element is that other sources and fluxes of any given quantity are considered. The problems of concern in this book do not involve interior mass sources, so the continuity equation,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \mathbf{u} = 0, \quad (2.26)$$

remains unchanged. The general transport equations for momentum and energy, in the nonrelativistic limit, are

$$\rho \left( \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) = -\nabla (p + p_R) + \nabla \cdot \underline{\boldsymbol{\sigma}}_\nu + \mathbf{F}_{\text{EM}} + \mathbf{F}_{\text{other}} \quad (2.27)$$

and

$$\frac{\partial}{\partial t} \left( \rho \epsilon + \frac{\rho u^2}{2} + E_R \right) + \nabla \cdot \left[ \rho \mathbf{u} \left( \epsilon + \frac{u^2}{2} \right) + p \mathbf{u} \right] = -\nabla \cdot \mathbf{H} - \mathbf{J} \cdot \mathbf{E} + \mathbf{F}_{\text{other}} \cdot \mathbf{u}. \quad (2.28)$$

Here (2.26) was used to simplify the left-hand side of (2.27); the more complex expression, involving the time dependence and divergence of the momentum density, is often more useful for computer simulations. The radiation pressure,

$p_R$ , and the viscous stress tensor,  $\underline{\sigma}_\nu$ , are further discussed below. The energy density of the radiation field,  $E_R$ , is usually ignorable and is also discussed further shortly. The reader may notice the lack of symmetry in (2.27), because the radiation pressure enters but the radiation momentum density does not. The radiation momentum density is insignificant except in relativistic systems, when it must be included. These equations can be derived either by taking moments of particle distribution functions or by reasoning about the behavior of small elements of fluid. Versions of these equations, including or excluding various specific source terms, can be found in any plasma physics or fluid dynamics text.

The force density due to the interaction of charges with the electromagnetic fields is

$$\mathbf{F}_{\text{EM}} = \rho_c \mathbf{E} + \frac{\mathbf{J} \times \mathbf{B}}{c} + \nabla p_R, \quad (2.29)$$

in which  $\rho_c$  is the density of charge, and as always throughout this book the electric field is  $\mathbf{E}$ , the magnetic field is  $\mathbf{B}$ , the current density is  $\mathbf{J}$ , and the speed of light is  $c$ . This form of the Lorentz force (with the  $c$ ) is expressed in Gaussian cgs units. These are often though not always the most convenient for practical work. Note that when a plasma is treated as a single fluid, the current flows within the plasma but the flow of current does not require motion of the single-fluid plasma. Of course, this is because the electrons carry nearly all the current but nearly none of the momentum. Similarly, the presence of a significant charge density does not require a significant accumulation of mass. The force density  $\mathbf{F}_{\text{EM}}$  is typically needed when intense laser light or other narrow-band radiation interacts with a plasma, and it will play a role in our discussions of laser-produced plasmas. In contrast, when thermal radiation interacts with dense plasmas,  $\mathbf{F}_{\text{EM}} \sim 0$ , because under these conditions the  $\mathbf{J} \times \mathbf{B}$  force, after appropriate analysis and averaging, equals  $-\nabla p_R$ . Writing (2.27) as we did is a compromise that allows us to avoid introducing new equations *ex nihilo* later in the text. In practice, we will take  $\mathbf{F}_{\text{EM}} = 0$  in radiation hydrodynamic systems and will let the terms involving  $p_R$  cancel when considering laser-plasma interactions.

The divergence of the energy flux,  $\mathbf{H}$ , which enters (2.28), is

$$\nabla \cdot \mathbf{H} = \nabla \cdot [\mathbf{F}_R + (p_R + E_R) \mathbf{u} + \mathbf{Q} - \underline{\sigma}_\nu \cdot \mathbf{u}], \quad (2.30)$$

in which the radiative energy flux is  $\mathbf{F}_R$  and the energy flux from thermal heat conduction is  $\mathbf{Q}$ . The penultimate term in (2.28),  $-\mathbf{J} \cdot \mathbf{E}$ , describes the volumetric heating by the currents driven by electromagnetic waves. This term is typically negligible except in any plasma penetrated by intense laser light. Note also that the terms on the left-hand side of (2.28) can be rearranged into any of the forms discussed previously, and in particular into the form of (2.14), with the addition only of source terms from the right-hand side of (2.27) and (2.28). Alternatively, by including all the terms involving radiation energy and radiation pressure one can develop an equation in the form of (2.13), with  $\epsilon$  replaced by  $(\epsilon + E_R/\rho)$  and  $p$  replaced by  $(p + p_R)$ , and with source terms on the right-hand side.

It helps develop understanding and intuition to discuss these equations while considering a dimensional analysis in which we identify a characteristic velocity of the system,  $U$ , and a characteristic dimension,  $L$ , which together give a timescale  $L/U$ . To make scaling arguments one replaces  $u$  by  $U$ ,  $\nabla$  by  $1/L$ , and  $(\partial/\partial t)$  by  $U/L$ . There is a sound reason for this. In any profile shaped as an exponential or linear function, the derivative is equivalent to division by whatever scale length is present in the profile. (In any power-law profile other than linear, the scale length is the distance variable itself, such as  $x$  or  $r$ , to within numerical factors.) These scale lengths are not the wavelength and frequency of local fluctuations but rather are the global scales that define the overall system evolution. Given this identification, one can say that the characteristic global, convective rate of change of momentum and energy are  $\rho U^2/L$  and  $\rho U^3/L$ , respectively. If one divides any given equation by the relevant one of these, then one obtains a dimensionless equation from which one can assess the relative contributions of the various terms.

The use of a scalar pressure,  $p$ , is a simplification that is usually justified. The exception is systems involving solid-state materials. In this case  $\nabla p$  must be replaced by the divergence of the pressure tensor. The material experiences forces, for example, when the force per unit area in one direction has a gradient in an orthogonal direction. The term involving pressure in the energy equation represents the work of compression or expansion, often referred to as  $p dV$  work. The fact that this work enters the energy equation in this way implies that the specific enthalpy,  $h = \epsilon + p/\rho$ , is often a useful variable in describing how hydrodynamic systems behave. If one takes the energy flux,  $\mathbf{H}$ , to be zero and assumes the medium to be a polytropic gas, one can recover (2.3) from (2.28). To evaluate the dimensionless scaling of the pressure term, one takes  $\nabla p \sim p/L$  and divides by  $\rho U^2/L$  to find its normalized amplitude, which is  $(p/\rho)/U^2$ . This is proportional to the inverse of the internal Mach number,  $U/c_s$ . Thus, pressure gradients are of decreasing importance as the internal Mach number increases.

The flow of heat is described in (2.30) as the divergence of a heat flux,  $\mathbf{Q}$ . The heat flux is very important in the heating of plasma by laser light, and in some of the phenomena observed in plasmas produced from gases at a low enough (less than atmospheric) pressure. It is not important in the behavior of plasmas at near solid density or (for reasons discussed in Chap. 10) in the behavior of most astrophysical plasmas. In many cases, the heat flux can be related to the gradient in fluid temperature,  $T$ , using an equation of state to relate  $T$  to  $p$  or  $\epsilon$ :

$$\mathbf{Q} = -\kappa_{\text{th}} \nabla T, \quad (2.31)$$

in which the coefficient of heat conduction is  $\kappa_{\text{th}}$ . In a stationary fluid in which only the temperature variation is important and  $\epsilon \propto T$ , this yields a diffusion equation,  $\partial T/\partial t \propto -\nabla^2 T$ , so the heat transport from such a description is essentially diffusive. For scaling arguments, it is useful to identify and calculate the kinematic coefficient of thermal diffusivity,  $\chi$ , which has purely

the dimensions of a diffusion coefficient (step size squared / collision time). The relation between  $\chi$  and  $\kappa_{\text{th}}$ , developed in *Landau and Lifshitz*, is

$$\kappa_{\text{th}} = \chi \rho c_p = \chi n k_B \gamma / (\gamma - 1), \quad (2.32)$$

in which  $c_p$  is the specific heat at constant pressure and  $n$  is the density of particles in the fluid, each of which is part of a distribution with the common temperature  $T$ . The second equality gives the result for a polytropic gas whose pressure is described by Boyle's law. The Boltzmann constant is  $k_B$ , which can be combined with  $T$  to give  $k_B T$  in energy units. In practical units, one has

$$\begin{aligned} \chi (\text{cm}^2 \text{s}^{-1}) &= 2 \times 10^{21} \frac{[T(\text{eV})]^{5/2}}{\ln AZ(Z+1)n_i(\text{cm}^{-3})} \\ &= 3.3 \times 10^{-3} \frac{A[T(\text{eV})]^{5/2}}{\ln AZ(Z+1)\rho(\text{g cm}^{-3})}, \end{aligned} \quad (2.33)$$

in which  $A$  and  $Z$  are the average atomic mass and ionic charge of the plasma ions,  $\ln A$  is the Coulomb logarithm discussed in Sect. 2.4, and the particle and mass density of the ions are  $n_i$  and  $\rho$ , respectively. These specific formulas are based on the analysis of processes dominated by Coulomb collisions in the book chapter by *Braginskii* (1965). Precise values in sufficiently dense plasmas might be different.

To evaluate the dimensionless scaling of the heat transport term, one finds  $\nabla \cdot \mathbf{Q} \sim \chi \rho (k_B T) / (A m_p L^2)$ , divides by  $\rho U^3 / L$ , and notes that  $k_B T / (A m_p) \sim U^2$ . The normalized amplitude of the heat transport term is thus the inverse of the *Peclet* number,  $Pe = UL / \chi$ . When  $Pe$  is large, heat transport can be neglected.

These equations include several terms describing the effects of radiation. Their derivation and more general forms are discussed in *Mihalas and Mihalas*. Here we define these terms and consider when they matter. In general, the radiative energy flux,  $\mathbf{F}_R$ .  $\mathbf{F}_R$  is in fact equal to  $\mathbf{S}$ , the Poynting flux, given by  $(\mathbf{E} \times \mathbf{B})c / (4\pi)$  (when we explicitly account for all charges and the group velocity of the light is  $c$ ), when  $\mathbf{S}$  is evaluated for all the radiation present. However, in practical applications one uses the traditional form of  $\mathbf{S}$  only when there are few waves in the problem, as for example in laser-plasma interactions. When there is broadband or line radiation from emission and absorption by dense plasma, one works instead with expressions for  $\mathbf{F}_R$  that formally represent the integral of the Poynting flux, averaged over appropriate time and spatial scales. For example, the radiative flux emitted by a blackbody at a temperature  $T$  is  $\sigma T^4$ . Fluids cool by emitting radiation. They emit blackbody radiation when they are sufficiently opaque. Otherwise, their cooling is often dominated by emission from atomic lines. We discuss this further in Chap. 6. The radiative energy flux is often significant in high-energy-density experiments. Note that  $-\nabla \cdot \mathbf{F}_R$  is the rate of absorption, per unit volume, of radiative energy by the fluid. In Chap. 8 we will consider cases in which the absorption of laser light or the absorption of x-rays are important.

The terms involving  $E_R$  and  $p_R$  are important much less often, and it is easy to show why. When the radiation and the fluid are in equilibrium with a temperature  $T$ , then one has

$$p_R = E_R/3 = 4\sigma T^4/(3c). \quad (2.34)$$

The ratio of radiation pressure to plasma pressure is of order

$$\frac{4m_p\sigma T^4}{3c\rho k_B T} = 0.05T^3/\rho, \quad (2.35)$$

in which the proton mass is  $m_p$  and on the right-hand side  $T$  is in keV and  $\rho$  is in  $\text{g}/\text{cm}^3$ . Plasmas often have densities of  $\sim 1\text{g}/\text{cm}^3$ , as does water, so one can see from (2.35) that temperatures above 1 keV are required for radiation pressure and energy to be important in the fluid dynamics. Radiation pressure is dominant over material pressure in the shocked material in supernovae (at somewhat lower density). The readers of this book may well be producing and studying radiation-dominated plasmas using facilities now under construction in the early 21st century.

Let us more formally explore the dimensionless scaling of the radiative terms. The normalized radiation pressure term, for radiation in equilibrium with the fluid, is  $p_R/\rho U^2$ , which is approximately the same ratio as in (2.35) (with  $k_B T/m_p \sim U^2$ ). The energy flux term in (2.30) is larger than the enthalpy ( $p_R + \epsilon$ ) term by roughly  $U/c$ , and has a normalized value of  $\sigma T^4/(\rho U^3) \sim m_p\sigma T^4/(\rho k_B T U) \sim 1/Bo$ , in which  $Bo$  is known as the Boltzmann number (see *Mihalas and Mihalas*) and is small when the energy flux due to radiation affects the dynamics significantly. Note that  $1/Bo$  is  $c/U$  times larger than the ratio in (2.35), which reflects the fact that radiative energy fluxes become significant at temperatures much lower than those required for radiative pressures to be significant. We give specific examples of this in Chaps. 6 and 7.

In some systems, the relative importance of radiation may need to be evaluated by other measures. One can construct a radiation Peclet number when one can identify a (kinematic) radiative thermal conductivity,  $\chi_r \sim \bar{\ell}c$ , where the mean free path  $\bar{\ell}$  might be due to bremsstrahlung interactions, to Compton scattering, or to atomic emission and absorption. Alternatively, one can compare the radiative cooling time, defined as the ratio of energy content to blackbody energy flux, to the hydrodynamic time,  $L/U$ . There is a further discussion of these points in Ryutov et al. (1999).

In general, fluids also possess internal friction. The collisions of the particles in the fluid resist its motion, a process known as viscosity. These effects are generally small for plasmas, but we will see when we discuss turbulence in Chap. 5 that they can have important consequences for the structures that develop. In general, collisional viscous effects create forces in a given direction due to gradients in velocity in orthogonal directions. This may be easiest to see by imagining a simple shear layer, in which the velocity is entirely in a

direction we label as  $z$ , but there is a gradient in velocity in the orthogonal direction,  $x$ . When collisions move particles in the  $x$  direction, they cause a net transport of momentum. This creates a force. The elements of the stress tensor are

$$\sigma_{vij} = \rho\nu \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3} \delta_{ij} \frac{\partial u_k}{\partial x_k} \right) + \zeta \delta_{ij} \frac{\partial u_k}{\partial x_k}, \quad (2.36)$$

in which  $\delta_{ij}$  is the Kronecker delta, the kinematic viscosity is  $\nu$ , and the second coefficient of viscosity (often ignorable) is  $\zeta$ . In the usual case that particulate viscosity dominates, the kinematic viscosity is approximately the mean free path squared divided by the collision time; the quantity  $(\rho\nu)$  is the *dynamic viscosity*. In vector notation, the stress tensor is

$$\underline{\sigma}_\nu = \rho\nu \left( \nabla \mathbf{u} + (\nabla \mathbf{u})^T - \frac{2}{3} (\nabla \cdot \mathbf{u}) \underline{\mathbf{I}} \right) + \zeta (\nabla \cdot \mathbf{u}) \underline{\mathbf{I}}, \quad (2.37)$$

in which  $\underline{\mathbf{I}}$  is the identity tensor and the superscript  $T$  designates the transpose. Most theories of turbulence are developed for incompressible fluids, which have  $\nabla \cdot \mathbf{u} = 0$ . In other cases, such as the damping of acoustic waves, the compressible terms are essential. The gradient of the viscous stress is experienced by the fluid as a force density.

The presence of the viscous stress also contributes to the energy content of the fluid. Energy is transported as the stressed fluid moves. The contribution of viscosity to the increase in total energy,  $(\nabla \cdot \underline{\sigma}_\nu) \cdot \mathbf{u}$ , includes both contributions to the rate of increase in internal energy, of the form

$$\rho\nu \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \frac{\partial u_i}{\partial x_j}, \quad (2.38)$$

and to the rate of increase in kinetic energy (the remainder). For readers interested in more detail, *Hinze* provides a derivation of this, while *Landau and Lifshitz* consider only the kinetic energy.

In practical units, the kinematic viscosity is dominated by the ions (for  $Z$  below about 6) and is given by

$$\begin{aligned} \nu_i (\text{cm}^2 \text{s}^{-1}) &= 2 \times 10^{19} \frac{[T(\text{eV})]^{5/2}}{\ln \Lambda \sqrt{AZ^4 n_i} (\text{cm}^{-3})} \\ &= 3.3 \times 10^{-5} \frac{\sqrt{A} [T(\text{eV})]^{5/2}}{\ln \Lambda Z^4 \rho (\text{g cm}^{-3})}. \end{aligned} \quad (2.39)$$

Here the definitions are those used for (2.33). In plasmas of a high enough temperature, the photon viscosity can be important (in this case one adds the kinematic viscosities). The kinematic photon viscosity (*see Jeans, Thomas*) is

$$\nu_{\text{rad}} (\text{cm}^2 \text{s}^{-1}) \approx \frac{\bar{l} c \sigma T^4}{\rho c^3} = 3 \times 10^{-9} \frac{A [T(\text{eV})]^4}{Z [\rho (\text{g cm}^{-3})]^2}. \quad (2.40)$$

Here  $\bar{\ell}$  is the photon mean-free path and other quantities have their standard definitions.

To evaluate the dimensionless scaling of the viscous effects in the momentum equation, one takes  $\nabla \cdot \underline{\sigma}_\nu \sim \rho\nu U/L^2$  and divides by  $\rho U^2/L$  to find the normalized amplitude, which is  $1/Re$ , where  $Re$  is the Reynolds number,  $Re = UL/\nu$ . (The viscous effects in the energy equation have this same scaling, as the extra factor of  $U$  from the viscous term in the energy equation divides out when one normalizes.) The Reynolds number is perhaps the most well-known dimensionless parameter, because it has proven very useful in characterizing qualitative regimes of turbulent behavior. When  $Re$  is large, viscous effects can be ignored in (2.27) to (2.30). However, turbulence phenomena may inherently involve viscous dissipation on some scale, a topic discussed further in Chap. 5.

The momentum and energy equations also include the electromagnetic force  $\mathbf{F}_{EM}$  and the plasma heating that results. The divergence of the radiative flux,  $-\nabla \cdot \mathbf{F}_R$ , which was already discussed, is the heating related to the  $\mathbf{J} \times \mathbf{B}$  force. The electrostatic force,  $\rho_q \mathbf{E}$ , produces volumetric heating that can be expressed as  $\mathbf{J} \cdot \mathbf{E}$ . In the simple case that the current is resistive and given by  $\eta \mathbf{J} = \mathbf{E}$ , where  $\eta$  is the resistivity, this power dissipation is also resistive and of magnitude  $\eta J^2$ . The heating need not be resistive in the general case, however. For problems with an electromagnetic component, one finds  $\mathbf{J}$ ,  $\mathbf{E}$ , and  $\mathbf{B}$ , in addition to the fluid quantities, by solving the Maxwell equations (see Sect. 2.3) in addition to the single-fluid equations. The dimensionless parameter that in most circumstances relates to the scaling of these forces is the magnetic Reynolds number, discussed in Sect. 2.3.2.

Finally, the force  $\mathbf{F}_{other}$  is present in (2.27) and (2.28) to allow for the inclusion of other forces such as gravity.

### 2.3.2 Magnetohydrodynamics

In many plasma applications the magnetic field plays an essential role in the dynamics, phenomena at frequencies a great deal larger than acoustic frequencies are unimportant, and all the distances that matter are a very large multiple of the ion orbit radius (see Sect. 2.5). In such cases, the fluid equations and the Maxwell equations can be reduced to a much simpler set of equations, known as the equations of magnetohydrodynamics, or the MHD equations. There are many cases in astrophysics for which MHD modeling can be highly valuable, including the study of stellar atmospheres, planetary magnetospheres, interplanetary and interstellar space, among others. Furthermore, the relativistic generalization of the MHD equations is effective for the description of pulsar magnetospheres, galactic jets, and other phenomena. In some of these systems the magnetic field is dynamically important and strongly affects the behavior of matter. Even when the magnetic field is not dynamically important, over large volumes it can contain substantial amounts of energy.

We will need to consider such “low-frequency” magnetic fields and their effects in some of the high-energy-density systems we discuss. The magnetic field is essential to the use of Z-pinches and the accelerators that drive them for high-energy-density physics. It is also important to the dynamics of relativistic high-energy-density systems.

The *MHD equations* in Gaussian cgs units are continuity,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \mathbf{u} = 0, \quad (2.41)$$

momentum,

$$\rho \left( \frac{\partial \mathbf{u}}{\partial t} \right) = -\nabla p + \frac{\mathbf{J} \times \mathbf{B}}{c}, \quad (2.42)$$

Ohm’s law,

$$\mathbf{E} + \frac{\mathbf{u} \times \mathbf{B}}{c} = \eta \mathbf{J}, \quad (2.43)$$

Faraday’s law,

$$\nabla \times \mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{B}}{\partial t}, \quad (2.44)$$

and Ampere’s law,

$$c \nabla \times \mathbf{B} = 4\pi \mathbf{J} \quad (2.45)$$

The terms dropped from the more general equations (specifically (2.16) and (2.18)) are small under the assumptions of MHD theory. The new equation here is Ohm’s law, which is expressed here under the same assumptions. Both a more general version of Ohm’s law and the conditions and variants of MHD theory are discussed in most texts on plasma physics.

This author is partial to the treatment in *Krall and Trivelpiece*. The resistivity is

$$\eta = \frac{\nu_{ei} m_e}{n_e e^2} = \frac{4\pi \nu_{ei}}{\omega_{pe}^2} \text{ sec}, \quad (2.46)$$

in cgs units; the conversion to SI units is  $1 \text{ sec} = 9 \times 10^9 \text{ ohm-m}$ . Here the plasma frequency  $\omega_{pe}$  and the electron-ion collision frequency  $\nu_{ei}$  are both defined in Sect. 2.4. The MHD equations are not a closed set of equations. As in the case of the Euler equations, an equation of state or some other assumption that permits one to relate density and pressure is necessary in order to solve these equations.

The implications of the MHD equations for motion of the magnetic field are important for the behavior of Z-pinches and of many astrophysical systems. If one substitutes for  $\mathbf{E}$  in Faraday’s law using Ohm’s law and replaces the current using Ampere’s law, one finds

$$\frac{\partial \mathbf{B}}{\partial t} = \frac{\eta c^2}{4\pi} \nabla^2 \mathbf{B} + \nabla \times (\mathbf{u} \times \mathbf{B}). \quad (2.47)$$

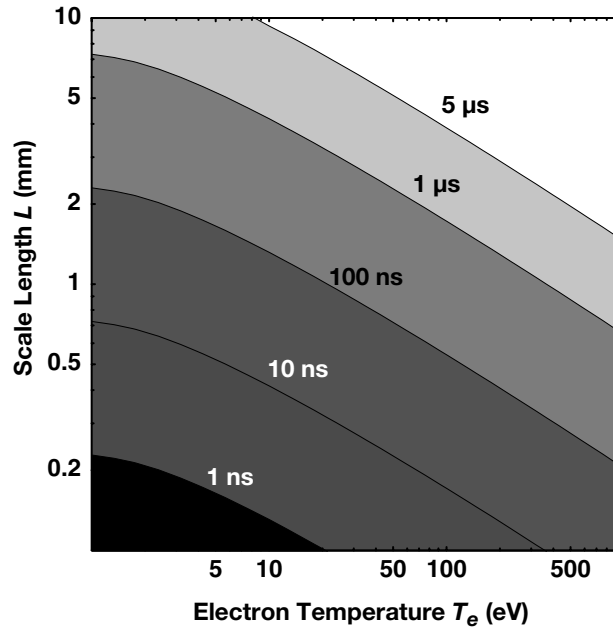
For a fluid at rest, with  $\mathbf{u} = 0$ , this is a pure diffusion equation for the magnetic field. (The notion that magnetic field can diffuse is confusing to



some students. It may help to note two things. First, the presence of a field implies the presence of current, generally carried by particles. Second, the particles are affected by collisions, which will tend to cause any current-carrying region to broaden.) In this case the magnetic diffusion time  $\tau_B$ , with a system spatial scale of  $L$ , is

$$\tau_B = \frac{4\pi L^2}{\eta c^2} = \frac{\omega_{pe}^2 L^2}{\nu_{ei} c^2} = 1.2 \times 10^{-8} \frac{L_{\text{mm}}^2 T_{\text{eV}}^{3/2}}{Z \ln \Lambda} \text{ sec}, \quad (2.48)$$

in which  $L_{\text{mm}}$  is  $L$  in mm and  $T_{\text{eV}}$  is  $T$  in eV, and the Coulomb logarithm  $\ln \Lambda$  is also defined in Sect. 2.5. The numbers implied by (2.48) are quite interesting and are illustrated in Fig. 2.2. Plasmas with sub-mm scale lengths tend to have 1–10 ns magnetic diffusion times, while plasmas with scale lengths of a few mm and temperatures above 100 eV have  $\mu\text{s}$ -scale diffusion times. High-energy-density systems that are magnetized can be found in either of these regimes.



**Fig. 2.2.** Contours of constant magnetic diffusion time, with contour levels from bottom to top of 1 ns, 10 ns, 100 ns, 1  $\mu\text{s}$ , and 5  $\mu\text{s}$ . For  $n_e = 10^{20} \text{ cm}^{-3}$  and  $Z = 0.63\sqrt{T_{\text{eV}}}$  (see Chap. 3)

If one uses the normalizing relations of the previous section to evaluate the dimensionless scaling of (2.48), one finds that the relative magnitude of the diffusion term scales with  $1/Rm$ , where  $Rm$  is the *magnetic Reynolds*

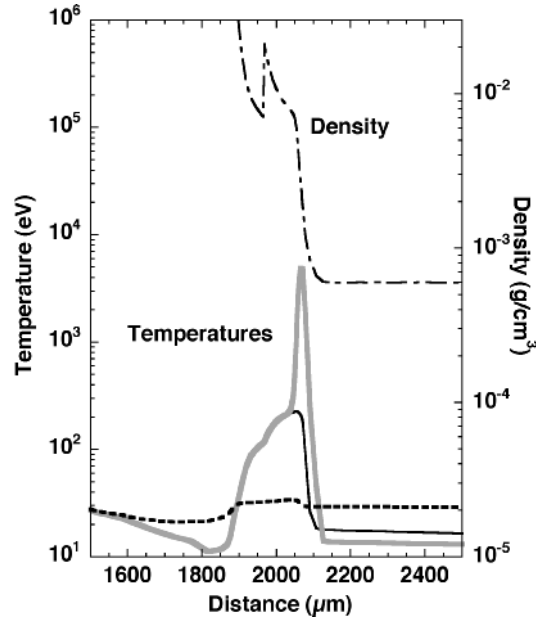
*number* defined as  $Rm = 4\pi UL/(\eta c^2)$ . The magnetic Reynold number is thus the ratio of the magnetic diffusion time to the hydrodynamic timescale ( $L/U$ ).

### 2.3.3 Single Fluid, Three Temperature

In principle one can identify a distinct “temperature” for the electrons, the ions, and the radiation. “Temperature” is in quotes here because this concept is routinely abused in practice in comparison to its pure definition in thermodynamics or statistical mechanics. The meaning of temperature in routine practice is “the value of the temperature of an equilibrium thermodynamic system that would have the same mean energy as that of the actual system being described.” The actual system, which might be an energy distribution of electrons or photons, typically is not in equilibrium and very often has an energy spectrum that departs significantly from the equilibrium energy spectrum. Identifying three temperatures in a plasma is a particularly paradoxical action, because the thermodynamic definition of temperature only strictly applies when they are all equal. Nonetheless, the “three-temperature” description of a single-fluid plasma is particularly useful, especially for computer simulations.

It is accurate to employ the single-fluid Euler equations with a single temperature when two conditions apply: radiation must be either negligible or dominant, and if radiation is negligible then the collisional coupling of the electron and ion temperatures must be strong. Under most circumstances in high-energy-density systems, the electrons are very strongly coupled to the ions by collisions, having the same temperature and a local density that is  $n_e = Zn_i$ , where  $n_i$  is the ion density and  $Z$  is the average charge. For any given density, as the temperature of a system increases the coupling of the electrons and the ions decreases. One of the first places one sees this is at a shock front, because shock waves directly heat the ions. The ion energy is shared with the electrons by collisions, and the electrons are in turn the primary source of radiation. Figure 2.3 shows how the ion temperature can deviate from the electron temperature at a shock front, after which collisions equalize the two over some distance.

In a three-temperature, single-fluid description, (2.1) and (2.2) are unchanged, although the pressure must now be determined by adding the contributions from the three species – electrons, ions, and radiation – independently. One then replaces (2.3) with one equation for each species. For each species, the temporal and convective rate of change of temperature (or perhaps energy density) are equal to the terms involving the sources of energy from the spatial flow of heat within the species, from exchanges of energy with other species, and from any external sources.



**Fig. 2.3.** At shock fronts, the ion, electron, and radiation temperatures may differ significantly. This shock wave is driven at  $\sim 260$  km/s through xenon gas of density  $0.006$  g/cm<sup>3</sup>. The ion temperature is shown in gray, the electron temperature is a solid black curve, and the radiation temperature is the dashed curve

### 2.3.4 Approaches to Computer Simulation

Because of its emphasis on conceptual descriptions, this book includes relatively little material on computer simulations. Yet it does include a number of examples produced from computer simulations, and many readers will proceed to work with extensively with them. The purpose of the present section is to provide some initial context regarding what computer simulations are and some alternative approaches to them. We will also see that these are limiting cases and that many other possibilities exist.

Those computer simulations of interest here seek to represent a physical system using a much smaller number of computational elements. When one includes the material particles and the photons (or electromagnetic properties, depending on the context), the simulations of interest in high-energy-density physics typically represent the behavior of systems having a number of elements within a few orders of magnitude of Avogadro's number. In contrast, at this writing computers are able to access of order 1 GByte per node and have of order 100 nodes. The implication is that the description of the physical system must be an approximate treatment of the behavior of aggregates of particles.

Fundamentally one desires that the simulation follow the evolution of the physical system in time and in space. The fundamental description of the system is always based on a set of differential equations like those we have discussed above. But the simulation must necessarily take one step backward in the calculus, and work with a *discretized* set of equations. This always involves dividing the system into components. In many cases these components are physical cells, in which case the distance between cells establishes the spatial increment used in defining derivatives. For example, if this distance in one direction is  $\delta x$ , then the derivative of pressure across the boundary from cell  $i$  to cell  $j$  is evaluated as  $\partial p/\partial x = (p(j) - p(i))/\delta x$ . We discuss some aspects of this below. Other aspects, such as whether to evaluate quantities in the center of a cell or at cell boundaries (and why), we leave to deeper discussions.

One also must establish the temporal increment used to determine how the value of the variables changes in time. This is known as a *timestep*. It must be small enough to give reasonably accurate dynamics yet large enough that the simulation will finish in a reasonable time. One constraint on the timestep is known as the *Courant condition*. The fastest wave of interest must cross no more than one cell in one timestep, otherwise the simulation will artificially retard the propagation of this wave. In cases of interest here, this wave is usually a sound wave. If it has speed  $c_s$ , then the Courant condition for the timestep is  $\delta t < \min(\delta x/c_s)$ .

A calculation (or portion of one) in which variables are advanced in time based on increments found from the differential equation using a timestep is described as an *explicit scheme*. Very fast waves, such as the light waves that make up the radiation, are often treated by an *implicit scheme*. In this case one realizes that the response of the radiation is very fast, reaching a steady state so quickly that the dynamic behavior does not affect the hydrodynamics. As a result, one can find new values of the hydrodynamic variables explicitly and then can solve for the new steady state of the radiation. This is only one example of the use of an implicit scheme. There are many other more complicated examples, but these are not common in high-energy-density physics.

The simulator also faces the problem of figuring out how to divide up the system of interest and deciding what equations to use to describe its evolution. The most natural choice might be to divide the space of interest into small regions. This is known as the *Eulerian* approach. For example, if the system is contained within a volume of a cubic mm, one might divide this volume into  $10^9$  cubic cells each  $1 \mu\text{m}$  on a side. One could describe the initial condition of the system by giving the variables a value for each cell.

On the other hand, we might choose to divide the matter into cells, so that each cell permanently followed the evolution of a given quantity of material. This is known as the *Lagrangian* approach. This would mean that the cells could move as the material moved. The equations solved in this case would not

be precisely those discussed above. Instead one would recast these equations using the Lagrangian mass variable, often written as  $m$ , and defined by  $dm = \rho dx$ .

Simple Eulerian codes have the strength that they can handle arbitrary motions of the material. Their main weakness is that they are inherently diffusive. Once material enters a cell it is treated as though it is spread evenly across the cell. Then in the next timestep some of this matter can move by another cell. The result of enhanced mass diffusion is that materials interpenetrate one another far faster than they would physically. The result of momentum diffusion is that the numerical viscosity is large, typically corresponding to a Reynolds number of about  $10^3$ . One way to minimize these effects is to locate many cells at regions where materials meet or where gradients are steep. This can be done by using a moving *grid* (the grid is the set of lines demarking the cell boundaries). A more advanced approach is to use an adaptive grid, which changes the distribution of the cells during the simulation. Examples of this at this writing include the FLASH hydrodynamic code developed at the University of Chicago and the BATS-R-US MHD code developed at the University of Michigan. A different advanced approach is to explicitly track any material interfaces in a problem and to treat them distinctly so that the materials do not interpenetrate. This is done for example by the FRONTIER code developed at SUNY Stony Brook.

Simple Lagrangian codes have the strength that they follow the motions of the actual material, allowing an accurate description of complex systems involving a number of components with different properties. Such systems are common in high-energy-density experiments. Such codes allow *no* diffusion, which is usually a strength but can at times be a weakness. They are outstanding tools for one-dimensional modeling of experiments. They have a major weakness in two or three dimensions, however, because they cannot follow swirling (vortical) motions. When the material tries to form a vortex, it tries to send matter from one zone through another zone. This could result in overlapping zones but usually causes one corner of a cell to overtake another corner, so that the new cell is no longer rectangular but instead looks like a twisted rectangle (called a “bow tie” because it looks like one). One example of a Lagrangian code, used to produce a number of figures in this book, is the commercially available code HYADES.

Simulators have been inventing improvements on these techniques for decades; the description above is necessarily sparse. One can for example make a code that incorporates both Lagrangian and Eulerian elements. Examples of such codes are the RAGE code developed at Los Alamos National Laboratory and the CALE code developed at the Lawrence Livermore National Laboratory.

In addition, beyond these core techniques the treatment of various specific physical mechanisms can be a major challenge. For example, accurately treating the absorption of laser light requires implementing additional physical

models along the lines of those described in Chap. 8. As another example, the treatment of thermal radiation or line radiation is a major issue in all these codes. We discuss some aspects of describing radiation in Chap. 6. The book by *Castor* addresses at length the difficult problem of treating radiation and hydrodynamics computationally.

A very different alternative, not useful for hydrodynamic systems but useful for the relativistic behavior of small number of particles is the *particle in cell* or PIC approach. In a PIC code one solves for the electric and magnetic fields using a discretized system of differential equations in some way, but one approximates the particles by using sample particles taken to represent the behavior of all the particles within a sphere of one Debye length (see Sect. 2.4) in radius. One explicitly follows the motion of all these sample particles within the environment of the electric and magnetic fields. A variation on PIC, usually described as a hybrid code, is a code that treats the electrons and perhaps some of the ions as a fluid, treating the remaining ions as PIC particles.

## 2.4 Plasma Theories

High-energy-density systems are nearly always plasmas, in the sense that they are ionized and that electromagnetic interactions at a distance can play a role in their dynamics, at least in principle. Unfortunately, the theory of plasmas, as covered in traditional texts such as *Krall and Trivelpiece*, has a range of validity that only partly overlaps the regimes of high-energy-density physics. Even so, plasma concepts have tremendous utility when they are valid. This motivates a discussion of these issues, before we consider briefly two traditional plasma descriptions: the two-fluid equations and kinetic theory.

### 2.4.1 Regimes of Validity of Traditional Plasma Theory

Traditional plasma theory faces the challenge of describing a system composed of mobile charged particles and capable of dramatic electrodynamic effects. The particles quickly scurry over to surround any exposed charge, yet also can carry currents that produce magnetic fields which can store immense energy. The eruptions on the surface of the sun are an example of the potential consequences. The shielding of exposed charges is one of the fundamental aspects of plasmas. Yet even as the charges try to cluster about one another, their thermal motions limit the clustering. The competition between these gives rise to a characteristic shielding distance, known as the *Debye length*. The Debye length is defined in Gaussian cgs units by

$$\lambda_D^{-2} = 4\pi e^2 \left( \frac{n_e}{k_B T_e} + \sum_{\alpha} \frac{n_{\alpha} Z_{\alpha}^2}{k_B T_{\alpha}} \right), \quad (2.49)$$

in which the sum is over all ion species, the subscript  $e$  designates electrons while  $\alpha$  designates an ion species,  $n$  is a number density,  $T$  is a temperature,  $Z$  is a number of unit charges,  $k_B$  is the Boltzmann constant, and  $e$  is the electronic charge ( $4.8 \times 10^{-10}$  statcoul here). On the one hand, when one considers fast enough timescales, the ions cannot move and the *electron Debye length*,

$$\lambda_{De} = \sqrt{\frac{k_B T_e}{4\pi n_e e^2}}, \quad (2.50)$$

(in the same units) becomes relevant. This is the only Debye length defined in the *NRL Plasma Formulary*, among other references. In addition, traditional plasma texts often assume all plasmas to be pure hydrogen, replacing the 4 with an 8 in (2.50). On the other hand, there are cases in dense plasmas when ion-ion shielding determines the behavior, as for example when the electrons cluster poorly because they are Fermi degenerate (Sect. 3.1.3). Then the *ion Debye length*,

$$\lambda_{Di}^{-2} = 4\pi e^2 \sum_{\alpha} \frac{n_{\alpha} Z_{\alpha}^2}{k_B T_{\alpha}}, \quad (2.51)$$

(in the same units) comes into play.

High-energy-density plasmas, like most plasmas, are quasi-neutral, so that

$$n_e = \sum_{\alpha} n_{\alpha} Z_{\alpha}. \quad (2.52)$$

In addition, in such plasmas collision rates are large (Sect. 2.4.2) so the temperatures of the particle species are usually equal and designated by  $T$ . When this is the case, one can use the standard definition of the effective charge,  $Z_{\text{eff}}$ , as

$$Z_{\text{eff}} = \frac{\sum_{\alpha} n_{\alpha} Z_{\alpha}^2}{\sum_{\alpha} n_{\alpha} Z_{\alpha}} = \frac{\sum_{\alpha} n_{\alpha} Z_{\alpha}^2}{n_e}, \text{ to write} \quad (2.53)$$

$$\lambda_D = \sqrt{\frac{k_B T}{4\pi n_e (1 + Z_{\text{eff}}) e^2}}, \quad (2.54)$$

again in Gaussian cgs units. This is a form we will use in later discussions. For calculations involving binary collisions,  $Z_{\text{eff}}$  is the appropriate average charge, while for calculations involving particle counting,  $Z = n_e/n_i$  is the appropriate average charge.

The Debye length arises quite naturally in the most-sophisticated developments of plasma theory. It also can be found from a simple calculation that can be used to highlight the limitations of traditional plasma theory for us. We consider a two-species plasma, in which the ions have charge  $Z$ , and we also suppose that the particles are distributed by classical statistics with a common temperature  $T$ . This implies that the density of particles with charge  $q$ , at a location with a potential  $\phi$  relative to the potential at some reference

location is proportional to  $\exp[-q\phi/(k_B T)]$ . Then the charge density  $\rho_c$  in the vicinity of an ion at  $\mathbf{x} = 0$  is

$$\rho_c = Ze\delta(0) - n_e e \exp\left[\frac{e\phi}{k_B T}\right] + n_i e Z \exp\left[\frac{-eZ\phi}{k_B T}\right]. \quad (2.55)$$

If we assume that  $|q\phi| \ll k_B T$  and that the plasma is quasi-neutral, then this becomes

$$\rho_c = Ze\delta(0) - \frac{e^2\phi}{k_B T} (n_e + n_i Z^2) = Ze\delta(0) - \frac{\phi}{4\pi\lambda_D^2}. \quad (2.56)$$

At this point we can write Poisson's equation in spherical coordinates, assuming that the charges are distributed with spherical symmetry, as

$$\frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{d\phi}{dr} \right) = -4\pi Ze\delta(0) + \frac{\phi}{\lambda_D^2}, \quad (2.57)$$

which (in cgs units) has the solution

$$\phi = \frac{Ze}{r} e^{-r/\lambda_D}. \quad (2.58)$$

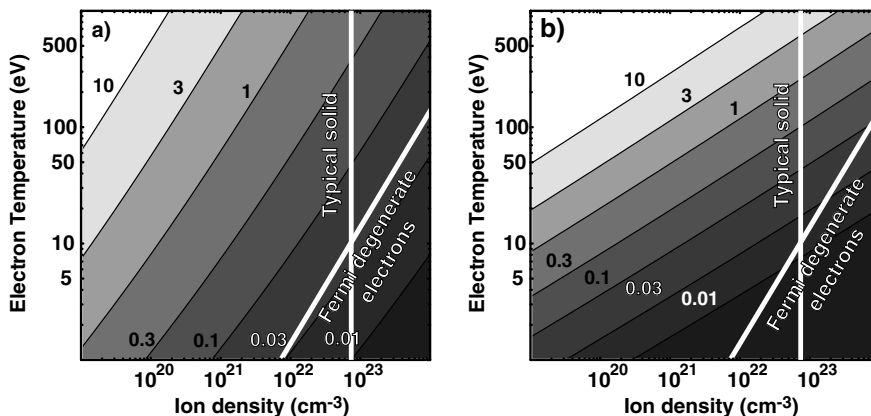
### Homework 2.6

Generalize the above derivation to a plasma with an arbitrary number of ion species, each of which may have a distinct temperature.

Equation (2.58) displays the standard result that the potential of any given charge falls away exponentially faster in a plasma than it would in vacuum. But what is relevant to our interests are two aspects of this derivation. First, (2.55) only makes sense in the end if there are numerous particles within a sphere whose radius is the Debye length. Second, the key assumption in this derivation is that  $|q\phi| \ll k_B T$ , which must be violated if the particles are cold enough. These turn out to be related, and we will explore them in turn.

The number of particles in a Debye sphere, in a quasi-neutral plasma, is  $n_e(1 + 1/Z)(4\pi/3)\lambda_D^3$ . The inverse of this, sometimes defined without the numerical coefficients, is a fundamental expansion parameter for traditional plasma theory (see *Krall and Trivelpiece*). A plasma is known as an *ideal plasma* when the number of particles in a Debye sphere can be taken to approach infinity. In this case collective effects, involving all the particles, remain, while effects relating to particle correlations vanish. Figure 2.4 shows the number of particles in a Debye sphere in the high-energy-density regime. There are not many. The number varies from tens of particles in the upper-left corner of the regime shown to less than 0.01 particles in the lower-left corner. In the lower-left corner the electrons are Fermi degenerate (Sect. 3.1.3), which





**Fig. 2.4.** Contours of the number of particles in a Debye sphere. The contours show 0.01, 0.03, 0.1, 0.3, 1, 3, and 10 particles, and increase to the upper left. (a) A high- $Z$  plasma with  $Z = 0.63\sqrt{T_{eV}}$  (see Chap. 3). (b) A low- $Z$  plasma with  $Z = 4$

reduces even further their ability to shield the ions. The ion density in a typical solid is also shown. It is evident from this figure that high-energy-density plasmas are almost never ideal plasmas.

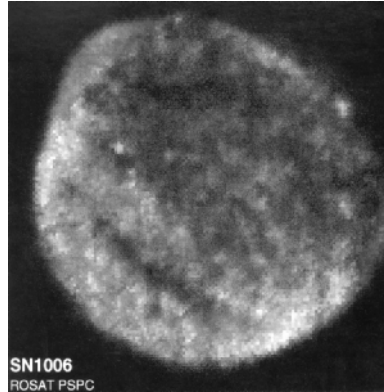
Now consider the assumption that  $|q\phi| \ll k_B T$ . We can take a typical value of  $\phi$  to be the electrostatic interaction of two particles at their average spacing. We find the average spacing by giving each particle a spherical volume of radius  $r_{av}$ , so that the average spacing is  $2r_{av}$ . Thus we take  $4\pi r_{av}^3/3 = 1/[n_e(1 + 1/Z)]$ . Then we find

$$\phi = \frac{k_1 Z e}{2r_{av}} = \frac{k_1 Z e}{2} \left( \frac{3}{4\pi n_e (1 + 1/Z)} \right)^{1/3}, \quad (2.59)$$

so that the assumption becomes

$$\frac{|q\phi|}{k_B T} = \frac{k_1 Z e^2}{2r_{av} k_B T} = \frac{k_1 4\pi n_e (1 + Z) e^2}{2k_B T} \frac{Z/(4\pi)}{r_{av} n_e (1 + Z)} = \frac{\lambda_D}{6r_{av}} g = \frac{g^{2/3}}{6} \ll 1, \quad (2.60)$$

where  $g$  is the inverse of the number of particles in a Debye sphere,  $1/g = n_e(1 + 1/Z)(4\pi/3)\lambda_D^3$ . Thus, the two requirements of the Debye-shielding analysis are intimately connected. It is no surprise that this assumption (2.60) is violated over about half the parameter space shown in Fig. 2.4. The ratio  $|q\phi|/k_B T$  is often known as the *strong coupling parameter*,  $\Gamma$ . *Salzman* discusses this parameter, which he calls the *plasma coupling constant*, at more length. Like the Debye length,  $\Gamma$  comes in different flavors depending upon whether one evaluates ion–ion coupling, ion–electron coupling, or electron–electron coupling. To be precise, one must evaluate  $Z$  and  $r_{av}$  for a specific, chosen set of particles. The most common type of  $\Gamma$  found in the literature is that for ion–ion coupling. Across much of the parameter space of Fig. 2.4, the ions are strongly coupled but the ions and electrons are not. In plasmas



**Fig. 2.5.** SNR 1006. X-ray image from the ROSAT satellite, the remnant of the widely observed supernova from A.D. 1006. The image convolves emission in several energy bands. The brighter emission at the upper left and lower right is attributed to cosmic ray acceleration. Image courtesy of University of Leicester, X-ray Astronomy Group <http://wave.xray.mpe.mpg.de/rosat/calendar/1997/jul>

with many electrons per ion, the result is that the ions will be well shielded from one another and their nominal strong coupling will not be so important. Even so, the pressure and energy of the plasma will depart from their ideal-gas values across much of this regime. We discuss this further in Sect. 3.3.2.

#### 2.4.2 The Two-Fluid Equations

Figure 2.5 shows the supernova remnant SNR 1006. At the edges of this image one can see the shock wave produced as the disturbance caused by the remnant propagates outward into the interstellar medium. As a first level of description, one can treat this shock wave as a hydrodynamic structure using the Euler equations, as we discuss further in Chaps. 4 and 5. And there are weaker shocks and other phenomena within the solar system that can be modeled with fair accuracy using the MHD equations. But in fact these are collisionless shocks. Structures in the magnetic and electric fields are essential to their existence. The electrons and ions interact with them very differently. In addition, there is a third group of particles that often are analyzed as a separate species – the energetic ions accelerated by such shocks, some of which eventually become cosmic rays. This is an astrophysical example of a system in which a multi-fluid treatment is essential to obtain an accurate description. In the laboratory, multi-fluid treatments are also required to describe the interaction of an intense laser with a plasma, discussed further in Chap. 8. Here we discuss two-fluid models, the scaling of collisional coupling, and the electron plasma oscillations that occur in most plasmas.

As density decreases or temperature increases, the collisional coupling of the electrons and ions becomes smaller. Eventually the electrons and ions

begin to act independently, and some phenomena appear in which there are important differences in their densities. Most of these phenomena can be successfully described using continuity and momentum equations like the following for each of the species in the plasma:

$$\frac{\partial n}{\partial t} + \nabla \cdot (n\mathbf{u}) = 0, \quad (2.61)$$

and

$$mn \frac{\partial \mathbf{u}}{\partial t} + mn\mathbf{u} \cdot \nabla \mathbf{u} = nq \left( \mathbf{E} + \frac{\mathbf{u}}{c} \times \mathbf{B} \right) - \nabla p - \sum_l mn(\mathbf{u} - \mathbf{u}_l)\nu_{jl}, \quad (2.62)$$

in which we would add a subscript (often  $e$  for electrons and  $i$  for ions) to  $n, v, m, p$ , and  $q$  to correspond to each species,  $\mathbf{E}$  and  $\mathbf{B}$  are the electric and magnetic fields, respectively, and we discuss the sum shortly. Equation (2.61) is obvious, but there are some new features in the right-hand side of (2.62) by comparison to (2.3.2). The electromagnetic effects now appear in the form of the complete Lorentz force density,  $nq(\mathbf{E} + \mathbf{u} \times \mathbf{B}/c)$  in Gaussian cgs units. As a result, and unlike the case of the MHD theory, the two-fluid theory describes phenomena in which there is a dynamic or static electric field. Examples include plasma oscillations, discussed below, and the Debye sheath that forms at bodies immersed in plasmas. In addition, the final term on the right-hand side describes the momentum exchange with other species. The sum is over the other species in the plasma, designated by  $l$ , and the rate of momentum loss through interaction of the species described by the equation and the other species is  $\nu_{jl}$ . (Here  $j$  would be replaced by  $e, i$ , or some other designation as appropriate.) We discuss this momentum loss, or drag, term further below.

Successful analysis using these equations depends upon having a qualitative sense of the differences between electron and ions. (I have yet to see a student in a qualifying exam who did not know the approximate ratio of electron to ion mass, but I regret to report that I did encounter one student who seemed to have no sense of the implications of this. He did not pass.) Two very important points are that the electrons nearly always move much faster than the ions, but the momentum of an ion is huge compared to that of an electron. A related point is that the radius of the ion orbits in a magnetic field is much larger than that of the electron orbits.

A simple implication of (2.61) and (2.62) is the presence of electron plasma waves in plasmas with weak enough collisions. Suppose that the final term in (2.62) can be ignored and that we are looking for very fast, fluctuating phenomena so that the ion density can be assumed to be fixed and unvarying. Also suppose there is no ambient  $\mathbf{E}$  or  $\mathbf{B}$ . Then by linearizing these two equations, taking the partial derivatives in time of the first and the divergence of the second, then simplifying, we find

$$\frac{\partial^2 n_{e1}}{\partial t^2} = \frac{n_{e0}e}{m_e} \nabla \cdot \mathbf{E}_1 + \frac{1}{m_e} \nabla^2 p_{e1}. \quad (2.63)$$

**Homework 2.7**

Derive (2.63).

This particular equation helps one see the physics of the wave we are finding. It is a purely longitudinal wave like an acoustic wave, in which the fluctuating electric field and compression by the electron pressure both cause the electron density to vary. The first term on the right-hand side can be evaluated from Poisson's equation (Sect. 2.2), which gives in this case

$$\nabla \cdot \mathbf{E}_1 = 4\pi(Zen_{io} - en_{eo} - en_{e1}), \quad (2.64)$$

in which the first two terms in parentheses cancel because the plasma is quasi-neutral. Then assuming the electrons behave as a polytropic gas with index  $\gamma_e$ , we obtain a wave equation

$$\left( \frac{\partial^2}{\partial t^2} + \omega_{pe}^2 - \frac{\gamma_e p_{eo}}{n_{eo} m_e} \nabla^2 \right) n_{e1} = 0, \quad (2.65)$$

in which we have introduced the electron plasma frequency,

$$\omega_{pe} = \sqrt{4\pi e^2 n_{eo} / m_e} = 5.64 \times 10^4 \sqrt{n_{eo}} \text{ rad/s}, \quad (2.66)$$

with  $n_{eo}$  in  $\text{cm}^{-3}$  and using Gaussian cgs units. Equation (2.65) describes waves known as electron plasma waves. By comparison with the derivation of acoustic waves in Sect. 2.1, one can see that the pressure term in (2.65) will introduce terms involving the wavenumber into the dispersion relation. For high-frequency waves which involve adiabatic compression of the electrons, there is only one degree of freedom and  $\gamma_e = 3$ . This result can also be confirmed using kinetic theory (Sect. 2.4.3). Then with  $p_{eo} = n_{eo} k_B T_e$ , where  $T_e$  is the electron temperature, one finds from 2.4.17 a dispersion relation

$$\omega^2 - \omega_{pe}^2 - 3 \frac{k_B T_e}{m_e} k^2 = 0. \quad (2.67)$$

Equation (2.67) is generally known as the Bohm–Gross dispersion relation. In the limit that  $T_e$  or  $k$  are small, one obtains the so-called cold-plasma oscillations, with  $\omega = \omega_{pe}$ . This emphasizes that plasma with weak collisions tend to sustain oscillations at  $\omega \sim \omega_{pe}$ . The discussion here is introductory. A lot more can be said about electron plasma waves and their interactions with other waves. The reader who needs to work with these waves seriously should consult plasma-physics books on their damping and laser–plasma-interactions books on their interactions.

We now return to the final term in (2.62) and discuss collisional momentum exchange between species. Note that this term gives a rate of change of momentum that is measured with respect to the momentum of the designated species. As a result, the coefficient  $\nu_{jl}$  is not symmetric in the exchange of  $j$  and  $l$ . This is trivial to visualize, if one imagines for example that one throws

bowling balls into a room full of bouncing ping-pong balls. The effect of the ping-pong balls is to make tiny, and perhaps negligible changes in the momentum of the bowling balls relative to their initial momentum. In contrast, the bowling balls make enormous changes in the momentum of those ping-pong balls that interact with it. Mathematically, with  $b$  for bowling balls and  $p$  for ping-pong balls, one can see that  $\nu_{bp} \ll \nu_{pb}$ . Similarly, for electrons and ions in (2.62),  $\nu_{ie} \ll \nu_{ei}$ . In fact, the final term in (2.62) is nearly always negligible in the ion equation, but often important in the electron equation. Let us consider further  $\nu_{ei}$ , which enters the electron equation.

First recall some of the fundamental relations involving collisional interactions. If particles of type  $a$  and density  $n_a$ , having a single, fixed relative velocity,  $v_{ab} = |\mathbf{v}_a - \mathbf{v}_b|$ , are interacting with particle of type  $b$  and density  $n_b$ , and the interaction cross section at this velocity is  $\sigma_{ab}$ , then the mean free path for this interaction is

$$\lambda_{mfp} = 1/(n_b\sigma_{ab}), \quad (2.68)$$

the interaction time is  $1/(n_b\sigma_{ab}v_{ab})$ , and the interaction rate is

$$\nu_{ab} = n_b\sigma_{ab}v_{ab}. \quad (2.69)$$

In many cases, including the one of interest here, the interaction cross section depends upon  $v_{ab}$  and  $v_{ab}$  is not fixed. In this case, describing the distribution in velocity of the two species by distribution functions  $f_a(\mathbf{v}_a)$  and  $f_b(\mathbf{v}_b)$ , normalized to unity so that e.g.  $\int f_z(\mathbf{v}_a)d\mathbf{v}_a = 1$  with the integral over all velocities, one has in general

$$\nu_{ab} = n_b \int \int f_b(\mathbf{v}_b)f_a(\mathbf{v}_a) \times \sigma_{ab}(|\mathbf{v}_a - \mathbf{v}_b|) \times |\mathbf{v}_a - \mathbf{v}_b| d\mathbf{v}_a d\mathbf{v}_b. \quad (2.70)$$

Next consider some of the specific properties of Coulomb collisions in plasmas. They involve the interaction of particles in the presence of an inverse-square-law force. This is the Rutherford scattering, with results that are typically derived in either classical mechanics or plasma-physics texts. The force between two isolated particles extends to infinity, but the presence of other particles creates a shielding effect, so that the collision only has an effect until the particle separation reaches a distance comparable to this shielding distance. The net result is that the cross section for momentum transfer is

$$\sigma_{ab} = 4\pi \ln \Lambda \left( \frac{q_a q_b}{m^* v_{ab}} \right)^2, \quad (2.71)$$

in which  $m^*$  is the reduced mass  $m_a m_b / (m_a + m_b)$  and  $\ln \Lambda$  is the Coulomb logarithm, which accounts for the effects of shielding. We refer the reader to any plasma-physics text for a partial discussion of  $\ln \Lambda$ , and to the book by *Shkarofsky et al.* for a complete one. Such discussions are lengthy, as a number of factors must be considered. In addition, in high-energy-density

systems the shielding distance often becomes so small that  $\ln \Lambda$  approaches its limiting small value of order 1. In high-energy-density research, it is generally sufficient to take

$$\ln \Lambda = \text{Max} [1, \{24 - \ln(\sqrt{n_e/T_e})\}], \quad (2.72)$$

with  $n_e$  in  $\text{cm}^{-3}$  and  $T_e$  in eV.

The most important point about (2.71) is that the cross section is proportional to  $1/v_{ab}^4$ , so that the contribution to the overall rate at each velocity is proportional to  $1/v_{ab}^3$ . Thus, pairs of particles having low relative velocities dominate the effects of Coulomb collisions, and high-velocity particles contribute little. One consequence of this is that all Coulomb processes, from momentum exchange to ionization to excitation, become much weaker as the plasma temperature increases.

The net result of the integral in (2.70), giving the change of electron momentum by interaction with ions, when evaluated for Maxwellian distributions of particles, is

$$\nu_{ei} = \frac{1}{3(2\pi)^{3/2}} \frac{Z\omega_{pe}^4}{n_e v_e^3} \ln \Lambda = 3 \times 10^{-6} \ln \Lambda \frac{n_e Z}{T_e^{3/2}} \text{ (1/s)}, \quad (2.73)$$

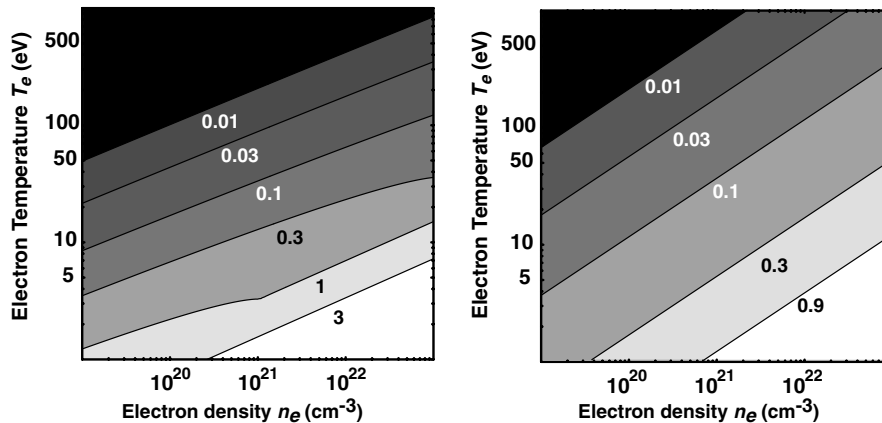
in which  $v_e = \sqrt{T_e/m_e}$  and on the right  $n_e$  is in  $\text{cm}^{-3}$  and  $T_e$  is in eV. Figure 2.6 shows contours of constant  $\nu_{ei}/\omega_{pe}$  as a function of  $n_e$  and  $T_e$ . Two different evaluations of  $Z$  are used. Part (a) shows results for  $Z = 3.5$ , typical of low- $Z$  materials such as plastic. Part (b) shows results for  $Z = 0.63\sqrt{T_e}$ , typical of higher- $Z$  materials as is discussed in Chap. 3. Wherever this quantity exceeds 0.1, any electron plasma wave will damp within 10 cycles, i.e., on femtosecond timescales. (The drag term at the end of (2.62) introduces a term proportional to  $\partial n/\partial t$  into the wave equation, which in turn introduces an imaginary term into the dispersion relation so that the implied frequency is no longer purely real, which leads to damping.) One can see that under most high-energy-density conditions this damping is very strong.

### Homework 2.8

Derive a replacement for (2.65), keeping an appropriate version of the drag term at the end of (2.62).

One is at times interested in the collisional mean free path. One may need for example to assess whether heat transport might matter or to compare the size of a computational zone to this distance. This can be estimated as

$$\lambda_{mfp} = v_e/\nu_{ei} = (1/\nu_{ei})\sqrt{T_e/m_e}. \quad (2.74)$$



**Fig. 2.6.** Curves of constant collisionality,  $\nu_{ei}/\omega_{pe}$ , as labeled, for low- $Z$  (left) and high- $Z$  (right) plasmas

### 2.4.3 The Kinetic Description

All of the above equations are strictly correct only if the velocity distributions of all the particles are Maxwellian. This means that the number of particles of species  $s$  within an interval  $dv$  around  $v$  is given by

$$f_s(v) = \left( \frac{m_s}{2\pi k_B T_s} \right)^{3/2} \exp\left( \frac{-m_s v^2}{2k_B T_s} \right), \quad (2.75)$$

in which the temperature and mass of species  $s$  are  $T_s$  and  $m_s$ , respectively. Any velocity distribution must be properly normalized. In this case the normalization is

$$\int f_s(\mathbf{v}) d\mathbf{v} = 1. \quad (2.76)$$

The reader should note that it is common for  $f_s(v)$  to be normalized to 1, as shown here, or to the particle density,  $n_s$ . In the literature the specific normalization is often not defined; one can even find papers that switch normalizations in the course of their work. If the particle distributions are not Maxwellian, but the interactions of interest are determined by the average energy that they carry, then the fluid and other equations above are accurate, though perhaps with some changes in the value of some coefficients and certainly with a nonthermodynamic definition of “temperature”, as discussed in Sect. 2.3.3.

In addition, there are circumstances in which it is an energetic “tail” on the distribution function that produces the phenomena of interest. At low energies the distribution usually has a Maxwellian shape. Whenever there are waves or instabilities that affect the particles, one frequently sees a surplus

of particles at energies above the thermal energy. This is the *tail*. Laser-plasma instabilities, discussed in Chap. 8, typically produce such exponential tails (though not for reasons that are well understood). In space systems and astrophysics, one typically encounters power-law tails. An important example of this is the distribution of cosmic rays, whose flux falls as  $1/v^3$ , implying a distribution function scaling as  $1/v^4$ . In other cases of interest, such as the transport of heat into a target (Sect. 8.1.5), the structure of the tail is more complicated.

When distributions of energetic particles, or any other deviation from Maxwellian distributions, are important to the dynamics of interest, then to investigate their effects one must work with the Boltzmann equation:

$$\frac{\partial f_s}{\partial t} + \mathbf{v} \cdot \nabla f_s + \frac{\mathbf{F}}{m_s} \cdot \nabla_v f_s = \left( \frac{\delta f_s}{\delta t} \right)_C, \quad (2.77)$$

in which  $\mathbf{F}$  is the sum of all forces acting on each particle and  $\nabla_v$  is the gradient operator in velocity space, sometimes written as  $\nabla_v = \partial/\partial\mathbf{v}$ . The symbol  $\mathbf{F}$  is typically the Lorentz force,  $q(\mathbf{E} + (\mathbf{v}/c) \times \mathbf{B})$ , but also would include any other forces that are present. Equation (2.77) is fundamentally a continuity equation relating the local rate of change of  $f_s$  in time, the flow of  $f_s$  within the six-dimensional phase space of  $\mathbf{x}$  and  $\mathbf{v}$ , and the source of  $f_s$  on the right-hand side. Particles suddenly appear in an element of velocity space as a result of collisions, that the right-hand side of (2.77) is the rate of change of  $f_s$  due to collisions. This term is discussed in detail in the book by *Shkarofsky et al.* When the right-hand side is zero, (2.80) is known as the Vlasov equation. It effectively describes many phenomena in collisionless plasmas.

## 2.5 Single-Particle Motions

Magnetic fields play a small role in most nonrelativistic high-energy-density systems. Even so, at the boundaries of such systems magnetic fields can be important. In addition, they are important in some astrophysical systems that one might hope to understand with the aid of high-energy-density experiments. Moreover, by using high-energy-density tools and working at lower mass density, one might hope to examine some issues in energetic, magnetized plasmas. And finally, relativistic high-energy-density systems often inherently involve magnetic fields. For all these reasons, understanding some simple aspects of particle motion in magnetic fields is important for those who work with high-energy-density systems. Those readers who have studied plasma physics have this knowledge. For those readers with no background in plasma physics, this section is included. In Chap. 11, some aspects of relativistic motion are discussed.

The motion of a single particle is in general described by Newton's second law, which in nonrelativistic form reads



$$m \frac{d\mathbf{v}}{dt} = q \left( \mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B} \right) + \mathbf{F}, \quad (2.78)$$

in which the particle velocity, mass, and charge are  $\mathbf{v}$ ,  $m$ , and  $q$ , respectively, and the nonelectromagnetic forces are designated by  $\mathbf{F}$ . Because of the cross product in the Lorentz force, it makes sense to write  $\mathbf{v}$  as a sum of components perpendicular and parallel to the magnetic field,  $\mathbf{B}$ , as

$$\mathbf{v} = \mathbf{v}_\perp + \mathbf{v}_\parallel. \quad (2.79)$$

The equation of motion along  $\mathbf{B}$  is then

$$m \frac{d\mathbf{v}_\parallel}{dt} = q\mathbf{E}_\parallel + \mathbf{F}_\parallel, \quad (2.80)$$

in which the  $\parallel$  subscript designates the component of a vector that is parallel to  $\mathbf{B}$ . Likewise the  $\perp$  subscript designates components perpendicular to  $\mathbf{B}$ .

The motion perpendicular to  $\mathbf{B}$  is more complex but fortunately can be separated into distinct elements. We find the first of these by assuming  $\mathbf{E} = \mathbf{F} = 0$ . The equation of motion perpendicular to  $\mathbf{B}$  is then

$$m \frac{d\mathbf{v}_\perp}{dt} = q \frac{\mathbf{v}_\perp}{c} \times \mathbf{B}. \quad (2.81)$$

One sees that the derivative of  $\mathbf{v}_\perp$  is inherently perpendicular to  $\mathbf{v}_\perp$ , and is constant in magnitude. This circumstance describes circular motion. One traditionally identifies the center of the circle as a line of magnetic field, and says that the particle “orbits” this field line. The radius of this *orbit* is known as the Larmor radius, and is given by an equation that sets the force involved in circular motion equal to that from (2.81), as follows:

$$\frac{mv_\perp^2}{r_L} = q \frac{v_\perp B}{c}, \quad (2.82)$$

which gives

$$r_L = \frac{mv_\perp}{qB/c} \text{ (cgs)} = \frac{mv_\perp}{qB} \text{ (SI)}. \quad (2.83)$$

The frequency of the orbit is known as the gyrofrequency and (in radians per second) is given in cgs units by  $qB/(mc)$  or in SI units by  $qB/m$ .

To find the next element of the particle motion, suppose  $\mathbf{E}$  is nonzero and write the particle velocity as

$$\mathbf{v} = \mathbf{v}_g + \mathbf{v}_d + \mathbf{v}_\parallel, \quad (2.84)$$

in which we understand that the gyromotion is fully included in  $\mathbf{v}_g$  and that the vector  $\mathbf{v}_d$  describes the new, “drift” motion due to the electric field but still perpendicular to  $\mathbf{B}$ . In this case the equation of motion becomes

$$m \frac{d\mathbf{v}_d}{dt} = q \left( \mathbf{E} + \frac{\mathbf{v}_d}{c} \times \mathbf{B} \right). \quad (2.85)$$

To find the effect of  $\mathbf{E}$  in the direction perpendicular to  $\mathbf{B}$ , we cross this equation with  $\mathbf{B}$ , obtaining with the aid of a vector identity

$$\frac{m}{q} \frac{d\mathbf{v}_d}{dt} \times \mathbf{B} = (\mathbf{E} \times \mathbf{B}) + \frac{\mathbf{v}_d}{c} B^2. \quad (2.86)$$

The solution of this equation, for constant fields so  $\mathbf{v}_d$  is constant in time, is

$$\mathbf{v}_d = c \frac{(\mathbf{E} \times \mathbf{B})}{B^2} \text{ cgs.} \quad (2.87)$$

Thus, the particles drift in a direction perpendicular to both  $\mathbf{E}$  and  $\mathbf{B}$ . An interesting aspect of this behavior is that both positive and negative particles drift in the same direction. A similar derivation shows that the drift velocity associated with an arbitrary force  $\mathbf{F}$  is

$$\mathbf{v}_d = \frac{c}{q} \frac{(\mathbf{F} \times \mathbf{B})}{B^2} \text{ cgs.} \quad (2.88)$$

If such a force is not charge-dependent, then the resulting drift will drive a current.

### Homework 2.9

Find the sizes and directions of the particle orbits. Explain from fundamental laws of electromagnetics why their direction is as it is. Show pictorially why the  $\mathbf{E} \times \mathbf{B}$  drift moves particles in the same direction.

In the common event that the magnetic field has a spatial gradient, this also produces a drift, known as the *grad B drift*. In the typical case the gradient is small along  $B$ , so that

$$\nabla B \cdot \mathbf{B} = 0. \quad (2.89)$$

Identifying the magnetic field at the center of the orbit as  $\mathbf{B}_o$ , the field locally experienced by the particle is then

$$\mathbf{B} = \mathbf{B}_o + \mathbf{r}_L \cdot \nabla B, \quad (2.90)$$

so that the equation of motion perpendicular to  $\mathbf{B}$  for  $\mathbf{E} = \mathbf{F} = 0$  is

$$\frac{d\mathbf{v}_\perp}{dt} = \frac{q}{m} \frac{\mathbf{v}_\perp}{c} \times \mathbf{B}_o \left( 1 + \frac{\mathbf{r}_L \cdot \nabla B}{B_o} \right). \quad (2.91)$$

If we now write  $\mathbf{v}_\perp$  as

$$\mathbf{v}_\perp = \mathbf{v}_d + \mathbf{v}_g = \mathbf{v}_d + \frac{q}{m} \mathbf{r}_L \times \mathbf{B}_o, \quad (2.92)$$

where we have written the orbital velocity in terms of  $\mathbf{r}_L$ , then (2.91) becomes, by substitution and using a vector identity,

$$\frac{d\mathbf{v}_\perp}{dt} = - \left( \frac{qB_o}{m} \right)^2 \mathbf{r}_L \left( 1 + \frac{\mathbf{r}_L \cdot \nabla B}{B_o} \right) + \frac{q}{m} \mathbf{v}_d \times \mathbf{B}_o. \quad (2.93)$$

This is still an instantaneous equation, but we are in fact interested in the average behavior over many particle orbits. We average this equation over an orbit, noting that

$$\langle \mathbf{r}_L (\mathbf{r}_L \cdot \nabla B) \rangle = \frac{1}{2} r_L^2 \nabla B. \quad (2.94)$$

Then by taking the cross product of the averaged equation with  $\mathbf{B}$ , as before, we find the grad-B drift velocity,

$$\mathbf{v}_d = \frac{1}{2} \frac{mv_\perp^2}{q} \frac{\mathbf{B} \times \nabla B}{B^3}. \quad (2.95)$$

This concludes our brief summary of charged-particle drifts in magnetic fields.

One other aspect of particle motion is worth mentioning. Because  $\nabla \cdot \mathbf{B} = 0$ , any change in the magnitude of  $\mathbf{B}$  along the direction of the initial field line is also accompanied by a change in some other component of  $\mathbf{B}$ . The simplest example occurs when initially straight magnetic field lines are squeezed together, for example by a magnetic coil, producing an inward radial component to  $\mathbf{B}$ . The Lorentz force due to this second component of  $\mathbf{B}$  either accelerates or decelerates the particle in its orbit and in its motion along  $\mathbf{B}$ . Since magnetic forces do no work, this does not change the total energy of the particle, but it does redistribute the energy between motion along  $\mathbf{B}$  and orbital motion.

One can analyze the behavior in more than one way. On the one hand, one can consider explicitly the forces on the particle and determine the particle motion. On the other hand, one can note that the particle has a magnetic moment because its motion represents the flow of current around a circular loop, and can determine the effects of the electric field induced around the loop due to changes in  $B$ . (This second calculation is easier to do.) In either case, one finds that the magnetic moment of the particle remains constant, a result that can be expressed as

$$\frac{1}{2} \frac{mv_\perp^2}{B} = \text{constant}. \quad (2.96)$$

The result of (2.96) is that as  $B$  increases, so does  $v_\perp$ . This continues until all the energy of the particle is carried by the orbital motion, at which point the particle will change directions and begin moving in the direction of decreasing  $B$ . A magnetic structure in which magnetic field increases to a maximum, causing the reflection of many of the particles incident upon it, is known as a magnetic mirror.

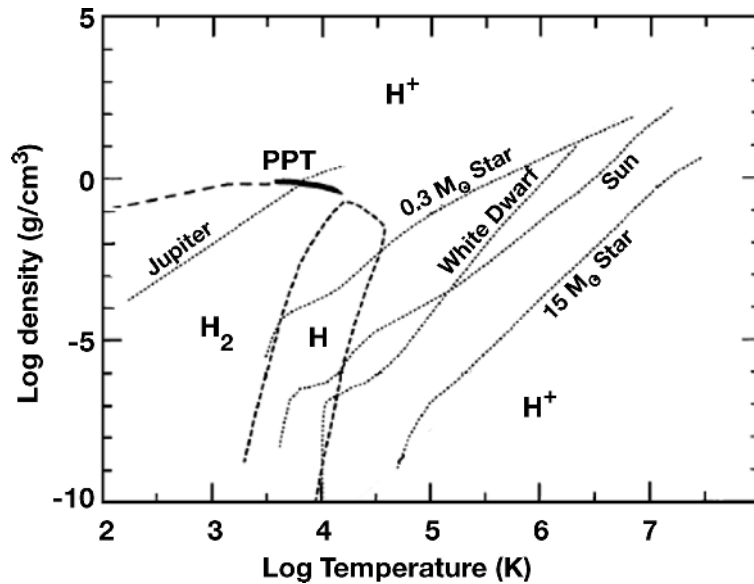
### 3 Properties of High-Energy-Density Plasmas

The discussion of energy in Sect. 2.1 was entirely based on the notion of a polytropic gas. The speed of sound waves, which we found by examining fluctuations in density and velocity, was found to depend upon the variation of pressure with density. These observations reveal the tip of an iceberg, and the iceberg is known as the closure problem. The fluid equations can be derived by taking moments of the velocity distribution of the particles, as is done for example in graduate courses in plasma physics. Here we designate particle velocities by  $v$  and fluid velocities by  $u$ . Thus the continuity equation is the moment taken with  $v^0$ , the momentum equation is the moment with  $v$ , the energy equation is the moment with  $v^2$ , the heat transport equation is the moment with  $v^3$ , and one can keep going. The closure problem arises because every moment equation contains terms involving the next higher moment. Equation (2.1) involves the momentum ( $\rho u$ ), (2.2) involves the energy density (as  $p$ ), and (2.3) would involve the heat flux had we not assumed it to be zero. Because we assumed the heat flux to be zero, (2.1) to (2.3) form a closed system of equations. In general, one obtains a closed system of fluid equations by assuming that some moment of the velocity distribution is either zero or a known function of lower moments. As another example, sometimes the energy equation is expressed as an equation for temperature and the heat flux is written as  $-\kappa_{\text{th}}\nabla T$ , which also produces a closed system of three equations.

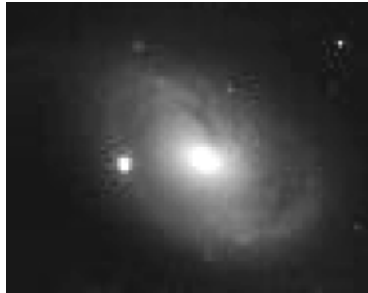
Both the polytropic gas and this last example illustrate an important point – to successfully solve the continuity, momentum, and energy equations, even numerically, one must typically understand the relation among internal energy, pressure, density, and temperature. The equations that specify these relations are known as equations of state, symbolized as EOS. We discuss them in this chapter. We begin by discussing simple equations of state, which are often useful in limited regimes and for estimates generally. We end this section with a more extensive discussion of electrons, because understanding their behavior is essential to understanding the properties of matter in high-energy-density systems. In Sect. 3.2 we take up two issues. We first discuss the degree of ionization, because high-energy-density plasmas are always somewhat ionized but only occasionally fully ionized. This enables us to address two of the issues raised in Chap. 2. These are how the

ions behave when the Debye length is less than the size of an atom, and how strong Coulomb interactions manifest themselves in thermodynamic behavior. This in turn will enable us in Sect. 3.3 to consider the thermodynamics of ionizing plasmas. By the end of this section we will understand the fundamental elements of the equations of state of high-energy-density plasma. At this point, we will have completely addressed the problem identified in Chap. 2 – that the assumptions of traditional plasma physics do not apply to high-energy-density plasmas. We proceed to discuss briefly some more complicated equations of state that are important to computer simulations (Sect. 3.4), to discuss the relation of EOS measurements in the laboratory to astrophysical questions (Sect. 3.5), and specific experimental methods for measuring EOS (Sect. 3.6).

Before turning to the details, consider this example of the relevance of EOS to astrophysics. Figure 3.1 shows a theoretical phase diagram for hydrogen, and also shows where three interesting objects lie in this diagram. The objects are Jupiter, a typical brown dwarf, and a typical dwarf star. The phase diagram is a model, and the location of the curves depends on the model. These might be wrong, but the range of pressures is correct. The phase diagram of hydrogen includes a region of molecular hydrogen, of atomic hydrogen, and of so-called *metallic hydrogen* in which the electrons are free to move and to conduct electricity. Metallic hydrogen carries the currents that



**Fig. 3.1.** Phase Diagram of hydrogen. The dark curve segment shows a theoretical plasma phase transition. Dotted curves show the theoretical path of various astrophysical objects. Adapted from Saumon et al., *Astrophys. J. Suppl.* 99, 713 1995



**Fig. 3.2.** A Type Ia supernova produced the bright spot of emission near the edge of this galaxy. Credit: Jha et al., Harvard Center for Astrophysics

sustain Jupiter’s large magnetic field. These regions have boundaries, which might on the one hand be gradual transitions and might on the other hand be abrupt phase transition. In this particular model, the atomic-to-metallic transition is a phase transition. Evidently a thorough understanding of the EOS will be essential to thoroughly understand astrophysical objects.

### 3.1 Simple Equations of State

Figure 3.2 shows an image of a Type Ia supernova explosion. This explosion is brighter than the entire galaxy that surrounds it, which is not uncommon. Current understanding is that a Type Ia supernova occurs when a white dwarf star, accumulating mass from its environment, reaches a total of just over 1.4 solar masses. This is enough for gravitational forces to overcome the pressure of the degenerate electrons (Sect. 3.2), which initiates the gravitational collapse of the star. However, the star does not fully collapse. Instead, the energy released as collapse begins heats the C and O that make up the white dwarf, which initiates the violent fusion burning that blows the star apart. The properties of the star as its explosion begins are very relevant to this chapter. Its outer layers are accurately described using the polytropic equation of state for an ideal gas (Sect. 3.1.1). To describe its core, one must use the Fermi-degenerate equation of state (Sect. 3.1.3). And the region heated by the fusion burning requires an equation of state for a radiation-dominated plasma (Sect. 3.1.2). These simple models introduce the relevant regimes and concepts; detailed treatment of white dwarf stars and Type Ia supernovae requires more-sophisticated models.

#### 3.1.1 Polytropic Gases

The polytropic equation of state (EOS) is a useful approximation under many circumstances. At a high enough temperature, any material will behave like

an ideal gas. In practice, once the temperature is far enough above the value required to fully ionize any material, its behavior is well described by a polytropic EOS. As we shall see, even radiation-dominated plasmas can be described that way. Moreover, for conceptual and analytic calculations we often use a polytropic description even when it is not precisely accurate. The following equations of state apply to a polytropic ideal gas:

$$p = \frac{\rho(1+Z)k_B T}{Am_p}, \quad (3.1)$$

$$c_s^2 = (\partial p / \partial \rho)_s = \gamma p / \rho, \quad (3.2)$$

and

$$\rho \epsilon = p / (\gamma - 1). \quad (3.3)$$

Here the specific internal energy is given by  $\epsilon$ , and  $k_B$  and  $T$  are the Boltzmann constant and the temperature, respectively. The average level of ionization is  $Z$ , the average atomic mass of the ions in the fluid is  $A$ , and the proton mass is  $m_p$ . In the present discussion, we assume both  $Z$  and  $\gamma$  to be constant. *This is often a poor assumption in high-energy-density physics*, as will become clear in later sections.

Equation (3.1) can be recognized as essentially Boyle's law. The importance of (3.2) was seen in the discussion surrounding (2.6) to (2.9), and we note that here the partial derivative is taken at constant entropy, designated here by the subscript  $s$ . If the gas is an ideal gas with  $n$  degrees of freedom, then  $\gamma = 1 + 2/n$ . Then these equations imply that the energy per particle is  $(n/2)k_B T$ , a result that is familiar from statistical mechanics. It will also prove useful to define the specific heat (at constant volume),  $c_V$ , for a polytropic gas. It is

$$c_V = \left( \frac{\partial \epsilon}{\partial T} \right)_\rho, \quad (3.4)$$

which from (3.1) is

$$(c_V)_{Z,\gamma} = \frac{(1+Z)k_B}{(\gamma-1)Am_p}, \quad (3.5)$$

in which we emphasize that this equation only applies when  $Z$  and  $\gamma$  are constant.

It is worthwhile to explore (3.3) further, as it is often used to define  $\gamma$ . Defining  $\gamma$  by this equation, and taking  $p$  and  $\rho$  as the independent thermodynamic variables so  $\epsilon = \epsilon(p, \rho)$ , one has

$$\left( \frac{\partial p}{\partial \rho} \right)_s = (\gamma - 1)\epsilon + (\gamma - 1)\rho \left( \frac{\partial \epsilon}{\partial \rho} \right)_s. \quad (3.6)$$

In addition, because constant entropy requires zero heat flow and the first law of thermodynamics implies

$$d\epsilon = dq - pd(1/\rho), \quad (3.7)$$

in which the differential heat flow is  $dq$  and the second term on the right is the mechanical or “ $pdV$ ” work, one has  $(\partial\epsilon/\partial\rho)_s = p/\rho^2$ , so one finds  $(\partial p/\partial\rho)_s = \gamma p/\rho$ . This demonstrates that defining  $\gamma$  by (3.3) is consistent with the sound speed as usually defined and with  $p \propto \rho^\gamma$  for an adiabatic process in a polytropic gas. However, if  $\gamma$  is not constant then (3.4) is incomplete and some more general approach is needed. We will explore this further in Sect. 3.3.

### 3.1.2 Radiation-Dominated Plasma

The properties of blackbody radiation and of systems in which radiation is important or dominant are discussed in Chap. 6 and 7. The radiation pressure  $p_R$  is  $1/3$  the radiation energy density and may be expressed as

$$p_R = \frac{4}{3} \frac{\sigma}{c} T^4, \quad (3.8)$$

where  $T$  is the temperature,  $c$  is the speed of light, and  $\sigma$  is the Stefan-Boltzmann constant familiar from blackbody emission. Because this pressure depends upon  $T$  to the fourth power, while material pressures depend upon  $T$  to the first power, at a high enough temperature the radiation pressure will be completely dominant. This is the case, for example, within matter shocked during supernova explosions and near neutron stars and black holes. The transition temperature can be determined by asking when the radiation pressure equals the material pressure. One finds

$$T(\text{keV}) = \frac{1}{1.6 \times 10^{-9}} \left( \frac{3k_B^4 c \rho (1+Z)}{4\sigma m_p A} \right)^{1/3} = 2.6 \left( \frac{\rho(1+Z)}{A} \right)^{1/3}, \quad (3.9)$$

in which  $\rho$  is in  $\text{g/cm}^3$ . Here, outside the parentheses,  $k_B = 1.6 \times 10^{-9}$  ergs/keV to find the temperature in energy units (keV). Within the parentheses, the units of energy and temperature in  $k_B$  must be consistent with those in  $\sigma$  and with the other units used there. For laboratory systems or within stars where  $\rho$  is within a few orders of magnitude of  $1 \text{ g/cm}^3$ , keV temperatures are thus required for radiation to dominate. At typical astrophysical densities much lower temperatures would be required, except that such systems tend to be “optically thin” (see Chap. 6), implying that the radiation pressure is far below the value given by (3.8).

To utilize simple equations in describing radiation-dominated plasmas, one desires to determine  $\partial p/\partial\rho$  for this case – that is, to determine how the radiation pressure varies with plasma density. This is often feasible, because in order for the radiation temperature to remain large enough that the system stays radiation-dominated, the mean free path for the radiation must



be small on the scale of the physical system of interest. This in turn implies that the material is strongly coupled to the radiation and will have the same temperature. In addition, because the material is strongly coupled to the radiation, changes in the density of the material involve changes in the volume containing a fixed amount of radiation. This prepares us to identify a polytropic index for the radiation-dominated plasma, as follows.

Standard arguments in statistical mechanics lead to an expression for the pressure of the photon gas as

$$p = - \sum_j \bar{\sigma}_j \frac{\partial \epsilon_j}{\partial V}, \quad (3.10)$$

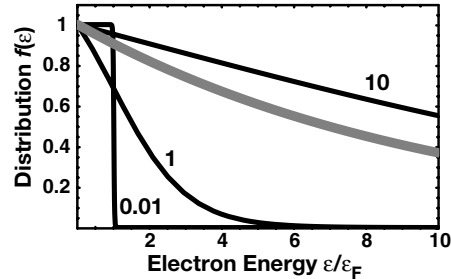
in which the sum is over all possible states  $j$ , the mean occupancy of each state is  $\bar{\sigma}_j = 1/[\exp(\epsilon_j/k_B T) - 1]$ , and the energy of each state is  $\epsilon_j$ . Equation (3.10) makes sense when one recalls that the pressure is the negative of the change in internal energy as volume increases. The energy of a state varies with volume as the wavelength of the light in that state is reduced or increased by the compression or expansion. One can see how by considering the simple example of a cubic box with an edge of length  $L$ , in which a given state has an integer number of wavelengths along each side of the box. The wavenumber of each state,  $k_j$ , is then proportional to  $1/L$ , so one has  $\epsilon_j = hck_j \propto L^{-1} \propto V^{-1/3}$ , where  $h$  is the Planck constant. Thus

$$-\frac{\partial \epsilon_j}{\partial V} \propto V^{-4/3} \propto \rho^{4/3} \quad (3.11)$$

and  $p \propto \rho^{4/3}$ , showing that  $\gamma = 4/3$  for a radiation-dominated plasma. The Euler equations can be applied to such a system using  $\gamma = 4/3$ . Further details on the justification of this can be found in the chapter on radiation hydrodynamics.

### 3.1.3 Fermi-Degenerate EOS

In ordinary plasmas it is the thermal pressure, experienced by the particles through Coulomb collisions, that resists compression of the plasma. This is a classical effect. But when plasma or other matter becomes dense enough, then quantum mechanical effects involving the electrons create pressure and resist compression. The electrons are subject to the Pauli exclusion principal, which prevents more than one of them from occupying the same quantum state. As we will see, this implies that the most energetic electron in cold, high-density matter can be quite energetic indeed. Matter in which most of the electrons are in their lowest-energy states is described as Fermi-degenerate matter. The EOS of Fermi-degenerate matter is of substantial importance in massive planets, white dwarf stars, and inertial fusion implosions or other high-energy-density experiments that compress solid matter. The fact that electrons are fermions has an impact over a broader range of conditions, as we



**Fig. 3.3.** Electron energy distributions in dense matter. The distribution function, normalized to be 1 at zero energy, is shown against energy, normalized to the Fermi energy, for  $k_B T = 0.01, 1,$  and  $10 \epsilon_F$ . The gray curve shows a Maxwellian distribution for  $k_B T = 10 \epsilon_F$

will see in Sect. 3.3. Fundamental derivations of the electron behavior can be found in any book on statistical physics, including for example the relevant volume by Landau and Lifshitz.

Figure 3.3 shows the energy distributions of free electrons in dense matter, for several temperatures. In very cold, dense matter the energy distribution is a step function – all the electrons are in the lowest accessible state. As temperature increases, some of these states are depleted and a tail of electrons develops at higher energy. The energy of the state whose occupancy is 50% is known as the Fermi energy. The Fermi energy at absolute zero,  $\epsilon_F$ , is

$$\epsilon_F = \frac{h^2}{2m_e} \left( \frac{3}{8\pi} n_e \right)^{2/3} = 7.9 n_{23}^{2/3} \text{ eV}, \quad (3.12)$$

in which  $m_e$  is the electron mass,  $n_e$  is the number density of electrons, and  $n_{23}$  is the electron density in units of  $10^{23} \text{ cm}^{-3}$ . This value ( $10^{23} \text{ cm}^{-3}$ ) is of order both the density of electrons in low- $Z$  plasmas with a mass density near  $1 \text{ g/cm}^3$  and the density of conduction electrons in a typical metal. In any material there may also be bound electrons, attached to specific atoms. These electrons do not contribute to the electron density  $n_e$  in (3.12). If we displayed the bound electrons on the scale of Fig. 3.3, they would appear as spikes at negative electron energy. We discuss the degree of ionization (and hence the relative numbers of free and bound electrons) in Sect. 3.2. Equation 3.12 has a number of consequences for physical systems of interest here. It implies that the electrons are not Fermi degenerate in plasmas with densities well below solid density, heated to temperatures of tens to hundreds of eV. In contrast, compressed plasmas at densities of more than 100 times solid density, produced in inertial fusion implosions, have a Fermi energy of hundreds of eV. Such plasmas are often cool enough that the EOS of the electrons is the Fermi degenerate EOS. The *degeneracy temperature*,  $T_d$ , above which the electrons can be approximated as free particles, is found by setting  $k_B T_d = \epsilon_F$ .

Despite its obvious differences from an ordinary gas, the equation of state of Fermi-degenerate matter is quite similar to that of an ideal polytropic gas with  $\gamma = 5/3$ . Equation 3.3 applies in both cases, so  $p = (2/3)\rho\epsilon$ . In addition, while the electron pressure in an ideal gas is  $p = n_e k_B T$ , the electron pressure in Fermi-degenerate matter is  $p_F = (2/5)n_e \epsilon_F$ . Evaluating this one finds

$$p_F = \frac{2}{5}n_e \epsilon_F = \frac{h^2}{20m_e} \left(\frac{3}{\pi}\right)^{2/3} n_e^{5/3}, \quad (3.13)$$

or in practical units

$$p_F = 0.50n_{23}^{5/3} = 9.9 \left(\frac{\rho}{A/Z}\right)^{5/3} \text{ Mbars}, \quad (3.14)$$

in which  $A/Z \sim 2$  and the units of density are cgs. The transition from (3.14) to (3.1) occurs approximately when  $T = T_d$ , although one can see in Fig. 3.3 that the electron distribution still departs significantly from a Maxwellian even at  $T = 10T_d$ .

### Homework 3.1

Inertial fusion designs typically involve the compression of DT fuel to about 1,000 times the liquid density of  $0.25 \text{ g cm}^{-3}$ . Assuming that this compression is isentropic and that the fuel remains at absolute zero, determine the energy per gram required to compress this fuel. Compare this to the energy per gram required to isentropically compress the fuel to this same density, assuming the fuel is an ideal gas whose final temperature is to be the ignition temperature of 5 keV.

The details of the partially degenerate matter at a temperature near the degeneracy temperature involve some straightforward numerical integrals. The ion density range of interest to high-energy-density physics spans  $10^{19}$  to  $10^{24} \text{ cm}^{-3}$ , but reaches  $\sim 10^{26} \text{ cm}^{-3}$  in compressed inertial fusion capsules. All of the electrons participate in Fermi-degenerate properties, so this corresponds to a range of electron densities from  $10^{19}$  to  $10^{26} \text{ cm}^{-3}$ , where the upper limit might correspond either to high- $Z$  matter at an ion density of  $10^{24} \text{ cm}^{-3}$  or to low- $Z$  matter compressed for inertial fusion. The electron temperatures of interest span 1 eV to 1000 eV. Let us examine the behavior of the electrons over this range of conditions.

The electron density is given by the integral over all momenta,  $\chi_e$ , of the probability that an electron will have a specific momentum. With the electron energy given by  $\xi_c = \chi_e^2/(2m_e)$ , this is

$$n_e = \frac{8\pi}{h^3} \int_0^\infty \frac{\chi_e^2 d\chi_e}{\exp\left[\frac{(-\mu + \xi_c)}{k_B T_e}\right] + 1}, \quad (3.15)$$

in which  $\mu$  is the chemical potential, which has energy units. Within this integral, the term equal to  $\chi_e^2 d\chi_e$  gives the scaling of the density of states while the remaining term gives probability that a certain state is occupied by an electron. Equation 3.15 can be put in the useful form

$$\Theta = \frac{T_e}{T_d} = T_e \left[ \left( \frac{8\pi}{3n_e} \right)^{2/3} \frac{2m_e k_B}{h^2} \right] = \left[ \frac{3}{2} F_{1/2} \left( \frac{\mu}{k_B T_e} \right) \right]^{-2/3}, \quad (3.16)$$

which defines the ratio of electron temperature to degeneracy temperature as  $\Theta$ . We also define in general  $F_n(\phi) = \int_0^\infty x^n [\exp(x - \phi) + 1]^{-1} dx$ . This will have further application below. Our parameter range of interest corresponds to  $\Theta = 10^{-3}$  to  $10^4$ .

The chemical potential is the internal energy required to add a particle to the system at constant entropy and constant volume. For a Fermi-degenerate system the chemical potential is positive; a new particle goes in at the Fermi energy even at zero entropy, so one must invest energy to put a new particle into the system. For a classical system  $\mu$  is negative: a new particle can be added at zero energy but to keep entropy constant the internal energy of the system must decrease. The limiting behavior of  $\mu/(k_B T_e)$  is of some interest. In the degenerate regime,  $\mu = \epsilon_F$  so

$$\frac{\mu}{k_B T_e} = \frac{\epsilon_F}{k_B T_e} = \frac{1}{\Theta}. \quad (3.17)$$

In the classical limit, designating the classical chemical potential as  $\mu_c$ , one has

$$e^{\mu_c/(k_B T_e)} = \frac{n_e h^3}{2(2\pi m_e k_B T_e)^{3/2}}, \quad (3.18)$$

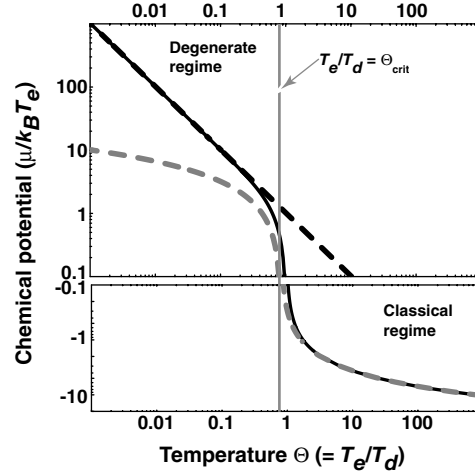
so

$$e^{\mu_c/(k_B T_e)} = \frac{4}{3\sqrt{\pi}\Theta^{3/2}}, \quad (3.19)$$

so  $\mu_c$  is zero when  $\Theta = \Theta_{\text{crit}} = 0.827$ . Atzeni and Meyer-ter-Vehn give a fit due to Ichimaru that spans both limits. This is

$$\frac{\mu}{k_B T_e} = -\frac{3}{2} \ln(\Theta) + \ln \left( \frac{4}{3\sqrt{\pi}} \right) + \frac{0.25054\Theta^{-1.858} + 0.072\Theta^{-1.858/2}}{1 + 0.25054\Theta^{-0.858}}. \quad (3.20)$$

One can vary  $\mu/(k_B T_e)$  and calculate the integral (3.16). Figure 3.4 compares the result of this calculation with the values implied by (3.17) and (3.19). The solid curve shows the actual value, with the gray, dashed curve showing the classical limit and the black, dashed curve showing the Fermi limit. The result is rather dramatic. The electron chemical potential has the classical value for  $\Theta > \Theta_{\text{crit}}$ , where it abruptly transitions to the degenerate value.



**Fig. 3.4.** The chemical potential is shown vs  $\Theta = T_e/T_e$ . In the classical regime,  $\mu$  is negative

### Homework 3.2

Argue conceptually that the contribution of the denominator in (3.16) at large  $\mu/(k_B T_e)$  is a step function. Evaluate this integral numerically to determine how rapidly it becomes a step function as  $\mu/(k_B T_e)$  increases.

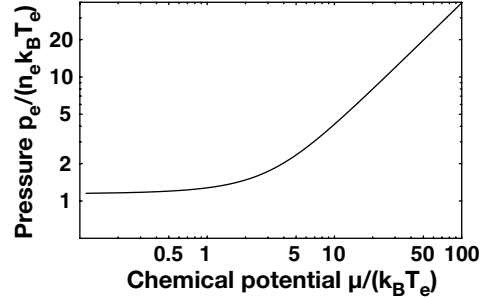
One can evaluate the electron pressure by averaging the energy of each state over the probability that the state is occupied. The general integral for the internal energy density,  $n_e \epsilon_e$ , where  $\epsilon_e$  is the specific internal energy per electron, and the pressure,  $p_e$ , is

$$n_e \epsilon_e = \frac{3}{2} p_e = \frac{8\pi}{h^3} \int_0^\infty \frac{\mathcal{E}_e \chi_e^2 d\chi_e}{\exp\left[\frac{(-\mu + \mathcal{E}_e)}{k_B T_e}\right] + 1}, \quad (3.21)$$

which can be written as

$$n_e \epsilon_e = \frac{3}{2} p_e = \frac{3}{2} n_e k_B T_e \Theta^{3/2} F_{3/2}\left(\frac{\mu}{k_B T_e}\right) = n_e k_B T_e \frac{F_{3/2}\left(\frac{\mu}{k_B T_e}\right)}{F_{1/2}\left(\frac{\mu}{k_B T_e}\right)}. \quad (3.22)$$

Figure 3.5 shows how the normalized pressure,  $p/(n_e k_B T_e)$ , increases with  $\mu/(k_B T_e)$ , for  $\Theta < \Theta_{\text{crit}}$ . The electron contribution to the pressure and internal energy is classical for  $\Theta > \Theta_{\text{crit}}$ . Despite the difference in the pressure, the electrons behave as a gas with  $\gamma = 5/3$  throughout. Under strongly Fermi-degenerate conditions, the electron pressure and energy completely dominate those of the ions. However, because of the energy associated with ionization, the electrons do not necessarily dominate the internal energy of the plasma throughout our regime of interest. We explore this further in Sect. 3.2.



**Fig. 3.5.** Normalized electron pressure *versus* chemical potential. This asymptotes to  $(2/5)\mu/(k_B T_e)$  at large  $\mu/(k_B T_e)$  and approaches 1 as  $\mu/(k_B T_e)$  approaches 0 (and is 1 in the classical regime)

### Homework 3.3

Show, in the limit as  $T_e \rightarrow 0$ , that  $n_e \epsilon_e = (3/5)n_e \epsilon_F$ .

For various applications, including inertial fusion, it is worthwhile to understand the heat capacity and entropy of electrons. For this purpose it helps to understand  $F_n(\phi)$  more thoroughly. One can show that  $F'_n(\phi) = n\phi' F_{n-1}(\phi)$ . In addition, if  $T_e$  is near zero, then  $F_n(\phi) = \phi^{n+1}/(n+1)$ . It is also useful to know that if  $\phi$  is zero, then  $F_{3/2} = 1.153$  while  $F_{1/2} = 0.678$ . In the classical limit (3.18) implies that

$$F_n\left(\frac{\mu}{k_B T_e}\right) = \frac{n_e h^3}{2(2\pi m_e k_B T_e)^{3/2}} \Gamma(1+n) = \frac{4}{3\sqrt{\pi}\Theta^{3/2}} \Gamma(1+n), \quad (3.23)$$

so in the classical limit  $F_{3/2} = \Theta^{-3/2}$  while  $F_{1/2} = (2/3)\Theta^{-3/2}$ .

Turning to the heat capacity, one finds

$$C_V = \left. \frac{\partial}{\partial T_e} (n_e \epsilon_e) \right|_{n_e} = \frac{3}{2} n_e k_B \left[ \frac{5}{3} \frac{F_{3/2}}{F_{1/2}} - \frac{\mu}{k_B T_e} + \frac{1}{k_B} \frac{\partial \mu}{\partial T_e} \right], \quad (3.24)$$

where  $C_V$  has units of energy per unit volume per unit temperature and the argument of both  $F_{3/2}$  and  $F_{1/2}$  is  $\mu/(k_B T_e)$ . In the classical limit this becomes  $C_V = (3/2)n_e k_B$ . In the degenerate limit and for small temperatures, one can expand the integrals to find  $C_V = (3/2)n_e k_B [\pi^2 k_B T_e / (3\epsilon_F)]$ . This is the electronic contribution, which is dominant for strongly degenerate matter. As any book on statistical physics will discuss, the ionic contribution, from the excitation of phonons, scales as temperature cubed.

The entropy per unit volume of the electrons,  $S/V$ , may be found from

$$\frac{S}{V} = \left( -\frac{1}{V} \frac{\partial(pV)}{\partial T_e} \right)_{\mu, V} = \frac{2}{3} \frac{\partial}{\partial T_e} (n_e \epsilon_e) \Big|_{\mu, V}, \quad (3.25)$$

in which  $-pV$  is one of the thermodynamic potentials discussed by Landau and Lifshitz in their volume on statistical physics. This implies

$$\frac{S}{V} = \frac{5}{2} n_e k_B \left[ \frac{2}{3} \frac{F_{3/2}(\frac{\mu}{k_B T_e})}{F_{1/2}(\frac{\mu}{k_B T_e})} - \frac{2}{5} \frac{\mu}{k_B T_e} \right], \quad (3.26)$$

which for the classical limit is

$$\begin{aligned} \frac{S}{V} &= n_e k_B \left( \frac{5}{2} + \ln \left[ \frac{2(2\pi m_e k_B T_e)^{3/2}}{n_e h^3} \right] \right) \\ &= n_e k_B \left[ \frac{5}{2} + \ln \left( \frac{3\sqrt{\pi} \Theta^{3/2}}{4} \right) \right], \end{aligned} \quad (3.27)$$

or for  $T_e \ll \epsilon_F$  is

$$\frac{S}{V} = \frac{3}{2} n_e k_B \left( \frac{\pi^2}{3} \Theta \right). \quad (3.28)$$

The entropy approaches zero as the temperature approaches absolute zero, as it should.

In the context of inertial fusion, one cares about the relation of pressure and entropy, because the shock waves produced during compression increase the entropy (see Chap. 4). Since the pressure is proportional to  $F_{3/2}$ , while density is proportional to  $F_{1/2}$ , (3.26) can be rearranged to obtain

$$p = \frac{2}{5} \frac{S}{V} T_e + \frac{2}{5} n_e \mu. \quad (3.29)$$

As  $T_e$  and  $S$  approach zero, this reduces to (3.13). One sees that the pressure is not sensitive to the value of the entropy until the entropy reaches a threshold value given by  $2n_e\mu/5$ . This is evident in Fig. 3.5, where we see that the pressure begins to depart from  $2\mu\epsilon_F/5$  when  $\mu \sim 5k_B T_e$  or  $\Theta \sim 0.2$  so  $T_e \sim 0.2T_d$ .

The quantity  $p/p_F$  is known in inertial fusion as the *degeneracy parameter*. It has important practical consequences as the fusion gain decreases for increasing  $p/p_F$ . In general  $p/p_F = 1$  for degenerate matter and increases with  $\Theta$ , equaling  $(5/2)\Theta$  in the classical regime. The practical importance of this quantity makes it useful to have approximate estimates of  $p/p_F$ . Atzeni and Meyer-ter-Vehn give the following fit for  $p/p_F$ :

$$\frac{p}{p_F} = \frac{5}{2} \Theta + \frac{0.27232\Theta^{-1.044} + 0.145\Theta^{0.022}}{1 + 0.27232\Theta^{-1.044}}. \quad (3.30)$$

### Homework 3.4

Derive 3.24 and 3.26 and discuss their differences.

## 3.2 Ionizing Plasmas

Mid- $Z$  and high- $Z$  ions in high-energy-density plasmas are rarely *fully stripped*, meaning that all their electrons have been removed. Only as temperatures approach and exceed 1 keV, or as compressions exceed ten times solid

density will one encounter completely stripped ions of any except very-low- $Z$  species. When it becomes routine to work far above solid density at temperatures of many keV, the materials may become fully stripped, although the increased role of radiation will provide ample new complications. We discuss some of these in Chap. 7. For the moment, it is clear that we must understand the behavior of partially ionized plasmas, which we will describe as *ionizing plasmas*, if we are to succeed in understanding high-energy-density phenomena. It also became clear in Sect. 2.4 that one must understand nonideal, strongly coupled plasmas if one is to describe plasma behavior throughout the high-energy-density regime.

One needs to estimate the degree of ionization for a variety of reasons. The most important is that their thermodynamic properties also depend upon ionization, as we discuss in Sect. 3.3. The internal energy of fully stripped ions also includes a major contribution from ionization. While the behavior of actual materials is complicated and difficult to calculate accurately, there are some simple models that can capture aspects of their behavior. These we discuss here.

The electron density is  $Zn_i$ , but the value of the average charge  $Z$  depends upon the temperature. To know  $Z$  precisely, one must evaluate the *ionization balance* to determine the relative populations,  $N_i$ , of the various ionization states. Then one has  $Z$  as a sum over ionization states,

$$Z = \frac{1}{N} \sum_i Z_i N_i, \quad (3.31)$$

in which the state populations can be either a number or a density, and  $N$  is either the total number of ions or the ion density  $n_i$ , respectively.

We will designate the various ionization states of a given species by their charge  $Z_i$ . The electrons in any given ion may reside in the ground state or in an excited state. These of course are designated precisely by the necessary quantum numbers, such as the principal quantum number,  $n$ , the spin quantum number,  $s$ , and the quantum number for orbital angular momentum,  $l$ . In the present discussion, we will occasionally have reason to specify the principal quantum number. We will often, however, ignore excited states and implicitly treat all ions as ground state ions. In most cases this is reasonable. The minimum excited state energy, with  $n = 2$ , has an energy above the ground state that is  $3/4$  of the ionization energy,  $E_i$ . On the one hand, if the ion is in an environment where  $E_i$  is well above  $T_e$ , as is common, then the excited state population is smaller than the ground state population by a factor smaller than  $\exp[-3E_i/(4k_B T_e)]$ , which is fairly small. On the other hand, if  $E_i$  is small relative to  $T_e$ , then it is more likely that the electrons striking the ion will deliver its outer electron into one of the indefinite number of free states as opposed to one of the few and definite excited states.

The exact ionization energy required to remove the outermost electron from a given ionization state does depend on the number and arrangement of



the remaining electrons, but we will ignore this here and adopt a *hydrogenic atom* analysis. In such a treatment, all atoms and all ions are treated as hydrogenic systems, having one electron and a nucleus with the appropriate net charge. This approach is more accurate as the net charge on the atom increases (so that the inner electrons are more tightly bound). This approach allows comparatively tractable computational models to work with a wide range of atoms and ionization states, giving qualitatively correct answers. In our work here we will primarily use the ionization energy associated with a hydrogenic atom model, which is energy  $E_i = Z^2 E_H$ , where  $E_H = 13.6$  eV and  $Z$  is the net charge on the atom (and thus is consistent with our use of “ $Z$ ” elsewhere).

The simple view of atomic structure we will use here is distinct from the computational “average atom model” (see Salzman). The computational model provides a physically consistent approach to the definition of an “average atom”, including both bound states and free electrons, that characterizes each element.

The density of ions will play an important role in our discussions of ionization, as this scales the electron density. A factor-of-two estimate of the typical ion density can be made by taking  $\rho = Z_{\text{nuc}}/4$  g/cm<sup>-3</sup> and  $A = 2Z_{\text{nuc}}$ . Then

$$n_i = \frac{\rho}{Am_p} \sim \frac{Z_{\text{nuc}}}{8Z_{\text{nuc}}m_p} = 7.5 \times 10^{22} \text{ g/cm}^3. \quad (3.32)$$

This density is indicated in several of the plots in the following.

### 3.2.1 Ionization Balance from the Saha Equation

Determining the exact degree of ionization is a difficult problem involving sophisticated calculations, but we can arrive at a reasonable approximation on very simple grounds. We can expect that the ionization energy of the ions in a plasma will have some typical relation to the electron temperature. If we approximate the ion as a hydrogenic ion of charge  $Z$ , then the ionization energy  $E_i = Z^2 E_H$ , where  $E_H = 13.6$  eV. Thus, we expect  $Z^2 E_H / (k_B T_e) \sim C^2$ , where  $C$  is a constant, so  $Z = C \sqrt{k_B T_e / E_H}$ , which is  $Z = 8.57C \sqrt{T_e}$  with  $T_e$  in keV. The problem is to find  $C$ . On the one hand, if we recall that Coulomb processes often are effective at energies of about  $3k_B T_e$ , as is the case for heat transport (see Chap. 8), then we would say  $C \sim \sqrt{3}$ , which is not far from the better estimates discussed next.

More sophisticated estimates of the ionization involve balancing ionization and recombination or assuming that the distribution of ions is in equilibrium. These turn out to be equivalent at high enough densities, but not at low densities. Griem and Salzman discuss in detail the dynamics that are involved, in their books. Here, we discuss the basic phenomena that are important for high-energy-density systems. In low-density plasmas, the archetype of which is the solar corona, collisional ionization is balanced by radiative recombination, establishing a situation known as *coronal equilibrium*. An additional

process, dielectronic recombination, is of increasing importance as the density increases, particularly in the range of densities found in magnetic fusion devices. But at the densities found in high-energy-density systems, the relevant balance is between collisional ionization and collisional (three-body) recombination. In equilibrium, collisional ionization and collisional recombination are equal by the principle of detailed balance. The relative populations of the ionization states are then given by the Saha equation, a fact which Griem uses to derive the rate coefficient for recombination.

At high enough density and temperature the distribution of ions, and the distribution of electrons within energy levels, approaches the equilibrium distribution given by the Saha equation. For an estimate of the ionization balance we will ignore the distribution of electrons among the excited states, and will focus only upon the distribution of ions among the ionization levels. A typical first estimate of the average ionization state would find the two states between which ionization and recombination are balanced. These, however, are just the states to which the Saha equation applies. Thus, we work here with the Saha equation to estimate  $Z$ . The Saha equation gives the ratio of the population of ions in state  $j$ ,  $N_j$ , to those in state  $k$ ,  $N_k$ , as

$$\frac{N_j}{N_k} n_e = \frac{g_j}{4g_k a_o^3} \left( \frac{k_B T_e}{\pi E_H} \right)^{3/2} e^{\frac{-E_{jk}}{k_B T_e}}, \quad (3.33)$$

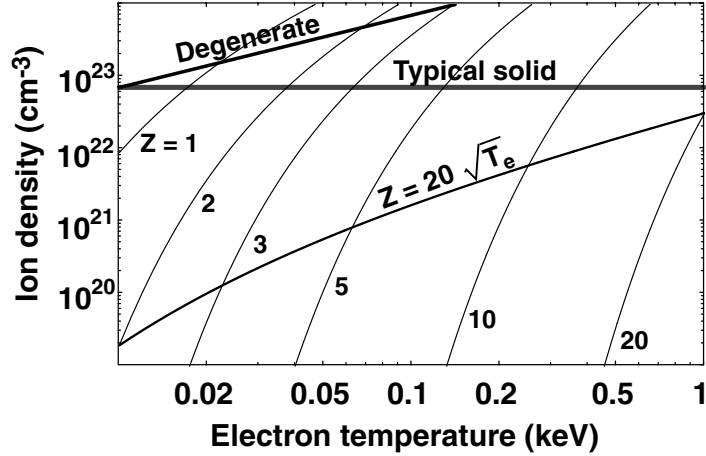
in which  $a_o = 5.29 \times 10^{-9}$  cm is the Bohr radius,  $E_{jk}$  is the energy required to go from state  $k$  to state  $j$ , and  $g_j$  and  $g_k$  are the statistical weights of states  $j$  and  $k$ , respectively. (To help interpret various references, it may help to know that  $E_H a_o^2 = h^2 / (8\pi^2 m_e)$ , ignoring a very small center-of-mass correction.) For simple calculations, the only practical choice is to assume that the ions are hydrogenic, so that the ionization energy from state  $k$  to state  $k+1 = j$ , in an isolated ion, is  $E_{(k+1)k} = Z_{k+1}^2 E_H$ . We will discuss below the consequences of the fact that the ions are not isolated. At a high enough temperature, this has a small effect on the average ionization.

We can determine a characteristic charge, not far from the actual average charge, from this equation as follows. There will be some value,  $Z_{bal}$ , not necessarily an integer, for which the ratio  $N_i/N_k = 1$  for two imaginary ionization states having charge  $Z_{bal} + 1/2$  and  $Z_{bal} - 1/2$ . Then  $Z_{bal}$  should be close to, but may not equal, the average charge  $Z$ . Recalling that  $n_e = Z n_i$ , we can solve for  $Z_{bal}$  to find

$$Z_{bal} = \sqrt{\frac{k_B T_e}{E_H}} \sqrt{\ln \left[ \frac{1}{n_e} \frac{g_j}{4g_k a_o^3} \left( \frac{k_B T_e}{\pi E_H} \right)^{3/2} \right]} - \frac{1}{2}, \quad (3.34)$$

which is

$$Z_{bal} = 19.7 \sqrt{T_e} \left[ 1 + 0.19 \ln \left( \frac{T_e^{3/2}}{n_{24}} \right) \right] - \frac{1}{2}, \quad (3.35)$$



**Fig. 3.6.** Ionization from the Saha equation. Curves of constant  $Z_{\text{bal}}$  are shown. The electrons are Fermi degenerate in the region above the line labeled “Degenerate”. The lower curve shows where  $Z_{\text{bal}}$  equals the approximate value  $20\sqrt{T_{\text{keV}}}$ .

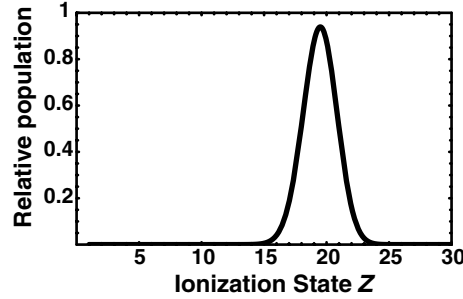
with  $T_e$  in keV and  $n_{24}$  being the electron density in units of  $10^{24} \text{ cm}^{-3}$ , and (3.35) assuming  $g_j = g_k$ . One might approximate this as  $Z_{\text{bal}} = 20\sqrt{T_e}$ , for  $Z_{\text{bal}} \leq Z_{\text{nuc}}n_i$ , where  $Z_{\text{nuc}}$  is the nuclear charge.

The first estimate is to assume  $Z = Z_{\text{bal}}$ , in which case one can either approximate  $n_{24}$  or solve (3.35), which becomes an implicit equation for  $Z$ , through the electron density (with  $n_e = Z_{\text{bal}}n_i$ ). In terms of the initial formulation of this problem above, the coefficient in (3.35) corresponds to  $C \sim 2.3$ , which is not far from our initial guess of  $\sqrt{3}$ . Figure 3.6 shows how  $Z_{\text{bal}}$  varies as ion density and temperature vary, solving implicitly for  $Z_{\text{bal}}$ . If the result were strictly  $20\sqrt{T_e}$ , the contours would be vertical. The curve crossing the plot shows where the solution for  $Z_{\text{bal}}$  does equal  $20\sqrt{T_e}$ . One can see that using  $20\sqrt{T_e}$  is accurate to about 50 % over most of the parameter space shown, with a greater error at ion densities above  $10^{23} \text{ cm}^{-3}$ . One would expect the ions to exist primarily in the one or two states for which ionization and recombination nearly balance, so the value of  $Z$  from (3.35) ought to be close to the actual average ion charge in the plasma.

### Homework 3.5

Make plots comparing  $Z_{\text{bal}}$  from (3.35) with the estimate  $20\sqrt{T_e}$  as a function of  $T_e$ , for ion densities of  $10^{19}$ ,  $10^{21}$ , and  $10^{23} \text{ cm}^{-3}$ . Discuss the results.

One can demonstrate that  $Z \sim Z_{\text{bal}}$ , when the ionization energies are as assumed above, as follows. One can use the definition of  $Z_{\text{bal}}$  to rewrite (3.32), for arbitrary  $j$  and  $k$ , as



**Fig. 3.7.** Relative populations of ionization states, for  $T_e = 1$  keV and  $Z_{\text{nuc}} = 30$

$$\frac{N_j}{N_k} = e^{-\frac{(E_{jk} - Z_{\text{bal}}^2 E_H)}{k_B T_e}}. \quad (3.36)$$

Note that this corresponds to a distribution of ions peaked around  $Z_k \sim Z_{\text{bal}}$ , since  $N_i < N_k$  for  $E_{jk}/E_H > Z_{\text{bal}}^2$ , and  $N_i > N_k$  for  $E_{jk}/E_H < Z_{\text{bal}}^2$ . Figure 3.7 shows the ratio  $N_j/N_1$  for  $T_e = 1$  keV and  $Z_{\text{bal}} = 20$ . Note that to obtain this one must apply (3.36) repeatedly, obtaining

$$N_j/N_1 = \prod_{k=1}^{j-1} N_{k+1}/N_k. \quad (3.37)$$

This gives a sum in the exponent that can be evaluated, as follows:

$$N_j/N_1 = \prod_{m=2}^j \exp\left[-\frac{m^2 - Z_{\text{bal}}^2}{k_B T_e / E_H}\right] = \exp\left[-\frac{(j-1)(6 + 5j + 2j^2 - 6Z_{\text{bal}}^2)}{6k_B T_e / E_H}\right]. \quad (3.38)$$

Figure 3.7 shows a plot of this distribution, which turns out to be very strongly peaked, with nearly all of the ions having a charge within a few unit charges of  $Z_{\text{bal}}$ . As it should, the peak of the distribution corresponds almost exactly to  $Z_{\text{bal}}$  as given by (3.35). One could formally evaluate the average charge using (3.38). For the ratio of ionization state populations, and a nuclear charge  $Z_{\text{nuc}}$ , one has

$$Z = \frac{\sum_{j=1}^{Z_{\text{nuc}}} j \frac{N_j}{N_1}}{\sum_{j=1}^{Z_{\text{nuc}}} \frac{N_j}{N_1}}. \quad (3.39)$$

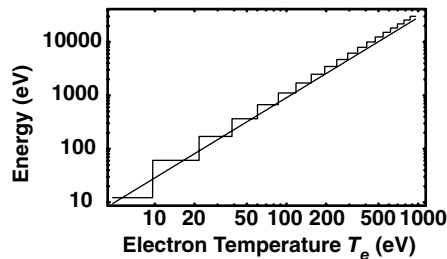
One can show that  $Z$  determined by this method is quite close to  $Z_{\text{bal}}$ .

### Homework 3.6

Carry out the evaluation in (3.39) and compare the result to  $Z_{\text{bal}}$ , for  $T_e = 1$  keV,  $Z_{\text{nuc}} = 30$ , and  $n_i = 10^{21} \text{cm}^{-3}$ .

It is worthwhile to emphasize that the fundamental basis for our estimate of  $Z$  is the Saha equation. However, the Saha equation is not an inviolate law of the universe, even for equilibrium systems. It is a consequence of statistical mechanics when the only important energies are the ionization and excitation energies. As plasmas become denser or colder, energies associated with the interaction of the particles become important. Some aspects of this are discussed in the next section. To some extent, these can be accounted for within the framework of the Saha equation. However, once quantum effects become essential to the behavior of the particles, whether through Fermi degeneracy or through ion-ion correlations, their partition functions change significantly and the Saha equation is no longer the relevant statement of equilibrium. The curve in the upper left corner of Fig. 3.6 shows where the electrons become Fermi degenerate based on the discussion of Sect. 3.1.3 (and assuming  $Z = 20\sqrt{T_e}$ , although the curve placement on such a log-log plot is not very sensitive to the specific assumption about  $Z$ ).

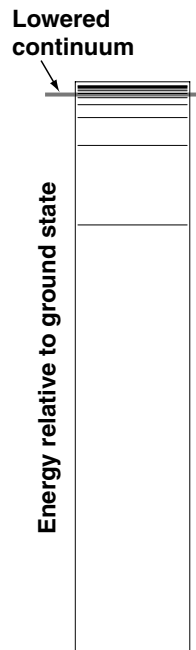
Following through on the question of when the electrons dominate the internal energy of high-energy-density plasmas, we can compare the total energy of ionization, which is part of the internal energy of the plasma, with the internal energy of the electrons. The ionization energy is the sum of  $Z_i^2 E_H$  over the ionization states up to  $Z$ . Here we will use the integer part of  $Z_{\text{bal}}$  as  $Z$  for this energy. The electron energy per ion is  $Zk_B T_e$ , where we will use  $Z = Z_{\text{bal}}$ . Figure 3.8 shows the comparison of these two energies. The ionization energy forms a staircase in such a model, though in reality the fact that several ionization states are present would smooth this out. The important conclusion is that, so long as the ion can keep on ionizing and the electrons are not Fermi degenerate, the ionization energy is the larger contribution to the internal energy. Only once the ions become fully stripped will the electron energy come to dominate. This is a major difference in comparison to low-density laboratory or space plasmas, in which the internal energy can usually be ignored.



**Fig. 3.8.** The increase of internal energy and ionization energy (the staircase) and electron kinetic energy (the line) in eV with increasing  $T_e$

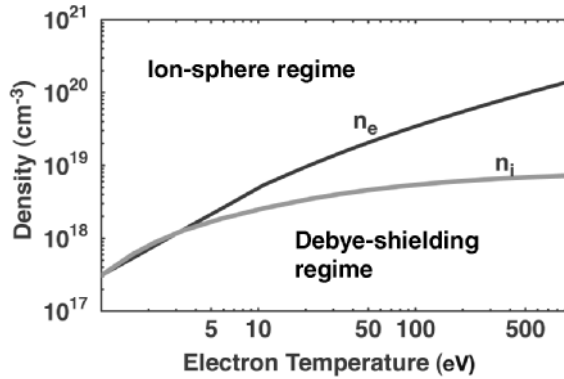
### 3.2.2 Continuum Lowering and the Ion Sphere Model

Equation (3.35) is generally accurate in the plasma that has ablated from the surface of a dense material. However, it becomes inaccurate in compressed, denser matter with a high nuclear charge and low temperature. One reason is that the electrons become Fermi degenerate. Another reason is that the ions in high-energy-density plasmas do not exist in isolation. Even though plasmas are charge-neutral on a volume-averaged basis, in detail the particles arrange themselves so that a particle with any given charge is closer on average to particles of the opposite charge. As a result, one would have to invest energy to pull the plasma apart, so that the particles were far enough away from one another that their interactions were negligible. That is to say, the potential energy of the plasma is negative relative to vacuum. The introduction of new particles or charges to the plasma, as occurs in ionization, lowers the potential even further. This effect is known as *continuum lowering*. This has consequences for the ions or atoms in the plasma – the vacuum energy levels having energies between the plasma potential and vacuum no longer exist. Figure 3.9 shows an energy level diagram to illustrate this point. With regard to ionization, the consequence is that the energy required to ionize is reduced relative to its value in vacuum.



**Fig. 3.9.** A lowered continuum can eliminate some excited states and reduce the ionization energy

The amount by which the continuum is lowered can be evaluated by determining the change in electrostatic potential energy produced by the ionization of an atom or ion. There are two approaches to this calculation, corresponding to two regimes of validity. For low-density plasmas, in which the Debye length exceeds the spacing of the ions, one can calculate the changes to the shielding potentials and the corresponding electrostatic energy introduced by ionization. Equivalent treatments of this regime can be found in Griem, Zeldovich and Raizer, and Krall and Trivelpiece. We will discuss only the case most relevant to high-energy-density plasmas, in which the spacing of the ions is more than a Debye length. This has the consequence that the shielding occurs in the vicinity of each ion individually. This will still be true if the electrons are Fermi degenerate, but the electron density will be more uniform in space than it would be otherwise. The fact that the shielding is local around each ion gives rise to the ion sphere model. Figure 3.10 shows that the boundary between the Debye shielding regime and the ion sphere regime lies at lower densities than those of primary interest in high-energy-density physics.



**Fig. 3.10.** Boundary between ion-sphere and long-range Debye-shielding regimes of continuum lowering. The ion density is shown for reference, inferred from  $Z = 20\sqrt{T_e}$  and  $n_e$

In the ion-sphere model, each ion is assumed to influence only a region within a radius  $R_o$  given by

$$\frac{4\pi}{3}R_o^3n_i = 1, \quad (3.40)$$

in which  $n_i$  is the particle density of the ions. Beyond this distance, the positive and negative charge densities, as seen by the ion, are equal, so these make no contribution to the electrostatic potential energy. Recalling that the typical ion density in  $\text{cm}^{-3}$  is  $7.5 \times 10^{22}$ , one can see that  $R_o \sim 10^{-8} \text{ cm} \sim 1 \text{ \AA}$

for solids, as one would expect since atoms are about  $1\text{\AA}$  in size. Within  $R_o$ , the charge due to the free electrons must balance that of the ion,  $Z_i$ , and for consistency (with the viewpoint of other ions) the average free electron density must equal that throughout the entire plasma, so

$$Z_i = \frac{4\pi}{3} R_o^3 n_e. \quad (3.41)$$

As discussed in Griem, an approximate calculation of the shift in the energy levels of the ion can be made by determining from Poisson's equation the electrostatic potential surrounding the ion, assuming a constant electron density, and by using the first-order perturbation theory of hydrogenic ions from basic quantum mechanics. One finds the principal quantum number of the highest remaining bound state to be

$$n_c = \sqrt{Z_i R_o / a_o}. \quad (3.42)$$

Zeldovich and Raizer find the same result from the semiclassical argument that the highest quantum number will be the one for which the semimajor axis of the orbit equals  $R_o$ . The corresponding reduction in ionization energy is

$$\Delta E \approx Z_i E_H a_o / R_o. \quad (3.43)$$

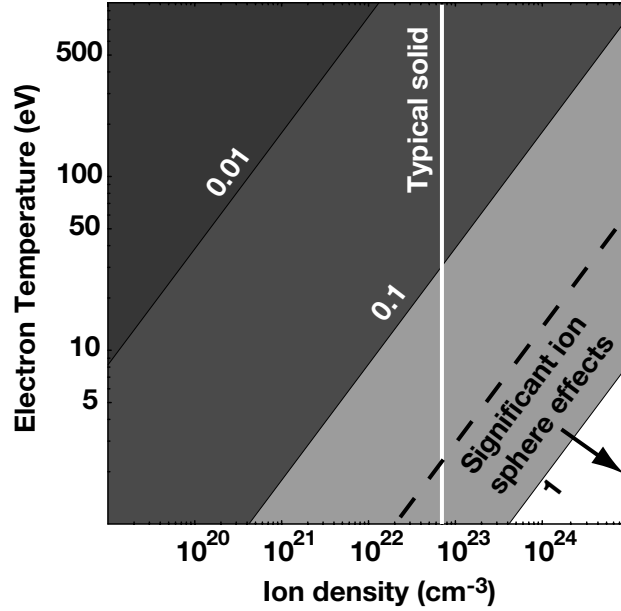
One can see that the relative importance of this effect decreases as the ionization energy in vacuum ( $\sim Z_i^2 E_H$ ) increases. Specifically, one finds

$$\frac{\Delta E}{Z_i^2 E_H} = \frac{a_o}{Z_i} \left( \frac{4\pi n_i}{3} \right)^{1/3} \approx 1.33 \left( \frac{n_{24}}{T_{eV}^{3/2}} \right)^{1/3}, \quad (3.44)$$

in which the approximation uses the ion density in units of  $10^{24} \text{ cm}^{-3}$  and uses the relation  $Z = 20\sqrt{T_e}$ , also converting  $T_e$  from keV to eV. When the shift in ionization energy becomes large, the relation  $Z = 20\sqrt{T_e}$  will cease to be accurate, but it can be used to map out the parameter space where the ionization energy shift becomes large. Figure 3.11 shows contours of the relative decrease in ionization energy in a parameter space of ion density and electron temperature. The ionization energy increases greatly as  $Z_i$  increases, so that a 30% change in this energy will make little difference in the average charge  $Z$  for low- $Z$  materials, and a difference of only a small number of ionization states for high- $Z$  materials. This may be very important for some applications such as x-ray lasers, but is not sufficient to make qualitative changes in the thermodynamic properties. One can see that the changes in ionization energy are not more than this for solid density materials above a few eV, and for once-shocked solids (corresponding to a density increase by a factor of 4 to 7) above 10 eV.

The generalization of (3.43), to include the behavior at lower densities where the Debye length,  $\lambda_D$ , exceeds  $R_o$ , is





**Fig. 3.11.** The relative decrease of ionization energy from (3.44). Contours show values of 0.01, 0.1, and 1

$$\Delta E \approx Z_i E_H \min \left( \frac{2a_o}{\lambda_D}, \frac{a_o}{R_o} \right). \quad (3.45)$$

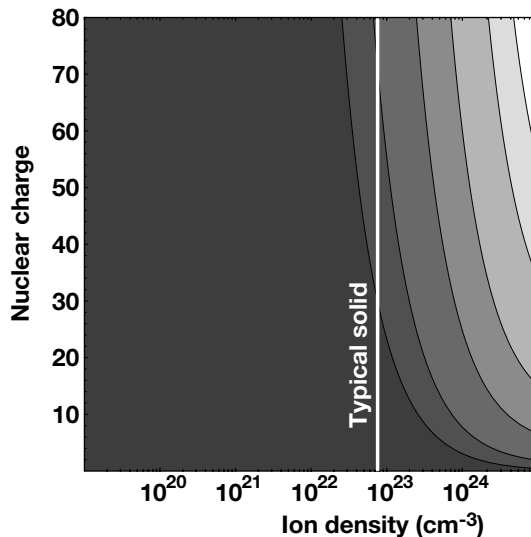
Figure 3.10 showed the boundary between the ion-sphere regime and the long-range Debye-shielding regime. One can see that essentially the entire realm of high-energy-density physics lies in the ion-sphere regime.

One can also see from (3.44) that there can be conditions where  $\Delta E$  is larger than the vacuum ionization energy, producing some ionization even at zero temperature. This effect is known by the somewhat misleading name of *pressure ionization*. (The name is misleading since only density enters. However, in dense matter and especially in Fermi-degenerate matter the pressure can be substantial even at zero temperature.)

### Homework 3.7

Plot the ratio of  $\Delta E$  to the ionization energy versus  $Z_i$  from 1 to 80 for ion densities of  $10^{19}$ ,  $10^{21}$ ,  $10^{23}$ , and  $10^{25}$   $\text{cm}^{-3}$ . Discuss the results.

One can go a bit further and determine the value of  $Z$  produced by pressure ionization for a hydrogenic atom, as follows. In this regime (high-density, low-temperature), all the bound states of the ion will be occupied. Thus, the difference between the nuclear charge,  $Z_{\text{nuc}}$ , and the ion charge  $Z_i$  (which equals the average ionic charge,  $Z$ ) is given by



**Fig. 3.12.** Pressure ionization. The contours show where  $Z = 1, 2, 5, 10, 20$  and  $30$ , based on the ion sphere model for hydrogenic ions

$$Z_{\text{nuc}} - Z_i = \sum_{n=1}^{n_c} 2n^2 = \frac{n_c}{3}(1 + n_c)(1 + 2n_c), \quad (3.46)$$

where (3.42) defines  $n_c$ . Figure 3.12 shows the resulting contours of  $Z$ . (This figure, based on the exact value of the sum in (3.46), differs from the figure in Griem, who approximates the sum.)

Thus, one can see that, to the extent that the ions behave like hydrogen, there is little pressure ionization at ordinary solid densities. Moreover, the amount of ionization is small for compression to only a few times solid density. The ionization increases rapidly for nuclear charges above 10 compressed to above ten times solid density. In the regime where the ions behave like hydrogen, one can take  $Z$  to be the maximum of the values implied by (3.46) and  $Z_{\text{bal}}$  from (3.35), as a crude rule of thumb. Near the transition between the two models, this will underestimate  $Z$ , because plasma effects will reduce the ionization energy of the next couple of ionization states significantly. To improve the estimate in this regime, one could use (3.45) to adjust the Saha equation. On the other hand, by the time  $T_e$  increases much above 10 eV, most materials will be in the Saha regime.

However, Fig. 3.12 can be quite misleading, because most ordinary materials, at temperatures of order an eV, do not behave like a simple hydrogenic model would predict. One might say, for example, that conductors have an effective ionization state corresponding to the number of free electrons per atom that exist in the conduction band. In the case of aluminum, for example, this is about 3 electrons per atom. One can reasonably describe this as

“pressure ionization”, recalling again that a better term would be “density ionization”. To some degree, the appearance of ionization where the hydrogenic model would not find it could be due to subtle quantum effects. But more important is the classical impact of the existence of multiple electrons around the nucleus. In a classical context, one would say that the inner electrons act to shield the outermost electrons from the nuclear charge. This effect is accounted for in the Thomas–Fermi model discussed in Sect. 3.4, which does find approximately the correct number of free electrons for aluminum. In full detail, the behavior of dense materials near and below an eV remains a subject of active research. While all materials behave in similar ways in a global sense, the exact density or temperature where certain transitions occur varies greatly. Quantum effects may be very important, especially in the regime known as *warm dense matter*, corresponding to densities of order solid density and temperatures below a few eV.

### 3.2.3 Coulomb Interactions

Throughout most areas of plasma physics, one can describe plasmas as ideal gasses. Boyle’s law gives the pressure and the polytropic index is near 5/3. Ionization may be present but creates only small changes in the thermodynamics. The dominance of hydrogen and helium in most astrophysical, space, and laboratory plasmas contributes to making them thermodynamically simple. (Indeed, it is the neutral hydrogen that creates complications in such environments.) This state of affairs, and traditional plasma theory, are made possible by two saving features of many plasma systems. The particles interact in pairs, and their interaction energy is small compared to their thermal energy. We saw in Chap. 2 that both of these saving features progressively abandon us as we enter the high-energy-density regime. The reader can find an excellent discussion of the departure from an ideal plasma in Krall and Trivelpiece, while *Griem* provides a connection with the more-recent literature. Much of the discussion in this section follows that of More et al.

Our task here is to see what more one can say about the dense-plasma regime of interest to us. We have seen that the electrons behave as an ideal gas (Sect. 3.1.3) but that their Coulomb interactions are substantial (Sect. 2.4), even though they typically interact at any moment only within the ion sphere in which they then reside. To evaluate the Coulomb effects, and to provide a context for our discussion of the Thomas–Fermi model in the next section, we begin with some further reminders regarding thermodynamic quantities.

The *Helmholtz free energy*  $F$  is fundamentally useful in determining quantities of interest. The units of free energy are energy units, though it may be expressed per particle, per unit mass, per unit volume, or as a total for a system of particles. Here, with an eye to our application in the charge-neutral plasma, we will work with specific quantities, defined as energy per unit total mass. If one considers the specific free energy of the electrons to be  $F_e$ , then

the electron pressure,  $p_e$ , the specific electron entropy,  $s_e$ , and the electron contribution to the specific internal energy,  $\epsilon_e$ , are

$$p_e = \rho^2 \frac{\partial F_e}{\partial \rho}, \quad (3.47)$$

$$s_e = -\frac{\partial F_e}{\partial T_e}, \quad (3.48)$$

and

$$\epsilon_e = -T_e^2 \frac{\partial}{\partial T_e} \left( \frac{F_e}{T_e} \right). \quad (3.49)$$

Equation (3.49) is convenient but also hides the fact that both free energy and entropy contribute to the internal energy. This can be seen explicitly by applying the chain rule to the derivative in this equation.

We determined in Sect. 3.1.3 that our primary interest is the regime in which the electrons behave as an ideal gas in an ion-sphere environment. In this regime the specific free energy (energy per unit total mass) is

$$F_e = \frac{Zk_B T_e}{Am_p} \left[ -1 + \ln \left( \frac{Zn_i h^3}{2(2\pi m_e k_B T_e)^{3/2}} \right) \right] - \frac{9}{10} \frac{Z^2 e^2}{R_o Am_p}, \quad (3.50)$$

in which  $R_o$  is once again the ion-sphere radius. In the context of a quasi-analytic model in the ion-sphere regime,  $Z$  in this equation is the number of free (ionized) electrons per ion. (When this equation is applied to the Thomas–Fermi model, discussed in Sect. 3.4,  $Z$  should be replaced by  $Z_{\text{nuc}}$  here.) The first term on the right-hand side of (3.50) is the standard expression for the specific free energy of an ideal gas of particle density  $n_i Z$ . The fact that  $Z$  can vary with temperature will produce some nonstandard consequences, as we will see. The argument of the logarithm in this term is  $\theta/\theta_{\text{crit}}$  by the definitions of Sect. 3.1.3, but the equation only applies when  $\theta < \theta_{\text{crit}}$  so the logarithm is negative.

The second term on the right-hand side of (3.50) is the Coulomb interaction term, which includes contributions from electron–electron and electron–ion interactions (which are larger). In this form, this is the “Madelung energy”, which one finds by determining the potential energy of  $Z$  free electrons uniformly distributed over the ion sphere, around a nucleus of charge  $Z$ . The electron-ion potential energy is  $-(3/2)Z^2 e^2/R_o$  while the electron-electron potential energy is  $+(3/5)Z^2 e^2/R_o$ . This is the term that drives conventional plasma physics out of business as density increases. The density enters this term through  $R_o$  but temperature and perhaps density can also enter through  $Z$ . Note that this is not the potential energy associated with interactions in an ideal plasma, discussed for example in Zeldovich and Raizer, which involves the Debye length. In the high-energy-density regime, the relevant length is the ion sphere radius.

From (3.50), one finds specific expressions for the pressure and internal energy of the electrons by applying (3.47) and (3.49) to (3.50). Taking  $Z$  to

be either  $0.63\sqrt{T_e}$  with  $T_e$  in eV or  $Z_{\text{nuc}}$  if the ion is fully stripped, one finds two results for pressure and internal energy of the electrons. Specifically,

$$p_e = 0.63k_B n_i T_e^{3/2} - 0.192e^2 n_i^{4/3} T_e \quad (3.51)$$

and

$$\rho\epsilon_e = 2.18k_B n_i T_e^{3/2} - 0.315k_B n_i T_e^{3/2} \ln \left[ \frac{n_i h^3}{T_e (m_e k_B)^{3/2}} \right], \quad (3.52)$$

or

$$p_e = Z_{\text{nuc}} k_B n_i T_e - \frac{6^{2/3} \pi^{1/3}}{10} e^2 n_i^{4/3} Z_{\text{nuc}}^2 \quad (3.53)$$

and

$$\rho\epsilon_e = \frac{3}{2} Z_{\text{nuc}} k_B n_i T_e - \frac{2^{2/3} 3^{5/3} \pi^{1/3}}{10} e^2 n_i^{4/3} Z_{\text{nuc}}^2. \quad (3.54)$$

Here all quantities are in cgs units (including  $e = 4.8 \times 10^{-10}$  statcoul except  $T_e$ , which is in eV). Also note that  $n_i = \rho/(Am_p)$ . In the context of the ionization model above, the system would transition from the first set of equations to the second set when  $T_e = (Z_{\text{nuc}}/0.63)^2$ . Neither set of equations would apply if ion sphere effects determined the degree of ionization, but this occurs at higher densities than those of our primary focus. Three of these equations have straightforward contributions from the kinetic energy and the Coulomb binding energy. In (3.52), the impact of the Coulomb binding energy on the contributions from free energy and entropy offset one another, eliminating the terms proportional to  $e^2$ . These four equations will be important in Sect. 3.3.2. In addition to these relations, one needs to account reasonably for the pressure and internal energy of the ions. We take this up next. Then, in Sect. 3.3.2, we evaluate the overall thermodynamic behavior in the high-energy-density regime. We will see there (in Figs. 3.13(a) and 3.15(a)) that the pressure drops by roughly a factor of two as one moves from the ideal-plasma regime at a high temperature and low density toward the Fermi degenerate regime. This decrease in pressure is due primarily to the Coulomb interactions just discussed.

### 3.3 Thermodynamics of Ionizing Plasmas

We finally know enough to develop a thermodynamic description of ionizing, high-energy-density plasmas. Before proceeding, it may help to summarize what we know from the previous sections. We know that the electrons are strongly degenerate under some conditions, but otherwise behave like an ideal gas throughout most of the parameter space of concern to us. We know this because we have evaluated their behavior as fermions, and *not* from traditional plasma physics arguments (which do not apply as this is a nonideal plasma). However, the treatment of electrons as fermions does not include their Coulomb interactions, which must be included in the free energy and

which lead to a reduced pressure. We have found that, typically, the ions carry most of the internal energy until they have become fully stripped, and that most elements are not fully stripped under typical conditions. We know that the ionic charges are shielded locally, so that the ions typically have little direct influence on one another except through the overall energy of compression. We have not considered a number of effects that have a small impact on the internal energy of the plasma, including the fact that the electrons cluster to some extent near the ions, the quantum-mechanical exchange energy between the electrons, and the spin-spin electron interactions. We will systematically apply the results just obtained below, after we discuss the problem of what to do when  $\gamma$  as defined previously is not constant. This will prepare us to consider the thermodynamic properties of ionizing plasmas, in Sect. 3.3.2.

### 3.3.1 Generalized Polytrropic Indices

Both ionizing and radiating plasmas, unfortunately, have pressures and internal energies that change in complex ways until after the plasma is fully ionized or completely radiation dominated. As a result, the assumption of constant polytropic index is a poor one for such systems. In this case, the question is whether there is any fairly simple way to treat the behavior of the system that might still allow simple models to be developed. Fortunately, one is able to do so, and several approaches are worked out in the literature. The best choice depends on the application. Here we follow Mihalas and Mihalas, identifying three generalized polytropic indices for specific contexts. First, one may need to express  $\epsilon$  as a function of  $p$  and  $\rho$ , so that the momentum and energy equations involve only pressure. In this case the effective polytropic index is as defined in (3.3),

$$\gamma = 1 + \frac{p}{\rho\epsilon}. \quad (3.55)$$

In this case  $\gamma$  is variable in principle. However, for the shock wave applications of Chap. 4, in which the fluid is assumed to be uniform on either side of an abrupt transition, this value of  $\gamma$  is very useful. For sound-wave applications, the derivation in Chap. 2 makes it clear that the relevant index is

$$\gamma_s = \left( \frac{\partial \ln p}{\partial \ln \rho} \right)_s. \quad (3.56)$$

On the other hand, for heat-transport applications, including radiative heat transport, one desires to relate temperature to the other parameters. In this case, we shall see that the index one needs is

$$\gamma_3 = 1 + \left( \frac{\partial \ln T}{\partial \ln \rho} \right)_s. \quad (3.57)$$

If  $\gamma$  as defined by (3.55) is constant, then  $\gamma = \gamma_s = \gamma_3$ . Finding useful expressions for these quantities, and in particular a useful equation for heat transport, takes one into the realm of thermodynamic functions. It is easy to get lost in the forest where one seemingly can take the partial derivative of anything with respect to everything. Our job here is not to visit all the trees in this forest, but rather to develop specific equations that we will use later. In particular, we will seek the generalization of (2.14), which was

$$\left(\frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla\right) p - c_s^2 \left(\frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla\right) \rho = 0, \quad (3.58)$$

with a source term on the right-hand side that will account for heat transport.

Remarkably, aside from some patience all the fundamental information we need to do this is a pair of equations from the first and second laws of thermodynamics,

$$d\epsilon - \frac{p}{\rho^2} d\rho = dq = T ds, \quad (3.59)$$

where  $d\epsilon$ ,  $dq$ , and  $ds$  are the specific internal energy, heat input, and entropy, respectively, and two mathematical relations, specifically

$$\left(\frac{\partial a}{\partial b}\right)_c = 1 / \left(\frac{\partial b}{\partial a}\right)_c, \quad (3.60)$$

and

$$\left(\frac{\partial a}{\partial b}\right)_c \left(\frac{\partial b}{\partial c}\right)_a \left(\frac{\partial c}{\partial a}\right)_b = -1. \quad (3.61)$$

As is usual in thermodynamic calculations, at any given moment we express the thermodynamic functions in terms of two independent variables chosen from the three quantities  $\rho$ ,  $p$ , and  $T$ . We proceed at first by expressing  $\epsilon$  as  $\epsilon(p, \rho)$ ; so from (3.59) we find

$$T ds = dq = \left(\frac{\partial \epsilon}{\partial p}\right)_\rho dp + \left[\left(\frac{\partial \epsilon}{\partial \rho}\right)_p - \frac{p}{\rho^2}\right] d\rho. \quad (3.62)$$

We also have, as  $ds$  is an exact differential,

$$dq = T ds = T \left(\frac{\partial s}{\partial p}\right)_\rho dp + T \left(\frac{\partial s}{\partial \rho}\right)_p d\rho. \quad (3.63)$$

The specific heats involve the use of  $T$  and  $\rho$  or  $T$  and  $p$  as the thermodynamic variables. Equation (3.63) implies that the specific heat at constant volume is

$$c_V = \left(\frac{dq}{dT}\right)_\rho = \left(\frac{\partial \epsilon}{\partial T}\right)_\rho, \quad (3.64)$$

while the specific heat at constant pressure is found by writing  $\epsilon$  as  $\epsilon(T, \rho)$  in (3.59) and then differentiating, to obtain

$$c_p = \left( \frac{dq}{dT} \right)_p = \left( \frac{\partial \epsilon}{\partial T} \right)_\rho + \left[ \left( \frac{\partial \epsilon}{\partial \rho} \right)_T - \frac{p}{\rho^2} \right] \left( \frac{\partial \rho}{\partial T} \right)_p. \quad (3.65)$$

We can use these definitions to evaluate the coefficients in (3.62). Multiplying (3.65) by  $(\partial T/\partial \rho)_p$ , and using the definition of  $c_V$ , one finds

$$\left( \frac{\partial \epsilon}{\partial \rho} \right)_p = \left( \frac{\partial \epsilon}{\partial T} \right)_\rho \left( \frac{\partial T}{\partial \rho} \right)_p + \left( \frac{\partial \epsilon}{\partial \rho} \right)_T = c_p \left( \frac{\partial T}{\partial \rho} \right)_p + \frac{p}{\rho^2}, \quad (3.66)$$

while from the chain rule

$$\left( \frac{\partial \epsilon}{\partial p} \right)_\rho = \left( \frac{\partial \epsilon}{\partial T} \right)_\rho \left( \frac{\partial T}{\partial p} \right)_\rho = c_V \left( \frac{\partial T}{\partial p} \right)_\rho. \quad (3.67)$$

This then gives for the heat input per (3.62)

$$dq = c_V \left( \frac{\partial T}{\partial p} \right)_\rho dp + c_p \left( \frac{\partial T}{\partial \rho} \right)_p d\rho = T ds. \quad (3.68)$$

To simplify this further note that

$$\frac{c_p}{c_V} = - \left( \frac{\partial p}{\partial \rho} \right)_s \left( \frac{\partial \rho}{\partial T} \right)_p \left( \frac{\partial T}{\partial p} \right)_\rho = \left( \frac{\partial p}{\partial \rho} \right)_s \left( \frac{\partial \rho}{\partial p} \right)_T, \quad (3.69)$$

obtained by substituting from (3.66) and (3.67) into (3.62) and taking  $(\partial p/\partial \rho)_s$ . Then (3.68) becomes

$$dq = c_V \left( \frac{\partial T}{\partial p} \right)_\rho \left[ dp + \left( \frac{\partial p}{\partial \rho} \right)_s \left( \frac{\partial \rho}{\partial p} \right)_T \left( \frac{\partial p}{\partial T} \right)_\rho \left( \frac{\partial T}{\partial \rho} \right)_p d\rho \right], \quad \text{or} \quad (3.70)$$

or

$$dq = c_V \left( \frac{\partial T}{\partial p} \right)_\rho \left[ dp - \left( \frac{\partial p}{\partial \rho} \right)_s d\rho \right], \quad (3.71)$$

using (3.61). This is the form we were seeking. The quantity in square brackets has the form of (2.14), as desired, with the isentropic sound speed given, from (3.69) and (3.65), by

$$\left( \frac{\partial p}{\partial \rho} \right)_s = \frac{c_p}{c_V} \left( \frac{\partial p}{\partial \rho} \right)_T = \left( \frac{\partial p}{\partial \rho} \right)_T - \frac{1}{c_V} \left[ \left( \frac{\partial \epsilon}{\partial \rho} \right)_T - \frac{p}{\rho^2} \right] \left( \frac{\partial p}{\partial T} \right)_\rho = \gamma_s \frac{p}{\rho}, \quad (3.72)$$

thus defining  $\gamma_s$ . This expression for the sound speed is readily evaluated from expressions for  $p$  and  $\epsilon$ , as is the overall factor that multiplies the square brackets in (3.71). Note that when one converts to an absolute expression for the heat input, one will need to multiply (3.71) by  $\rho$ . One can similarly use (3.62) and (3.63), along with an expansion of  $Tds$  as a function of  $T$  and  $\rho$ , to determine that



$$(\gamma_3 - 1)^{-1} = \rho c_V \left( \frac{\partial T}{\partial p} \right)_\rho. \quad (3.73)$$

This is worth knowing in order to interpret a number of the equations in Mihalas and Mihalas. In addition, note that so long as  $(\partial p / \partial \rho)_T = p / \rho$  and  $(\partial \epsilon / \partial \rho)_T = 0$ , one has  $\gamma_s = \gamma_3$ . This is the case in the following section but not for the radiating plasmas considered in Chap. 7. We now have the tools necessary to describe the behavior of an arbitrary plasma, so long as we know its pressure and internal energy as a function of density and temperature. We take up this problem next. Following that, in Sect. 3.3.2, we evaluate the thermodynamic parameters.

### Homework 3.8

Derive (3.73) from relations (3.60) to (3.64).

### 3.3.2 Pressure, Energy, and Their Consequences

We are now prepared to discuss the specific thermodynamic properties of ionizing gases, and to lay the groundwork we will need to deal with radiating gases in Chap. 7. We can represent the electron contributions using (3.51) to (3.54). We will see that these will work poorly when the electrons become Fermi degenerate, and we know that pressure ionization may increase  $Z$  at temperatures below some value when the ion density exceeds  $10^{24} \text{ cm}^{-3}$ . In most cases, the electrons dominate the pressure and the kinetic energy. We can take the ion contribution to the pressure,  $p_i$ , and specific kinetic energy,  $\epsilon_{ik}$ , to be

$$p_i = \frac{\rho k_B T_i}{A m_p} \quad (3.74)$$

and

$$\epsilon_{ik} = \frac{3 k_B T_i}{2 A m_p}, \quad (3.75)$$

knowing that the Coulomb binding energy associated with close packing of ions has been included in the electron formulae. In addition, in the context of our convention that the initial material state is a low (or zero) but positive energy state, the ions also contribute energies of ionization and excitation. We ignore excitation here, for reasons discussed near the start of Sect. 3.2, and once again use a hydrogenic model, describing the internal energy of the ions as

$$\epsilon_{ii} = \frac{R}{A m_p} = \frac{k_B E_H}{A m_p} \sum_{k=0}^Z k^2 = \frac{k_B E_H}{6 A m_p} Z(1+Z)(1+2Z), \quad (3.76)$$

where the maximum allowed value of  $Z$  is  $Z_{\text{nuc}}$  and we define  $R$  as the internal energy per ion.

**Homework 3.9**

The value of  $R$  used here ignores the internal energy in excited states (as well as the energy lost by radiation during ionization, which would properly have to be treated by more general equations). Again assuming hydrogenic ions, estimate what fraction of the internal energy is present in excited states, and how this varies with  $Z$ .

Given the above results for the ion contributions we can give the total pressure and energy, in the ionizing regime and the fully stripped regime, for  $T_e = T_i$ , as follows:

$$p = 1.6 \times 10^{-12} n_i T_e \left( 1 + 0.63\sqrt{T_e} - 2.76 \times 10^{-8} n_i^{1/3} \right) \quad (3.77)$$

and

$$\rho\epsilon = 1.6 \times 10^{-12} n_i \times \left[ 1.43\sqrt{T_e} + 4.20T_e + T_e^{3/2} \left( 1.3 - 0.315T_e^{3/2} \ln \left[ \frac{n_i}{10^{23}T_e} \right] \right) \right], \quad (3.78)$$

or

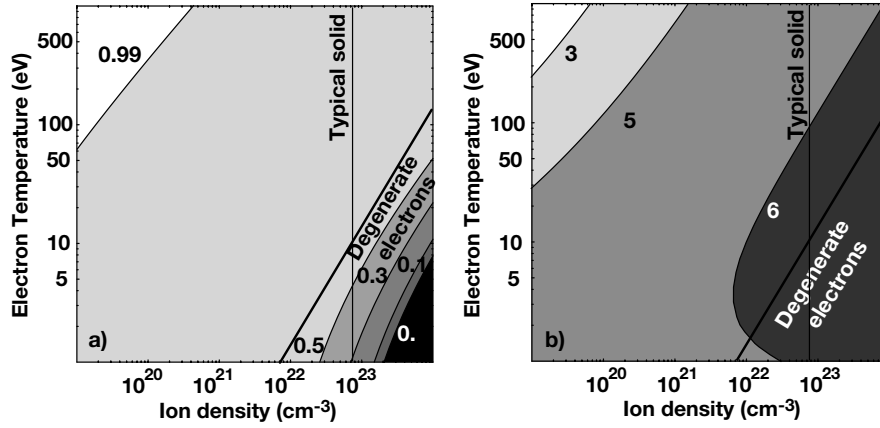
$$p = 1.6 \times 10^{-12} n_i \left( T_e(1 + Z_{\text{nuc}}) - 6.96 \times 10^{-8} n_i^{1/3} Z_{\text{nuc}}^2 \right) \quad (3.79)$$

and

$$\rho\epsilon = 2.4 \times 10^{-12} n_i \left( 825 + T_e(1 + Z_{\text{nuc}}) - 1.39 \times 10^{-7} n_i^{1/3} Z_{\text{nuc}}^2 \right). \quad (3.80)$$

These equations are in cgs units except that  $T_e$  is in eV. Also note that  $n_i = \rho/(Am_p)$ . We will first explore the implications of the first two equations for an element of high enough  $Z$  to keep ionizing. Then we will consider carbon as an example of an element that can be fully stripped.

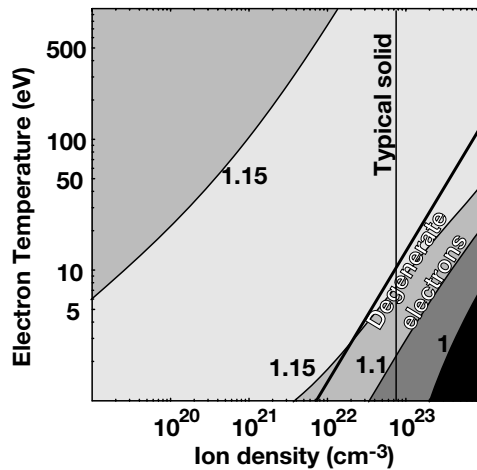
Considering a high- $Z$  element, it is informative to compare the pressure and energy from (3.77) and (3.78) with their ideal-gas equivalents, which are  $p = n_i(1 + Z)k_B T_e$  and  $\rho\epsilon = (3/2)n_i(1 + Z)k_B T_e$ , respectively. Figures 3.13 (a) and (b) show the ratio of the more-complete estimates in (3.77) and (3.78) to these ideal-gas values. One can see that the model for the pressure fails badly in the Fermi-degenerate region, which is no surprise. Otherwise, the pressure across the space of Fig. 3.13 (a) is typically between 50% and 100% of the ideal-gas value. Thus, the ideal-gas value is a factor-of-two estimate for nondegenerate electrons, though it may overestimate the pressure somewhat. In contrast, Fig. 3.13 (b) shows that the internal energy is a multiple of the ideal gas value, despite the decrease associated with heating new electrons. The calculation here would indicate that the internal energy is about five times the ideal-gas value. Quantitatively, this is larger than standard calculations would produce, and certainly ionization energies of the outer



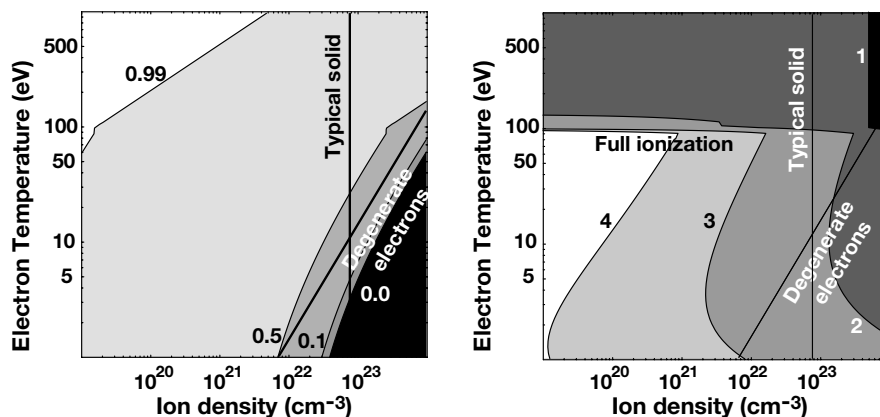
**Fig. 3.13.** (a) The pressure for an ionizing high- $Z$  element is shown, normalized to the ideal gas pressure. The contours increase from the lower right, and are at 0, 0.1, 0.3, 0.5, and 0.99. (b) The internal energy density for an ionizing high- $Z$  element is shown, normalized to the ideal-gas value. The contours increase from right to left, and are at 3, 5, and 6

electrons in high- $Z$  elements are reduced compared to the hydrogenic values. The qualitative conclusion is accurate, however. The internal energy is increased relative to the ideal-gas value.

With an increased internal energy and a decreased pressure, the value of  $\gamma$  inferred from (3.55) must decrease. Figure 3.14 shows the values of  $\gamma$



**Fig. 3.14.** The value of  $\gamma$  inferred from (3.55) for an ionizing high- $Z$  element is shown. The contours show 1 (lower right), 1.1, and 1.15. The value never reaches 1.2

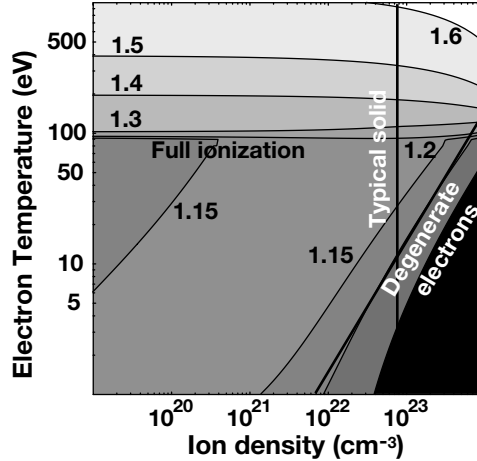


**Fig. 3.15.** (a) The pressure for carbon is shown, normalized to the ideal-gas pressure. The contours increase from the lower right, and are at 0.1, 0.5, and 0.99. (b) The internal energy density for carbon is shown, normalized to the ideal-gas value. The contours are labeled, and range from 1 to 4

obtained from (3.77) and (3.78). Here again, the quantitative value ( $\sim 1.15$ ) should not be taken too seriously but the qualitative point, that  $\gamma$  should be reduced substantially compared to the ideal-gas value of  $5/3$ , should be real. Standard EOS evaluations for xenon ( $A = 130$ ,  $Z_{\text{nuc}} = 54$ ), for example, give  $\gamma \sim 1.2$  to  $1.3$ . We will see in Chap. 4 that this implies increased compression by shocks.

Now consider carbon, an element with six electrons that can become fully stripped at modest temperatures. Using our estimate that  $Z = 0.63\sqrt{T_e}$ , Carbon will ionize fully at  $T_e = 91$  eV. At higher temperatures, the internal energy still includes the energy of ionization, but this contribution does not increase any further. To estimate the properties of carbon, we use 3.77 and 3.78 until  $T_e = 91$  eV, then 3.79 and 3.80 at higher temperatures. This produces Figs. 3.15 (a), (b), and 3.16. Carbon has a much smaller nuclear charge than xenon, so the Coulomb binding effects are much smaller. The main consequence, shown in Fig. 3.15 (a), is that the pressure is within 1% of the ideal-gas pressure over most of the relevant parameter space. In contrast to the relative uniformity of the pressure, the internal energy has a definite structure, as Fig. 3.15 (b) shows. At temperatures where carbon is not fully stripped, the ionization energy is a dominant factor and the internal energy substantially exceeds the ideal-gas value. As ion density increases, though, this effect becomes smaller. Then, once the temperature has increased enough to fully strip the material, the internal energy rapidly decreases toward the ideal-gas value.

This behavior leaves its footprint on the inferred  $\gamma$ , shown in Fig. 3.16. At temperatures below 91 eV, one sees behavior very like that of Fig. 3.14.



**Fig. 3.16.** The value of  $\gamma$  inferred from (3.55) for carbon is shown. The contours are labeled, except for  $\gamma = 1$  and 1.1 and on the lower right

The inferred value of  $\gamma$  is generally between 1.15 and 1.2. Then, once the element becomes fully stripped,  $\gamma$  begins to increase, although in this model it does not reach the ideal-gas limit of  $5/3$  by even at a temperature of 1 keV. This begins to become artificial, however, as radiation will begin to reduce  $\gamma$  toward  $4/3$  by the time  $T_e$  reaches 1 keV.

We can now proceed to infer other thermodynamic quantities from (3.77) to (3.80). In particular, we can evaluate the specific heat at constant volume,  $\gamma_s$ , and  $\gamma_3$ . For the ionizing plasma case of (3.78), we find

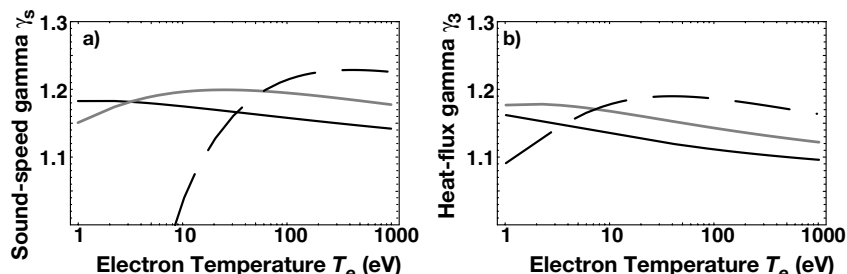
$$c_V = \frac{1.6 \times 10^{-12}}{Am_p} \left( 4.2 + 0.715/\sqrt{T_e} + 27.3\sqrt{T_e} - 0.473\sqrt{T_e} \ln[n_i/T_e] \right), \quad (3.81)$$

while for the fully stripped case of (3.80), we have

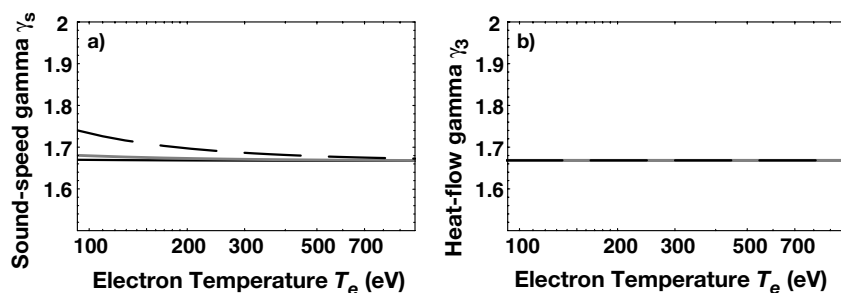
$$c_V = \frac{2.4 \times 10^{-12}(1 + Z_{\text{nuc}})}{Am_p}. \quad (3.82)$$

We can then apply (3.72) and (3.73) to find  $\gamma_s$  and  $\gamma_3$ . This produces rather messy expressions, but they are readily evaluated by computer. Figures 3.17 and 3.18 show the results. We see that these indices are small ( $\sim 1.2$ ), somewhat variable, and close in magnitude to  $\gamma$  in the ionizing regime. Figure 3.18, plotted for carbon, shows that these two indices are very close to  $5/3$  throughout the fully ionized regime.

This completes our discussion of specific models of equations of state. In the following chapters, we will typically take  $\gamma = 4/3$  or  $5/3$  for our examples. We will not need to distinguish among the different polytropic indices until we work with radiation hydrodynamics in Chap. 7. But it should be clear



**Fig. 3.17.** Values of (a)  $\gamma_s$  and (b)  $\gamma_3$  in the ionizing regime. The curves are for  $10^{19} \text{ cm}^{-3}$  (solid),  $10^{21} \text{ cm}^{-3}$  (gray),  $10^{23} \text{ cm}^{-3}$  (dashed). The model fails at a low temperature and high density as the electrons become degenerate



**Fig. 3.18.** Values of (a)  $\gamma_s$  and (b)  $\gamma_3$  in the fully stripped regime are shown, for carbon. The curves are for  $10^{19} \text{ cm}^{-3}$  (solid),  $10^{21} \text{ cm}^{-3}$  (gray),  $10^{23} \text{ cm}^{-3}$  (dashed)

from the above that  $\gamma$  can be substantially less than  $5/3$ , that these dense plasmas are *not* ideal-gases, and that it is not so easy to know just what the equation of state is. To close this chapter, the next section discusses tabular equations of state.

### 3.3.3 The EOS Landscape

At this point it may be useful to summarize what we have learned about the equation of state in high-energy-density systems. Figure 3.19 provides this summary. The specific lines in the figure are drawn for an ionizing plasma, assuming  $A = 2Z_{\text{nuc}}$ , but the relative orientation of the various elements in this log-log space is not sensitive to these assumptions. At the upper left is the ideal-plasma regime. Examples are hot enough coronal plasmas, as for example in the laser-heated zone in front of a dense target, or the plasma generated in z-pinchs during their implosion (see Chap. 8). At the lower right pressure ionization becomes important, as occurs when solids are sufficiently compressed by shock waves, and the electrons are Fermi degenerate, which

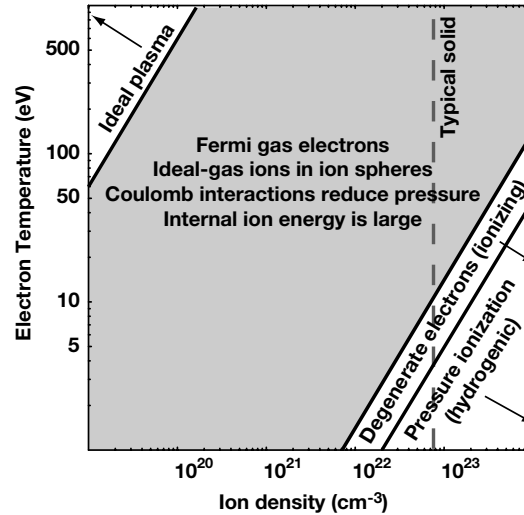


Fig. 3.19. The landscape of EOS for high-energy-density plasmas

determines the pressure needed to compress solid-density matter including the fuel for inertial fusion.

In between these two limits is the realm of many experiments in the early 21st century. Here the matter is partly ionized but probably is not fully stripped, the ions live in the privacy of their own ion spheres but represent much of the internal energy in the system, and the electrons are a Fermi gas whose pressure is reduced by Coulomb interactions.

### 3.4 Equations of State for Computations

The chapter thus far has made it evident that equations of state in the dense-plasma regime are complicated. The appeal of using a polytropic index, at the expense of detailed accuracy, is quite clear. Indeed, this will be our approach throughout much of the text. But if one is to try to simulate these systems with computers, then one would hope to be more accurate. It is evidently a great challenge to accurately simulate the behavior of materials at high energy density. One has Coulomb energy corrections, degenerate electrons, pressure ionization, and continuum lowering, among other effects. But in addition, to be fully accurate one would need to include in addition to several effects that we mentioned but did not incorporate, such as the impact of bound electrons. One would also need to handle the transitions between regimes more accurately. But the actual problem is worse than this, because high-energy-density matter nearly always evolves out of and is adjacent to matter that is not at high energy density, but rather is in a solid or liquid state. So realistic computations must also be able to account for these states of

matter and for their transition to hotter and perhaps denser conditions. A particularly difficult example at this writing is that of the behavior of the wires in z-pinch plasmas (see Chap. 8). These begin as solids, ablate and (perhaps) explode, creating the material that the z-pinch accelerates inward. Modeling this dynamics is a severe challenge.

In this section we discuss two approaches to this problem. The first is the Thomas–Fermi model and its extension to QEOS. The second is the use of EOS tables.

### 3.4.1 The Thomas–Fermi Model and QEOS

We first discuss the *Thomas–Fermi model* and lament its simple complexity. It is highly unfortunate that this model requires serious numerics for its use, because otherwise it would be the model of choice for quasi-analytic analysis in the ion–sphere regime. Based on a few very simple relations, the Thomas–Fermi model accurately includes the effects of ionization, excitation, Fermi-degeneracy, Coulomb interactions, self-consistent electron density structure, and to some extent ion–ion coupling. In various versions it may also include various quantum-mechanical effects such as those of shell structure. There is a nice summary of the Thomas–Fermi model in Salzman and more detail in Eliezer, Ghatak, and Hora.

The Thomas–Fermi model is a self-consistent combination of the ion–sphere model and the treatment of the electrons as fermions. The key to its power is that it demands that the electrostatic interaction of the electrons as fermions and the nucleus be self-consistent within this context. It naturally accommodates, in a classical context, the increase in electrostatic energy associated with increasing density or temperature. This allows one to ignore ionization and excitation as separate processes. They are accounted for, on average, by the expansion of the heated electrons or the lowering of the continuum as conditions change. The model can be formulated, in a simple form, as follows. It assumes spherical symmetry.

The electric potential,  $\Phi(r)$ , is given by the Poisson equation,

$$\nabla^2\Phi(r) = 4\pi en_e(r) - 4\pi Z_{\text{nuc}}e\delta(r), \quad (3.83)$$

with the boundary condition that  $\partial\Phi/\partial r = 0$  at the boundary of the ion sphere,  $r = R_o$ , which follows from the net charge neutrality of each ion sphere. The electron density is given by the generalization of (3.15) to include a varying potential energy,

$$\begin{aligned} n_e(r) &= \frac{8\pi}{h^3} \int_0^\infty \frac{\chi_e^2 d\chi_e}{\exp[(-\mu - e\Phi(r) + \mathcal{E}_e)/(k_B T_e)] + 1} \\ &= \frac{4\pi(2m_e k_B T_e)^{3/2}}{h^3} \int_0^\infty \frac{\sqrt{x} dx}{\exp[x - (\mu + e\Phi(r))/(k_B T_e)] + 1}. \end{aligned} \quad (3.84)$$



Thus, the nature of electrons as fermions is accounted for. The net neutrality of each ion sphere sets a constraint on the density,

$$Z_{\text{nuc}} = 4\pi \int_0^{R_o} n_e(r)r^2 dr, \quad (3.85)$$

which determines the chemical potential. These three equations are all that must be solved to describe the system. Once computational mathematics programs evolve beyond root finding to profile finding, this model may become simple to implement.

Now, supposing we have solved the above equations, we consider how the results may be used. The potential is only defined in the above to within an arbitrary constant, although the choice of this constant will affect the value of  $\mu$ . It is conventional to choose  $\Phi = 0$  at the boundary of the sphere. As a result, the potential throughout the sphere becomes increasingly positive as the density increases. This is how this model captures the effects of ion interactions. The charge state is calculated as

$$Z = \frac{4\pi}{3} R_o^3 n_e(R_o), \quad (3.86)$$

which amounts to assuming that the free electrons flow freely between ions and thus establish the density at the ion sphere boundary.

Some other thermodynamic quantities are as follows. The electron pressure is

$$p_e(r) = \frac{8\pi(2m_e)^{3/2}(k_B T_e)^{5/2}}{3h^3} \int_0^\infty \frac{x^{3/2} dx}{\exp[x - \mu/(k_B T_e)] + 1}, \quad (3.87)$$

and the electron kinetic energy in each ion sphere is given by

$$K_e = \frac{4\pi(2m_e)^{3/2}(k_B T_e)^{5/2}}{h^3} \times \int_0^{R_o} \int_0^\infty \frac{x^{3/2} dx}{\exp[x - (\mu + e\Phi(r))/(k_B T_e)] + 1} dr. \quad (3.88)$$

This is not, however, the entire energy, because the Coulomb energy of attraction remains to be accounted for. This can be calculated directly, with the interaction energy of the electrons and the nucleus, per atom, being  $U_{en}$  while the energy per atom of the interactions among the electrons is  $U_{ee}$ . One has

$$U_{en} = -4\pi Z_{\text{nuc}}^2 e^2 \int_0^{R_o} n(r)r dr \quad (3.89)$$

and

$$U_{ee} = \frac{e^2}{2} \int_0^{R_o} \int_0^{R_o} \frac{n(r)n(r')}{|r-r'|} d^3r d^3r'. \quad (3.90)$$

With these definitions, the total specific internal energy is

$$\epsilon = (K_e + U_{en} + U_{ee})/(Am_p). \quad (3.91)$$

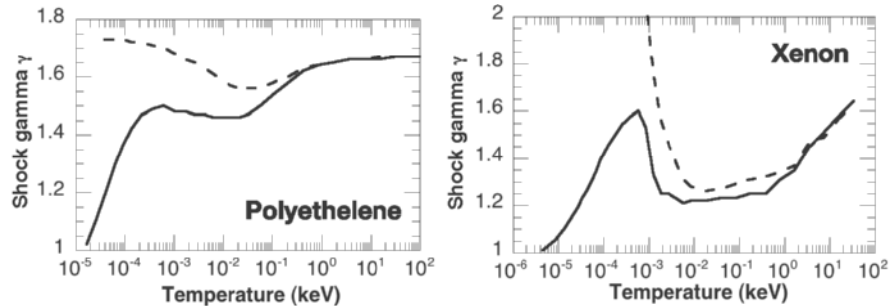
To avoid potential confusion, we should note that the zero of the energy scale here for the electrostatic energies is *not* consistent with the conventions used in all the other discussions in this book. In those other discussions, the implicit point of view is that a state of zero energy and pressure is an unbound neutral gas nominally at zero temperature (but without quantum effects). At higher temperatures, positive energy is invested to ionize it. At high densities, the Coulomb interactions of the ionized gas provide some binding energy and reduce the energy input that would otherwise be required. In contrast, the state of zero energy in a conventional Thomas–Fermi model has all the particles dispersed to infinity. To convert the Thomas–Fermi result to the standard scale, one would have to add the ionization energy necessary to totally strip the atom in vacuum to the quantity in parentheses in (3.60). This would matter if, for example, one were evaluating an effective polytropic index as a means of comparing models.

Finally, there are a class of computer models known as QEOS *models*, which stands for quotidian equation of state, where quotidian means “everyday” or “routinely usable”. The Thomas–Fermi model is often incorporated into these (see for example the description in More et al). Such models are likely to include additional terms or equations intended to account for the solid, liquid, and gaseous states and for the transitions between them. They can be a useful way to bridge the wide range of parameters that simulations must deal with.

### 3.4.2 Tabular Equations of State

Another approach to describing materials for simulations is to use a tabular EOS. The idea behind a tabular EOS is that one can work with experimental data, molecular dynamics simulations, and the best possible models. From them one can construct a table giving two of the thermodynamic variables ( $\rho$ ,  $p$ ,  $\epsilon$ , and  $T$ ) as a function of the other two. As is true of all the models we have discussed, this is necessarily done in equilibrium. Then a computer code can interpolate from the tables to find the properties it needs with adequate accuracy.

One challenging aspect of constructing such a table is the need for thermodynamic consistency. The table will show how some thermodynamic quantities vary when others are held constant. These variations must be thermodynamically consistent. As one does work on the material or adds heat to it, the changes of state that result must be consistent with the first law of thermodynamics. If this were not the case, then the computer code using the table would mysteriously create or absorb energy in an unphysical way. Achieving thermodynamic consistency in practice, while merging models that cover adjacent regimes, can be very difficult. One can check for thermodynamic consistency by applying the first law of thermodynamics to the table.



**Fig. 3.20.** For polyethelene ( $C_1H_1$ ) on the left and xenon ( $A = 131, Z = 54$ ) on the right, these figures show the inferred  $\gamma$  from the SESAME table. The lower curve is at  $0.1 \text{ g/cm}^3$  density, while the upper curve is at solid density. Credit: Carolyn Kuranz

One way to do this is to evaluate the local deviation from the first law of thermodynamics. Landau and Lifshitz show in Vol. 5 that one can write the first law of thermodynamics as  $d(\rho\epsilon)/dV + p - T(dp/dT) = 0$ . One can evaluate this quantity throughout a candidate EOS table and display the results as curves or a contour plot.

The most widely used EOS tables are the SESAME tables, available from the Los Alamos National Laboratories. These tabulate specific pressure (pressure per unit density) and specific energy as functions of density and temperature, over several orders of magnitude in density and in temperature. We show two examples in Fig. 3.20. In each case, we have used the equation of state to plot  $\gamma$ . The range of temperatures in the table is shown. The densities shown are solid density (dashed) and  $0.1 \text{ g/cm}^3$ , which are relevant to laboratory work in high-energy-density physics. One sees first that the behavior at low temperatures is quite different. This reflects the presumed development of a gaseous state (and perhaps even clusters) at low densities, with many degrees of freedom, which forces  $\gamma$  close to 1. In contrast, the solid becomes more ordered as temperature decreases. From traditional thermodynamics, one would expect  $\gamma$  to approach 3 at low temperatures if the solid forms a lattice with tightly bound planes. In the tables,  $\gamma$  sometimes exceeds 3 at low temperatures.

At the highest temperatures, the materials seem to approach  $\gamma = 5/3$ , which would correspond to a fully stripped ideal-gas. We comment more on this below. At intermediate temperatures, between a few eV and 100 eV for polyethelene and a few eV and 1000 eV for xenon,  $\gamma$  is reduced. This is as expected from the previous discussion in this chapter. Indeed, the result for xenon is not far above the value we inferred for an ionizing, high- $Z$  material. The value of  $\gamma$  for polyethelene, on the other hand, is not so far below  $5/3$ . One might be skeptical as to whether this decrease is in fact large enough.

These figures also provide one example of the limitations of these tables. If the high-temperature states (above 1 keV) were truly in equilibrium, as is assumed, then the presence of the radiation field would be driving  $\gamma$  to  $4/3$ . So these tables ignore the radiation field. The problem is that they have to make some specific assumptions, though in this case they do not assume true equilibrium. Real systems do vary greatly with regard to the coupling of the radiation field and the matter. There is no way that one table can account for this. Any given computer code may or may not handle it well.

There are other problems with the use of EOS tables in particular, and equilibrium models in general, in simulations of real systems. Real systems are almost never in equilibrium. They are often in steady state, or nearly in steady state, but not in equilibrium. A good example is a plasma that expands from a hot surface but is not actively heated. The expanding plasma cools, and after a time its properties slowly evolve. Even so, on the scale of tens of ns that often applies, the ions and electrons may not recombine and the plasma certainly will not reach its equilibrium state. The EOS table, on the other hand, presumes the plasma is instantaneously in equilibrium. Thus, if it reaches a condensation temperature, the table will make it condense, no matter how unrealistic this may be. This, and theoretical equilibrium phase changes in general, can be a source of abrupt density changes in simulations that are completely unreal. There are times when an ideal-gas model with fixed  $\gamma$  provides a much more realistic approach to simulating a time-varying system. The main point is that one must pay attention, think about what one sees, and not assume that the code reveals truth.

In addition, you may have noticed that some of the equations above would produce regimes where the pressure from a given model became negative. This happens with the models used for the EOS tables as well. In some cases, this is sensible. For example, the only realistic way to incorporate tension in a material, in the context of a hydrodynamic model, is by adding negative terms to the pressure. If the material is tightly enough bound and cold enough, it may be sensible in this sense to treat the pressure as negative. However, the existence of negative pressure regions in EOS tables can create serious problems when simulating real, nonequilibrium systems. In the example of the previous paragraph, for example, the plasma expanding from a surface may have a temperature and density that would correspond to a condensed state with tension in equilibrium, yet in actual fact may be more accurately treated as an ideal-gas. In some contexts, it is sensible to modify the EOS tables to destroy the tension regimes and maintain positive pressure. When the EOS table works well, it will do a better job of reproducing the dynamics than any simpler model can. But it can not be counted on to always work well. It is very often sensible to compare simulations using EOS tables for various similar materials and also using a fixed  $\gamma$  to help determine which aspects of the observed dynamics are due to the specifics of the EOS table.

Finally, tables do not typically exist for novel materials, such as low-density foams. These materials are not microscopically uniform. They are unlikely to behave like a uniform, low-density material. Indeed, in experiments to date the uniform-density models fail to accurately predict phenomena such as shock-wave propagation in such foams. Whether in the end new tables or some other approach proves the best for working with them remains to be seen.

### 3.5 Equations of State in the Laboratory and in Astrophysics

A moment's thought will show that equation of state (EOS) properties are quite important in astrophysics. In gravitationally bound objects, such as planets, white-dwarf stars, or neutron stars, the interior pressure is determined primarily by gravity. However, to know the density, and hence the volume of the material in any given pressure range one must know the equation of state. Direct astronomical measurements can determine the mass, and sometimes the size, of such objects, and may be able to learn about the surface composition from spectroscopy. But there is usually neither direct nor indirect information relating to the interior. (An exception is the Sun, for which seismology is possible and productive, producing data that greatly constrain the EOS.)

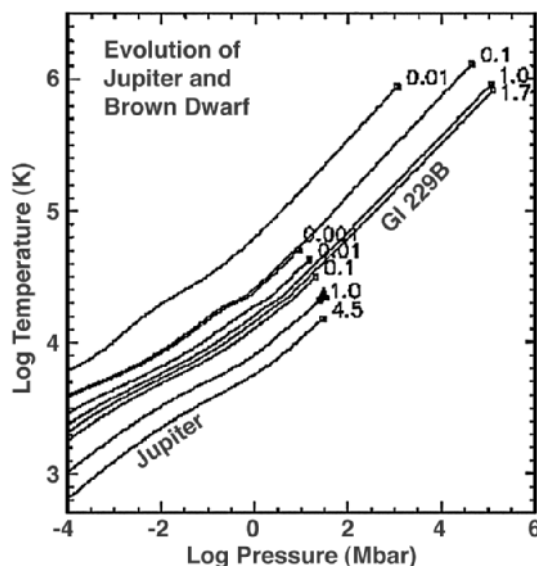
Assuming that one knows the EOS, one can construct a model of a planet in which the known mass of the planet is distributed in radius as gravitational pressure and the EOS dictate, based on assumptions about what the composition of the planet is. Uncertainties in the EOS make this more difficult. In the case of Jupiter, for example, it is an interesting question whether an entire planet of its size and mass might be composed of hydrogen or whether there must be an ice and rock core. This certainly has implications for theories of planet formation. With sufficient knowledge of the hydrogen EOS, one will be able to answer this question. At the turn of the 21st century, such knowledge was insufficient.

In addition, the EOS affects one's ability to understand magnetic fields, as we discussed briefly with reference to Fig. 1.3. Planetary magnetic fields are produced by interior currents, known as dynamos. The theory of planetary dynamos unfortunately requires complex three-dimensional calculations. Nonetheless, the possibilities for magnetic field generation are constrained by the locations where the planetary interior is conducting, and this is constrained by the EOS. Here again Jupiter provides an interesting way to frame the puzzle. Jupiter has an extremely strong magnetic field, producing very-large-scale effects within the solar system. At the surface of Jupiter, hydrogen is an insulator. The nature of the hydrogen EOS will determine how close to the surface of Jupiter currents can flow and what volume of the planet

can participate in the dynamo. This will constrain the possibilities for the production of Jupiter's magnetic field.

### 3.5.1 The Astrophysical Context for EOS

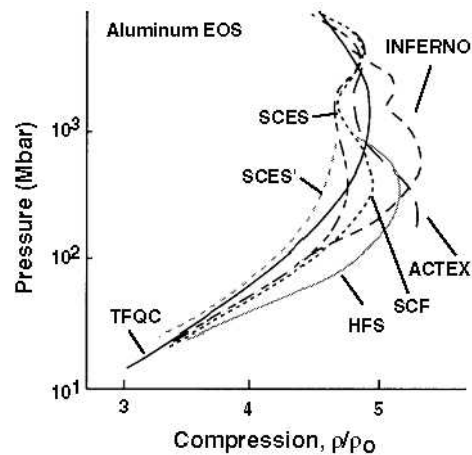
To illustrate the importance of EOS, consider Jupiter in more detail. Figure 1.3 shows a schematic of its interior (for more discussion, see Guillot et al.). Jupiter has an outer envelope of dielectric molecular  $\text{H}_2$ , believed to transition to metallic atomic hydrogen at a radius of  $0.75R_J$  and pressure of  $p \sim 2$  Mbar, and ending in an ice-rock core at a core pressure of  $\sim 40$  Mbar. The mass of Jupiter is  $M_J \sim 10^{-3}M_S$ , (where  $M_S$  is the mass of the sun) and its intrinsic radius is  $R_J = 7.2 \times 10^4$  km. Model calculations for the interior of Jupiter are shown as temperature–pressure ( $T - p$ ) profiles as a function of age in Fig. 3.21. Profiles for the brown dwarf Gl229B are also shown in this figure. Under these conditions, molecular hydrogen ( $\text{H}_2$ ) dissociates to atomic hydrogen and ionizes deeper in the mantle, changing from a dielectric to a conductor. The pressure and temperature in the mantle of Jupiter near the surface are in the range of 1–3 Mbar at temperatures of a fraction of an eV. Deeper in the interior, the pressure and temperature increase, rising to 40 Mbar at a couple of eV at the center. (The corresponding numbers for the brown dwarf Gl 229 are similar in the mantle, but it has four orders of magnitude higher pressures in the core,  $p_{\text{core}} \sim 10^5$  Mbar.)



**Fig. 3.21.** Temperature–pressure profiles in Jupiter and brown dwarf Gl 229B, for various ages, from models in Hubbard et al. (1997)

One of the key questions about the interior of Jupiter is whether there is a sharp boundary between the molecular hydrogen mantle and the monatomic hydrogen core, caused by a first-order plasma phase transition. This has significance for the exact internal structure, as the discontinuities caused by such a phase transition tend to inhibit convective heat transport, modifying the thermal profile of the planetary interior. This also affects the degree and rate of gravitational energy-release due to sedimentation of He and heavier elements. Jupiter and Saturn's atmospheres are observed to contain less helium than is believed to have been present at their formation. This is thought to be due to a H-He phase separation. The presence of a helium-poor outer region, and helium-rich inner region is important, both because it has implications for the amount of heavier elements contained deeper in the interior of the planet, and also because of the gravitational energy released as heat during helium sedimentation. Helium sedimentation is required to explain Saturn's intrinsic heat flux, and may also be significant in Jupiter. The important point in the present context is that all of the detailed issues of hydrogen behavior are quite uncertain at present. The nature of the transition from molecular to monatomic hydrogen, the existence of a metallic phase, the possibility of a H-He phase separation, and other factors are not known.

The EOS of elements heavier than H and He, relevant to Earth-like planets, is even more complex at ultrahigh pressures. To illustrate this, we show in Fig. 3.22 a plot of a number of different theoretical models for the behavior of Al at very-high pressures and compressions,  $p > 10$  Mbar,  $\rho/\rho_0 > 3$ . These models calculate the shock *Hugoniot*, which is the locus of the points in pressure and density that can be reached from a single initial condition by means



**Fig. 3.22.** Various theoretical models of the shock Hugoniot of Al, as described in the text. Note the considerable uncertainty, which only experiments can resolve. Adapted from Avrorin et al. (1987)

of shock waves of varying strength. The various models (see Avrorin et al. and Trunin) exhibit significant differences. The simplest and most widely used of the models is the statistical Thomas–Fermi model with quantum corrections (TFQC), shown by the solid curve. This model does not include atomic shell structure, but rather treats the electron states as a continuum. The self-consistent field (SCF), Hartree–Fock–Slater (HFS), and INFERNO models treat the electron shells quantum mechanically, but differ in their handling of close-packed levels corresponding to energy bands. The semiclassical equation of state (SCES) model treats both the discrete electron shells and the energy bands semiclassically. The ACTEX model is an ionization equilibrium plasma model which uses effective electron–ion potentials fitted to experimental spectroscopic data. These models typically include the nuclear component using the ideal-gas approximation. An exception is a Monte Carlo treatment of the thermal motion of the nuclei implemented in one of the versions of the semiclassical equation of state model (SCES).

The oscillations in the theoretical pressure versus compression curves shown in Fig. 3.22 result from the pressure ionization of the K- and L-shell electrons of Al. At pressures of 100–500 Mbar, ionization of the L-shell electrons occurs as the high compression forces neighboring atoms sufficiently close together to disrupt the  $n = 2$  electron orbital. When the shock places the material in a state where these electrons are becoming free, more of the energy flowing through the shock must go into internal energy. This leads to a larger density increase, exactly as we discuss in Sect. 4.1. Hence, at the onset of pressure ionization of a new shell in a model, the postshock density increases more rapidly with postshock pressure, behavior known as a “softer” EOS. This pressure-ionization effect on the EOS is qualitatively similar to that due to molecular dissociation of  $N_2$  and  $D_2$ , which has been experimentally observed at lower pressures (see Nellis). Once ionization from the shell is complete, the effect is a “hardening” of the EOS, as the fraction of the energy flowing through the shock that is converted to internal energy decreases. This is why, above  $\sim 1$  Gbar, some of the  $p - \rho$  curves turn back toward lower compression. A similar softening–hardening oscillation is predicted at pressures of 3–5 Gbar due to ionization of the K-shell electrons, though the magnitude of the effect is smaller due to the lower number of K electrons. How real such oscillations in the Hugoniot are is unclear at this writing. If the actual process of liberating new electrons develops more gradually than it does in the model, this may smooth out the response and avoid the oscillation.

### 3.5.2 Connecting EOS from the Laboratory to Astrophysics

The EOS describes the equilibrium properties of any large aggregation of atoms of a given type. Even microscopic quantities of matter typically include enormous numbers of atoms. As a result, measurements using aggregations of matter that are very small on a human scale can provide results which apply directly to aggregations of matter on a planetary or stellar scale. In this sense,



it is straightforward to make a laboratory measurement that applies directly to astrophysics.

Unfortunately, however, laboratory measurements can only achieve a limited range of pressures and densities by comparison with those existing in astrophysical systems. It would be desirable to be able to scale the equation of state in pressure and density, so that laboratory measurements could be applied to a wider range of astrophysical conditions. This is possible but unnecessary in the case of simple equations of state, such as an ideal-gas or a radiation-dominated system. In more-complex cases, however, the dynamics of the material is specific to the material conditions. The chemical structure of a material is not easily scaled to other conditions, and processes such as dissociation and ionization occur only at specific energies. Thus, laboratory measurements can only address astrophysical issues in EOS at pressures they can actually achieve.

Given the technologies of the early 21st century, it seems likely that the pressures employed for EOS studies during this period will be in the range of 1 to less than 100 Mbars. These are suitable for addressing issues in planetary equations of state. One can expect this to be the primary focus of such studies.

### 3.6 Experiments to Measure Equations of State

This final section is somewhat problematic. On the one hand, it seems natural to discuss measurements of EOS in the chapter on EOS. On the other hand, knowledge from Chap. 4, Chap. 8, and other areas is necessary to really understand how these measurements work. In the end, the author elected to put this section here where it would be easiest to find. If you are new to this field, then you will need to read other parts of the book before you will be able to make much sense of the details in what follows.

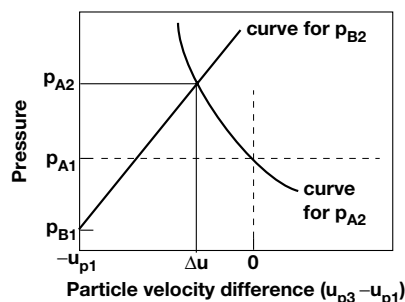
Experimental measurements of EOS have been an active area of research for many years. Much of this research has used flyer plates (see Chap. 4) to create shock waves with known postshock (“particle”) velocities. This enabled the EOS along the shock Hugoniot (defined with reference to Fig. 3.22) to be inferred. These experiments typically used either gas guns or rail guns to launch the flyer plates, and were limited to pressures below 1 Mbar prior to roughly the turn of the century. Hydrogen was a very active area of study in such experiments, because of its application to the interiors of gas giant planets (see Nellis et al.). Some experiments were done using nuclear weapons to launch flyer plates at a higher velocities, producing higher pressures. These experiments were in the high-energy-density regime, but were certainly not “laboratory” experiments in the usual sense. Equation-of-state experiments at a high energy density can be accomplished in the laboratory using a number of techniques. We discuss some of these here.

### 3.6.1 Direct Flyer-Plate Measurements

If one can produce a cold flyer plate of a given material, one has a great advantage for measuring the EOS of that material. To do so, one collides the flyer plate with another layer of the same material. One then knows, as discussed in Sect. 4.1.5, that the postshock fluid velocity is half the initial flyer plate velocity, and one knows that it is constant until after a shock emerges from the flyer plate or the impacted layer. One can measure the flyer plate velocity and the collision time optically. One then determines the shock velocity by measuring the time of emergence of the shock from the impacted layer of material. This can be done optically or by more-sophisticated techniques. Thus one obtains a direct and simultaneous measurement of the shock and postshock velocities, and is able to infer density and pressure as discussed in Sect. 4.1.2. Because Aluminum is a very practical material for flyer plates, the EOS of Aluminum is now very well known.

### 3.6.2 Impedance Matching

Once one has a material with a known EOS, one can use this to advantage in determining the EOS of other materials. This technique is commonly known as *impedance matching*, although it actually depends on measuring the difference in the degree to which two materials impede the shock. The analysis used in impedance matching is as follows, with reference to Fig. 3.23. One begins by producing a known and steady shock, with a known postshock fluid velocity,  $u_{p1}$ , in a first material, material  $A$ . It is essential that the shock be steady to very high ( $\sim 1\%$ ) accuracy, otherwise one finds large errors in the resulting EOS. One then allows this shock to enter a layer of the sample to be measured, of material  $B$ . This produces a transmitted shock through the sample and either a reflected shock or a rarefaction in the first material. One



**Fig. 3.23.** How impedance matching works. The initial shock in material  $A$  is located by the dashed line. The curves for  $p_{A2}$  and  $p_{B2}$  shows how the pressure varies with velocity. Their intersection gives the values of pressure and of  $\Delta u = u_{p3} - u_{p1}$  for the EOS measurement

measures the shock velocity,  $u_{sB}$ , in the second material, initially at pressure  $p_{B1}$  (often negligible). One knows the density of this material,  $\rho_{Bo}$ , so one knows that its postshock pressure,  $p_{B2}$ , lies along a line given by (4.9), which implies  $p_{B2} = p_{B1} + \rho_{Bo}u_{sB}u_{p3}$ , in which  $u_{p3}$  is the (not yet known) postshock fluid velocity of material B in the laboratory frame. Note that  $p_{B2}$  is an increasing function of  $u_{p3}$ .

As discussed in Sect. 4.1.6, one also knows that  $u_{p2} = u_{p3}$  and that  $p_{A2} = p_{B2}$ , in which  $u_{p2}$  and that  $p_{A2}$  are the fluid velocity and pressure, respectively, in material A at the interface with material B while the shock crosses material B. Given sufficient knowledge of the equation of state of material A, one can plot a curve giving  $p_{A2}$  as a function of  $u_{p2} = u_{p3}$ . One finds that  $p_{A2}$  is a decreasing function of  $u_{p3}$ , because a decrease of  $u_{p3}$  corresponds to an increase in the velocity at which the reflected shock separates from the interface, and thus to a higher reflected-shock pressure. In the event that the reflected wave is a rarefaction rather than a shock, it remains the case that  $p_{A2}$  is a decreasing function of  $u_{p3}$ , as one can verify by reviewing Sect. 4.2. The net result is that the curves for  $p_{A2}$  and  $p_{B2}$  cross at only one point (actually an area whose size is determined by the uncertainties). Because  $p_{A2} = p_{B2}$ , this point determines both  $p_{B2}$  and  $u_{p3}$ , giving us the EOS at one point. EOS results are often plotted in a space of pressure versus density. The postshock density is related to  $u_{p3}$  by  $\rho/\rho_{Bo} = (1 - u_{p3}/u_{sB})^{-1}$ . The entire ensemble of such  $p-\rho$  points, produced using compression by a steady-state shock, is referred to as the *principal Hugoniot* or the shock Hugoniot.

One way to apply the initial pressure to material A, so that one can know its value with high accuracy, is to make material A be Al and to apply the pressure by using an Al flyer plate to strike it. This has been accomplished using magnetically launched flyer plates from pulsed power machines, discussed in Sect. 8.3.3. Figure 3.24 shows a drawing of the experimental system used

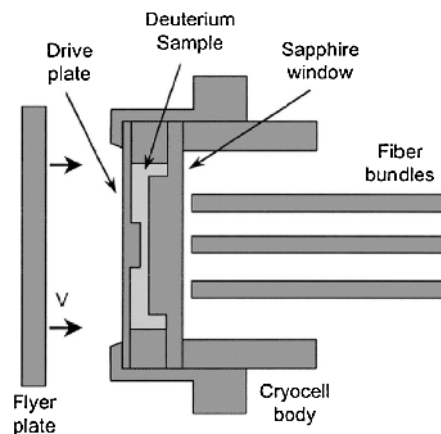


Fig. 3.24. Flyer-plate driven impedance matching experiment. From Knudson et al.

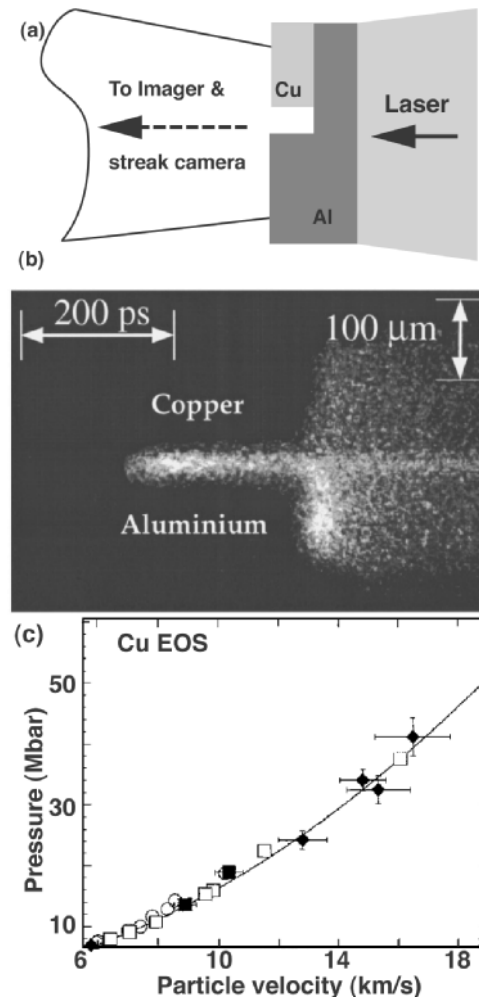
(see Knudson et al.) to determine the EOS of  $D_2$  by this method, at pressures above 1 Mbar. The Al flyer plate impacted a “drive plate”, also of Al, producing an interface velocity of half the flyer-plate velocity and a shock pressure ( $p_{A1}$ ) known from the Al EOS. The flyer plate was thick enough to sustain this pressure throughout the experiment. When the shock in the Al reached the  $D_2$ , it drove a transmitted shock through the  $D_2$  (and a rarefaction into the Al). The diagnostics measured the emergence of the transmitted shock after it had propagated through one of two thicknesses of  $D_2$ , thus giving  $u_{sB}$ , which with the density of  $D_2$  defines the slope of the curve for  $p_{B2}$  in Fig. 3.23. From this the authors obtained the equation of state, to a sufficient accuracy to exclude some models.

An alternative way to apply the pressure to material  $A$  is to use a laser or other radiation source to do so. In this case the initial shock velocity in material  $A$  is not as well known, so one must also measure it. Figure 3.25 shows a measurement of the EOS of Copper by this method. A steady shock wave was driven through a stepped aluminum plate using a pressure source (in this case from laser ablation; see Chap. 8). A sample of Cu abuts part of the surface of the thinner step. The optical emission produced upon the emergence of the shock from each region is detected using an optical streak camera. Such data are shown in Fig. 3.25b. What matters is the time difference between the signal in the middle (from the thin Al step) and the signals through the two thicker layers. An experimental complication is that the edges of the samples affect the shock propagation (slowing it down from the edges inward). This limits how thick one can make the samples, which limits the accuracy of the measurement.

The emergence of the shock is detected from both the thin and the thick Aluminum steps. This determines the shock velocity in the Al, from which the known Al EOS implies the postshock fluid velocity and the pressure  $p_{A1}$  in the Al. The time of emergence of the shock from the Cu sample then determines the shock velocity in the sample,  $u_{sB}$ . Then one applies the analysis illustrated in Fig. 3.23 to find the pressure and postshock fluid velocity in the Cu, shown in part Fig. 3.25c.

### 3.6.3 Other Techniques

Several other techniques have been demonstrated in recent years at this writing. Here we provide a summary and a few early references. The first of these involves the measurement of the reflected shocks in the sample material (Mostovych et al. 2001). In a system like that shown in Fig. 3.24, the shock wave reflects after it traverses the  $D_2$ . By measuring the properties of the reflected shock, or of later reflections, one can obtain a more accurate test of EOS models than one can by measuring a single shock.



**Fig. 3.25.** Laser-driven impedance-matching experiment. (a) Sketch. (b) Spatial profile of emission with time increasing to right. (c) Inferred EOS of Cu. Parts (b) and (c) are from Benuzzi et al.

As a second alternative, techniques have been developed for shockless compression of samples, opening up new terrain on the plot of pressure versus density. Such isentropic compression experiments (ICE) were first developed on a pulsed power facility (Asay et al. 1999). In this case, slowly increasing magnetic pressure was used. This allowed one to measure the time variation of the velocity of an (unshocked) surface. Such measurements have for example observed the fcc to hcp phase transition in Fe, showing that this technique can be used to study the kinetic, time-resolved evolution of a phase transition. One can attempt similar experiments using a laser as the energy source

(Edwards et al. 2003), by creating a flow of material that gently accumulates against the surface of a sample.

A third technique is possible if the shock in a transparent material is strong enough to induce pressure ionization at the shock front. Then a probe laser can be reflected directly off the shock front (not the interface) (Celliers et al. 2000). Converting the reflected signal into an interferogram, via a technique called VISAR (Barker and Hollenback 1972), allows the shock velocity to be measured along with shock reflectivity. The reflectivity can in turn be used to infer the conductivity of the shocked material, thereby determining whether the dielectric-to-metallic transition has occurred.

Fourth, one can measure shock breakout without a streak camera, using a chirped laser pulse (Benuzzi-Mounaix et al. 1999). The chirp, produced by passing a broadband laser pulse through a long fiber optic, sets a one-to-one correspondence between spectral component ( $\lambda$ ) and time, namely,  $t_{\text{sbo}} = a + b\lambda_{\text{sbo}}$ . Using this chirped pulse as the probe on the rear surface of the target, and imaging the reflected pulse into a static spectrometer, one can obtain with a high time resolution the shock breakout time by noting the discontinuity in the spectrum.

Fifth, one can measure the solid-state shock compression of moderately strong shocks ( $p < 1$  Mbar) for single crystal samples using time-resolved Bragg diffraction (Loveridge-Smith et al. 2001). As the lattice is shocked, the lattice spacing ( $2d$ ) gets smaller, and the position of the Bragg peak shifts according to Bragg's law. This directly gives the compression. By simultaneously diffracting off orthogonal lattice planes, one can observe the 3D crystal response in the compressed state, that is, the relaxation from a shocked 1D to a compressed 3D state.

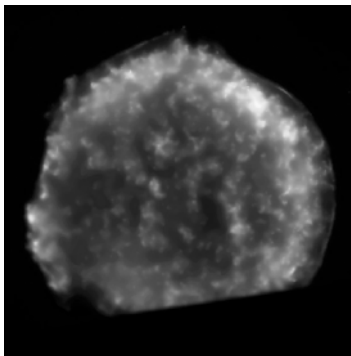
Sixth, in a completely different regime, one can use ultrashort pulse ( $\sim 100$  fs) lasers to probe the EOS of pressure-ionized matter at solid density, in the absence of any hydrodynamic motion or expansion. The laser pulses are so brief that the samples have no time to hydrodynamically expand, and one is probing the plasma behavior at ambient density. This technique has been demonstrated on a carbon sample (Nantel et al. 1998).

## 4 Shocks and Rarefactions

The word “shock” is used very widely in common experience. One is shocked by an unexpected event; a wounded victim goes into shock; and one shocks a material by suddenly cooling it. A “shock wave” is a sudden transition in the properties of a fluid medium, involving a difference in flow velocity across a narrow (ideally, abrupt) transition. In high-energy-density physics, nearly any experiment involves at least one shock wave. Such shock waves may be produced by applying pressure to a surface or by creating a collision between two materials. In astrophysics, nearly every sudden event produces a shock wave. Yet in common experience one encounters very few shock waves. We hear thunder after lightning, which is a long-term consequence of the shock wave produced by the lightning channel, but as we shall see below one would hope never to directly experience this shock wave. Most of us hear sonic booms infrequently, but they are the only shock wave of human origin we typically encounter.

We have more direct experience with rarefactions or “rarefaction waves,” in which a fluid begins to move, expanding and becoming less dense, with the edge of the moving region propagating into an initial body of fluid. Household drafts may be due to rarefactions, which can occur in a house, for example, when a gust of wind drops the pressure at an open door, by the Bernoulli effect. Rarefactions also have real practical uses, notably in refrigeration where they are used to produce expansion cooling. It is also true that nearly every high-energy-density experiment involves at least one rarefaction wave.

Moreover, most high-energy-density experiments involve at least one interface, where the density (and perhaps the equation of state) changes. Whenever a shock wave or a rarefaction wave reaches an interface, there are transmitted and reflected waves in response. In each of these two directions, these waves might be either shock waves or rarefaction waves, so that there are four possible responses. Which of these four occurs depends on the details. One can find systematic discussions of this in books on shock physics. As we proceed to consider various cases, we will encounter specific examples. It should be clear that shock waves, rarefaction waves, and their interactions merit a serious examination, which we undertake in this chapter.



**Fig. 4.1.** An image of the Tycho supernova remnant. Credit: National Aeronautics and Space Administration, Chandra X-ray Center, Smithsonian Center for Astrophysics

#### 4.1 Shock Waves

Figure 4.1 shows an image of the supernova remnant known as Tycho. The remnant shown in the image has sharp edges, where spectral measurements show that the temperature reaches 20 million degrees. This is one of many examples of an astrophysical shock. The magnetic field in such a shock is not dynamically important, except that it localizes the particles as discussed in Chap. 11. For this reason, a laboratory experiment can hope to produce dynamics similar to those of this shock. (There are caveats – it is possible that the dynamics of the shock in the remnant causes the magnetic field to grow, and it is also possible that cosmic ray acceleration at the shock has an effect on the shock itself. Both are active areas of research at this writing.) In contrast, the weaker shock waves produced by the sun are very much affected by the magnetic field. We will not take up magnetized shocks here, leaving this subject for books on space plasma physics.

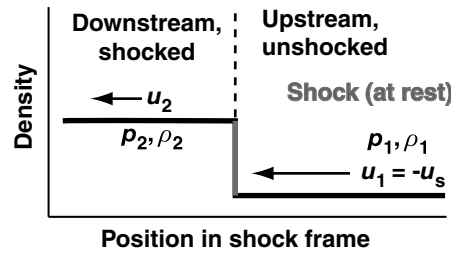
Even the simpler, unmagnetized shock waves are at first glance mysterious. Why would a fluid decide to abruptly change its properties? When we make music louder, the energy flux of the sound waves carrying energy to our ears increases. Why does the energy flux of sound waves not just increase as necessary to transport as much energy as is needed? The fundamental answer is that sound waves move at the sound speed, and that the largest pressure modulation they can transport is of order the initial pressure of the fluid. This pressure is of order  $\rho c_s^2$ , so the largest energy flux the sound waves could imaginably carry is of order  $\rho c_s^3$ . But one can readily force a fluid to carry more energy than this, either by rapidly moving its boundary or by releasing energy within it. For example, the pressure in a singly ionized plasma at a temperature of 1 eV and a density of  $1 \text{ g/cm}^3$  is of order a million atmospheres (1 Mbar). In high-energy-density experiments, much larger pressures are easy to obtain. The plasma cannot respond to such pressures



by radiating sound waves. Instead, a shock wave forms. We discuss its basic properties in this section.

#### 4.1.1 Jump Conditions

The shock wave does three things. First, it carries energy forward at the shock velocity, which is supersonic. Second, it heats and accelerates the medium as it passes, so that the fluid behind the shock carries kinetic energy and thermal energy (these are equal in strong shocks, in the frame of reference in which the upstream fluid is at rest, known as the *laboratory frame*). Third, the shock wave heats the fluid behind the shock so that the motion of the shock wave relative to the heated fluid is subsonic. As a result, changes in the source of the energy are communicated to the shock front at the (new, higher) sound speed. For the original fluid, though, the arrival of the disturbance comes as a shock. Figure 4.2 shows a schematic diagram of a shock wave, in the frame of reference in which the shock is at rest, known as the *shock frame*. We will work consistently from a viewpoint in which the shock moves from left to right so that in the shock frame the fluid flows from right to left.



**Fig. 4.2.** Diagram of an isolated, steady shock, in a reference frame that moves with the shock. Here  $u_1$  and  $u_2$  are  $< 0$  and we define the shock velocity as  $u_s = -u_1$  in this reference frame

Establishing a discontinuity does not in any way relieve the system from the conservation of mass, momentum, and energy, however. To explore this, we begin with the Euler equations in conservative form (with a scalar pressure and explicit internal energy terms):

Continuity

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho \mathbf{u}), \quad (4.1)$$

Momentum

$$\frac{\partial}{\partial t} (\rho \mathbf{u}) = -\nabla \cdot (\rho \mathbf{u} \mathbf{u}) - \nabla p, \text{ and} \quad (4.2)$$

Energy

$$\frac{\partial}{\partial t} \left( \frac{\rho u^2}{2} + \rho \epsilon \right) = -\nabla \cdot \left[ \rho \mathbf{u} \left( \epsilon + \frac{u^2}{2} \right) + p \mathbf{u} \right]. \quad (4.3)$$

Now consider a planar disturbance and integrate any one of these equations across a small region that may include an abrupt change in parameters. In the notation of Sect. 2.1 for a general equation in conservative form, we will have

$$\int_{x_1}^{x_2} \frac{\partial}{\partial t} \rho_Q dx' = - \int_{x_1}^{x_2} \frac{\partial}{\partial x} \Gamma_Q(x') dx' = \Gamma_Q(x_2) - \Gamma_Q(x_1). \quad (4.4)$$

The integral on the left approaches zero as  $x_2 - x_1$  becomes infinitesimal, but the fluxes  $\Gamma_Q$  need not. Instead, in the limit that  $x_2 - x_1 \rightarrow 0$ , one has  $\Gamma_Q(x_2) = \Gamma_Q(x_1)$ . This analysis evidently applies to a fixed location within one's coordinate system, and this is part of the importance of the shock frame, in which the shock location remains fixed at some  $x$  (typically  $x = 0$ ). Applying (4.4) to (4.1)–(4.3), one finds the *jump conditions* for a shock wave, which are

$$\rho_1 u_1 = \rho_2 u_2, \quad (4.5)$$

$$\rho_1 u_1^2 + p_1 = \rho_2 u_2^2 + p_2, \text{ and} \quad (4.6)$$

$$\left[ \rho_1 u_1 \left( \epsilon_1 + \frac{u_1^2}{2} \right) + p_1 u_1 \right] = \left[ \rho_2 u_2 \left( \epsilon_2 + \frac{u_2^2}{2} \right) + p_2 u_2 \right], \quad (4.7)$$

in the event that the motion is one-dimensional. More generally, for shock jump conditions we need integrate in the direction of propagation of the shock wave. The two vector quantities,  $\mathbf{u}$  and  $\nabla p$ , may or may not have components transverse to this direction. If all we care about is a single shock wave interacting with a planar interface in a planar system, then we go into the shock frame by choosing a reference frame that is moving in the transverse direction, so as to eliminate the transverse components of  $\mathbf{u}$ . This, however, is not always feasible in practice. We discuss shocks with finite transverse fluid velocity, known as *oblique shocks*, in Sect. 4.1.5.

In a fluid described by the Euler equations, the shock jump must be infinitesimal in width. When one introduces additional phenomena that are always present at some level in actual fluids, such as viscosity, then the shock transition becomes gradual. However, so long as viscosity or other effects are only important near the shock front, then in steady state the jump conditions apply equally well to locations that are far enough from the shock front. Shocks that involve radiation are discussed in Chap. 7. Equations (4.5)–(4.7) can be manipulated to find relations that are convenient in a given context. We consider some of these relations in the next section.

Before proceeding, though, note that there is a seemingly paradoxical aspect to our description of shocks so far. On the one hand, we described a shock wave as something that heats and compresses the medium that flows into it. On the other hand, (4.5)–(4.7) are symmetric in the exchange of the indices. From their point of view the matter flowing into a discontinuity could be heated and compressed, or alternatively could be cooled and made less dense (rarefied). An abrupt transition in which matter was cooled and

rarefied would be described as a *rarefaction shock*. From the point of view of the conservation equations, a rarefaction shock could exist. However, it is forbidden by the Second Law of Thermodynamics, as we will see in Sect. 4.1.4.

#### Homework 4.1

Add a gravitational force density and gravitational potential energy to (4.2) and (4.3) and derive the modified jump conditions.

#### Homework 4.2

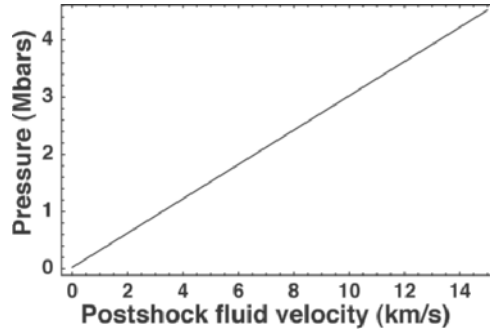
Suppose that during the shock transition significant energy is lost by radiation. Write down the modified jump conditions.

#### 4.1.2 The Shock Hugoniot and Equations of State

A shock wave can place a material in a new state, whose properties depend, for example, on the amount of internal energy the material requires at a given pressure and density. By varying the initial density, pressure, and velocity of the material, one can access a continuous sequence of final states. One of the primary methods used to determine the equation of state involves measurements using shock waves. These measurements determine points along the *Rankine–Hugoniot relation*, which is traditionally identified as the function  $p(p_1, 1/\rho_1, 1/\rho_2)$ . The inverse of the density is the *specific volume*, often written as  $V = 1/\rho$ . The use of the postshock (downstream) density is an arbitrary choice. One can use any of the postshock parameters, and indeed one sees Rankine–Hugoniot curves plotted in various ways. (Figures 3.22 and 3.25 show examples.) Let us consider how measurements can determine the Rankine–Hugoniot relation. This relation is also often called the shock Hugoniot, even though Rankine’s work (in 1870) came 17 years before Hugoniot’s (in 1887).

One often can manage to measure the shock velocity and the postshock fluid velocity. The shock velocity can be determined, for example, by measuring when the shock emerges from varying thicknesses of shocked material. This can often be done using emission from the surface, which is strongly heated by the shock wave. Measurements of the postshock fluid velocity use targets in which the shock wave crosses an interface whose motion can be measured, for example, using the Doppler shift of light reflected from it or measuring its shadow with x-ray radiography.

Such measurements are typically done in the inertial frame of the laboratory where the upstream fluid is at rest. In this case, the postshock fluid velocity,  $u_p$ , that one measures is the difference between the incoming and outgoing velocities in the shock frame, shown in Fig. 4.2. Thus  $u_p = u_1 - u_2$  or  $u_2 = u_1 - u_p$ . Then from (4.5) and (4.6) we can find



**Fig. 4.3.** A useful practical application of (4.9). Here the independent variable is the postshock fluid velocity in the lab frame,  $u_p$ , in km/s, the ordinate is postshock pressure in Mbars, and the curve is for  $\rho_1 u_1 = 30 \text{ g cm}^{-3} \text{ km s}^{-1}$

$$\frac{\rho_2}{\rho_1} = 1 + \frac{u_p}{u_1 - u_p} \quad \text{and} \quad (4.8)$$

$$p_2 - p_1 = \rho_1 u_1 (u_1 - u_2) = \rho_1 u_1 u_p. \quad (4.9)$$

It is a very neat trick to determine the thermodynamic state of the fluid from two measurements (of  $u_1$  and  $u_p$ ), but this is the power of shock Hugoniot measurements. However, determining  $u_p$  is often not easy. We discussed some of the experimental approaches to shock Hugoniot measurements in Sect. 3.6. Figure 4.3 illustrates (4.9), showing how measurements of  $u_1$  and  $u_p$  determine  $p_2 - p_1$ . Researchers doing shock Hugoniot measurements with flyer plates often work with this equation using graphs like that of Fig. 4.3a, which allow one to think directly in terms of the measured quantities.

#### 4.1.3 Useful Shock Relations

The jump conditions are sometimes useful as they stand, but there are also useful alternative solutions of these equations. There is an upstream fluid, within which the distance from any fluid element to the shock decreases with time, and which we will designate by the subscript 1. There is also a downstream fluid, within which the distance from any fluid element to the shock increases with time, and which we will designate by the subscript 2. The most useful equations relate specific properties of the upstream fluid to those of the downstream fluid and the *upstream Mach number*,  $M_u$ . This Mach number is defined as the ratio of the rate at which the upstream material and the shock approach one another to the sound speed in the upstream material. The rate at which the upstream material and the shock approach one another is often called the *shock velocity*, which is also the velocity of an isolated shock when the upstream fluid is at rest in the inertial frame of reference of the laboratory. We will designate this as  $u_s$ , which will equal  $|u_1|$  in the simple case we now discuss. However, when one analyzes complex

systems with several shocks, one must think carefully to properly identify the upstream Mach number for each one.

To obtain useful solutions of (4.5)–(4.7), we work in the *shock frame*. In the (one-dimensional) shock frame, one has three equations and four variables in the upstream state, so one needs an equation of state to relate  $\epsilon$  to  $p$ . It is most useful to assume the fluid to be a polytropic gas so that  $\rho\epsilon = p/(\gamma - 1)$ . Thus, in terms of the various polytropic indices discussed in Chap. 3 we are working with the shock gamma. This one term in the energy equation is the source of all the factors involving  $\gamma$  that appear in the following. However, for high-energy-density physics we should note that the polytropic index sometimes differs greatly across a shock. We may, for example, start with a cold, highly-ordered crystal for which  $\gamma$  approaches 3 and shock it into an ionizing, plasma state for which  $\gamma \sim 4/3$ . The equations (4.5–4.7) still apply in such a case, with  $\gamma$  evaluated appropriately on each side of the interface. In what follows, we provide both the traditional results obtained when  $\gamma$  is taken to be unchanging (and is not subscripted) and also results in which  $\gamma$  is subscripted and applies separately to the fluid on the two sides of the interface.

Thus, the velocity  $u_1$  is the shock velocity, and the upstream Mach number is  $M_u = -u_1/c_{s1} = u_s/c_{s1}$ , which for a polytropic gas is  $u_s\sqrt{\rho_1/(\gamma_1 p_1)}$ . Solving these equations for the ratio of the pressures, one can show that

$$\frac{p_2}{p_1} = \frac{\rho_2(\gamma + 1) - \rho_1(\gamma - 1)}{\rho_1(\gamma + 1) - \rho_2(\gamma - 1)} \quad \text{or} \quad (4.10)$$

$$\frac{p_2}{p_1} = \left[ \frac{\rho_2(\gamma_1 + 1) - \rho_1(\gamma_1 - 1)}{\rho_1(\gamma_2 + 1) - \rho_2(\gamma_2 - 1)} \right] \frac{(\gamma_2 - 1)}{(\gamma_1 - 1)}.$$

Figure 4.4a shows this pressure ratio as a function of the density ratio  $\rho_2/\rho_1$ . Alternatively, if one does the algebra by hand, for constant  $\gamma$ , it is easier to relate the pressures to the specific volumes (the inverse of density) as

$$\frac{p_2}{p_1} = \frac{V_1(\gamma + 1) - V_2(\gamma - 1)}{V_2(\gamma + 1) - V_1(\gamma - 1)}. \quad (4.11)$$

Note that the postshock pressure implied by these two equations diverges when the denominator becomes zero at a specific density ratio. One can rearrange (4.10) to find the density ratio in terms of the pressures, which is

$$\frac{\rho_2}{\rho_1} = \frac{p_2(\gamma + 1) + p_1(\gamma - 1)}{p_1(\gamma + 1) + p_2(\gamma - 1)} \quad \text{or} \quad (4.12)$$

$$\frac{\rho_2}{\rho_1} = \left[ \frac{p_2(\gamma_2 + 1) + p_1(\gamma_2 - 1)}{p_1(\gamma_1 + 1) + p_2(\gamma_1 - 1)} \right] \frac{(\gamma_1 - 1)}{(\gamma_2 - 1)}.$$

This makes it very clear that as  $p_2$  becomes  $\gg p_1$ , the density approaches a fixed density ratio given by

$$\frac{\rho_2}{\rho_1} = \frac{(\gamma + 1)}{(\gamma - 1)}. \quad (4.13)$$

This density ratio is the physical limit that can be produced by a single shock in a polytropic gas, and only the postshock value of  $\gamma$  enters. Shocks encountered in high-energy-density physics often have density ratios near this limit, and thus are *strong shocks* as defined shortly. Low atomic number materials, subject to strong enough shocks, may behave this way with  $\gamma \sim 5/3$ . Thus, the density ratio  $\rho_2/\rho_1$  may be 4 to 1. One may encounter differences if the internal energy of the shocked material is a significant fraction of the thermal energy density, as for example in materials that are ionizing as discussed in Chap. 3. In this case, the jump conditions [specifically (4.6) and (4.7)] imply that  $u_2$  must be smaller than it would be otherwise – the increased internal energy comes from the kinetic and thermal energies. As a result, (4.5) implies that  $\rho_2$  must be larger than it would be otherwise. (In some cases, radiation lost during ionization can have a similar effect.) There is still a limiting density ratio, but it is affected by the properties of the material. Xenon in particular is a gas that both absorbs a lot of ionization energy and radiates strongly under typical experimental conditions. Accordingly, a strong shock in xenon produces a larger density jump than is produced in a lower- $Z$  gas such as nitrogen. The way that this enters the mathematics is that the value of  $\gamma$  is smaller in such a material, just as we saw in Chap. 3. In terms of an effective polytropic index, xenon typically would have  $\gamma \sim 1.2$  to 1.3 at densities above atmospheric density. At lower densities, weakly ionized xenon can store a great deal of energy in excited states. If one accounts for this by adjusting  $\gamma$ , then  $\gamma$  can be driven down below approximately 1.1.

Returning to (4.12), the traditional approach is to find useful expressions involving  $M_u$ , for example, by substituting for  $p_2$  from (4.9). We then can obtain for the density ratio

$$\frac{\rho_2}{\rho_1} = \frac{M_u^2(\gamma + 1)}{M_u^2(\gamma - 1) + 2}. \quad (4.14)$$

This ratio exhibits the behavior we expect, tending to the value of 4.13 as  $M_u$  becomes large. In general, the limit in which  $M_u$  is large and only terms in the largest power of  $M_u$  need to be kept is referred to as the *strong shock limit*. In a similar way, we find the pressure ratio

$$\frac{p_2}{p_1} = \frac{2\gamma M_u^2 - (\gamma - 1)}{(\gamma + 1)}, \quad (4.15)$$

which increases indefinitely as  $M_u$  increases.

Unfortunately, in real high-energy-density systems  $M_u$  is often very poorly known. This is because the upstream temperature might be room

temperature, at which the system is typically prepared, a significantly smaller temperature through cooling in vacuum, or a significantly higher temperature because of small levels of heating by radiation or electrons in advance of the shock. The uncertainty in  $M_u$  can easily be a factor of several. However, the stronger the shock the less this matters. One can show this by working with (4.5), (4.6), and (4.10), dividing the pressures by  $\rho_1 u_s^2$ . Then one can define

$$S = \sqrt{1 + \left(\frac{\gamma_2 p_1}{\rho_1 u_s^2}\right)^2 + \frac{2p_1(\gamma_1 - \gamma_2)}{\rho_1 u_s^2(\gamma_1 - 1)}}. \quad (4.16)$$

Note that  $S$  approaches 1 as shock velocity increases. (For  $\gamma_1 = \gamma_2$ ,  $S = 1 - \gamma_2 p_1 / (\rho_1 u_s^2)$ .) This allows one to write the density ratio as

$$\frac{\rho_2}{\rho_1} = \left[ \frac{(\gamma_2 + S) + \frac{\gamma_2 p_1}{\rho_1 u_s^2}}{(\gamma_2 - 1) + 2 \frac{\gamma_1 p_1}{\rho_1 u_s^2} \frac{(\gamma_2 - 1)}{(\gamma_1 - 1)}} \right]. \quad (4.17)$$

The corresponding density ratio is shown for various values of  $\gamma_1$  and  $\gamma_2$  in Fig. 4.4a. While the eventual density ratio reached in a strong shock is not affected by  $\gamma_1$ , the ratio of  $\rho_1 u_s^2 / p_1$  required to approach this value is affected. The downstream pressure can similarly be written as

$$p_2 = \frac{2}{(\gamma + 1)} \rho_1 u_s^2 \left[ 1 - \frac{(\gamma - 1)p_1}{2\rho_1 u_s^2} \right]$$

or

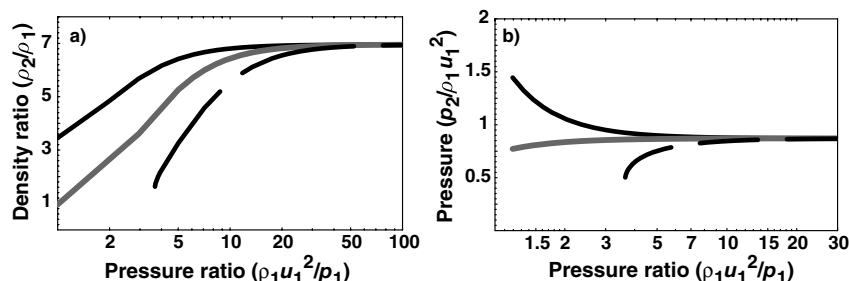
$$p_2 = \frac{\rho_1 u_s^2}{(\gamma_2 + S) + \frac{\gamma_2 p_1}{\rho_1 u_s^2}} \left[ 1 + S \left( 1 + \frac{p_1}{\rho_1 u_s^2} \right) + \frac{p_1}{\rho_1 u_s^2} \left( \frac{2(\gamma_1 - \gamma_2)}{(\gamma_1 - 1)} + \frac{\gamma_2 p_1}{\rho_1 u_s^2} \right) \right]. \quad (4.18)$$

The first form is useful for quick estimates assuming a single value for  $\gamma$ . Even more useful is the realization that  $2/(\gamma+1)$  is of order 1 so for strong shocks  $p_2 \sim \rho_1 u_s^2$ , which is easy to remember and to evaluate. Figure 4.4b shows the dependence of  $p_2/(\rho_1 u_s^2)$  on  $\rho_1 u_s^2/p_1$ .

One can proceed to obtain a similar expression for the temperature, taking  $p_2 = (Z_2 + 1)k_B T_2 \rho_2 / (Am_p)$ , where the electrons are assumed to fully equilibrate with the ions and not to be degenerate. One finds

$$k_B T_2 = \frac{Am_p}{(1 + Z_2)} u_s^2 \frac{2(\gamma_2 - 1)}{\left( (\gamma_2 + S) + \frac{\gamma_2 p_1}{\rho_1 u_s^2} \right)^2} \left( 1 + \frac{2\gamma_1 p_1}{(\gamma_1 - 1)\rho_1 u_s^2} \right) \times \left[ \frac{(1 + S)}{2} + \frac{p_1}{\rho_1 u_s^2} \left( \frac{S}{2} + \frac{(\gamma_1 - \gamma_2)}{(\gamma_1 - 1)} \right) + \frac{\gamma_2}{2} \left( \frac{p_1}{\rho_1 u_s^2} \right)^2 \right]. \quad (4.19)$$

Here  $Z_2$  is the average ionization of the postshock state. In the strong shock limit we find



**Fig. 4.4.** (a) The ratio of postshock to preshock density depends as shown on the ratio of  $\rho_1 u_1^2$  to preshock pressure, from (4.17). Here  $\gamma_2 = 4/3$ . The values of  $\gamma_1$  are 3 (black curve),  $4/3$ , (gray curve), and 1.1 (dashed curve). (b) The ratio of postshock pressure to  $\rho_1 u_1^2$  depends as shown on the ratio of  $\rho_1 u_1^2$  to preshock pressure, from (4.18). This ratio asymptotes to  $2/(\gamma+1)$ . Here  $\gamma_2 = 5/3$ . The values of  $\gamma_1$  are 3 (black curve),  $4/3$ , (gray curve), and 1.1 (dashed curve)

$$k_B T_2 = \frac{A m_p}{(1 + Z_2)} u_s^2 \frac{2(\gamma_2 - 1)}{(\gamma_2 + 1)^2}. \quad (4.20)$$

In shocks with weak collisionality, because of low density or high temperature, the immediate postshock temperature of the ions can be found by setting  $Z_2 = 0$  and  $\gamma_2 = 5/3$  in this equation. This also applies to shocks in atomic neutral gasses, which is why one may encounter  $k_B T_2 = (3/16) A m_p u_s^2$  in various places as a standard expression. However, in a highly ionized plasma with strong collisional coupling of electrons and ions, it is evident that this standard expression can greatly overestimate the temperature.

### Homework 4.3

Determine from energy arguments how to generalize (4.20) for a two-species plasma.

Strong shocks have some additional properties that are worthwhile to know. As always, the velocity ratio is the inverse of the density ratio, in this limit being

$$\frac{u_2}{u_1} = \frac{(\gamma - 1)}{(\gamma + 1)}. \quad (4.21)$$

From this one can find the postshock particle velocity,  $u_p$ , in the lab frame (in which the upstream fluid is at rest; see the discussion near (4.8)). This is

$$u_p = u_s + u_2 = \frac{2}{(\gamma + 1)} u_s, \quad (4.22)$$

which is  $(3/4)u_s$  for  $\gamma = 5/3$  and  $(6/7)u_s$  for  $\gamma = 4/3$ . So the postshock fluid velocity is approximately 80% of the shock velocity in typical materials. It



is easy to show that the postshock fluid velocity becomes closer to the shock velocity as the density jump increases. If we examine the postshock pressure  $p_2$  in the strong-shock limit, we can substitute for  $\rho_1$  and  $u_s$  to find

$$p_2 = (\gamma - 1) \frac{\rho_2 u_p^2}{2}, \quad (4.23)$$

so one can see that, in a frame of reference with a stationary upstream fluid, a strong shock in a polytropic gas produces a postshock fluid in which the kinetic and internal energies are equal [because  $\rho\epsilon = p/(\gamma - 1)$ ]. Using (4.23), we can look again at some of the velocities in the strong shock limit. The sound speed in the shocked fluid is  $\sqrt{\gamma p_2/\rho_2} = \sqrt{\gamma(\gamma - 1)/2} \times u_p$ , which for  $\gamma = 5/3$  is  $\sqrt{5/9}u_p$ . Thus, in the laboratory frame (and for  $\gamma < 2$ ) the flow is supersonic, as the sound speed is less than  $u_p$ . However, the source of pressure that sustains the shock must move at  $u_p$ , and the distance between it and the shock increases at speed  $u_2$ , which can easily be shown to be  $(\gamma - 1)/2 \times u_p$ . This is  $(1/3)u_p$  for  $\gamma = 5/3$ . Thus the separation of the pressure source from the shock is subsonic. This last statement is completely equivalent to saying that, in the shock frame, the downstream fluid moves subsonically.

#### Homework 4.4

Appendix B shows a derivation of (4.10)–(4.15). For  $\gamma_1 = \gamma_2$ , derive (4.18) and (4.20). Using a computational mathematics program is suggested.

#### Homework 4.5

Derive from (4.10) and (4.12) a general expression for  $T_2$ , valid for weak and strong shocks, for  $\gamma_1 = \gamma_2$ . Express the result in physically clear parameters, so the relation among the terms is evident. Check your result by finding it as a limit of (4.19) and by finding (4.20) as a limit from it.

We can also look at some typical parameters. A high-energy-density experiment may produce a shock wave in a plastic material having  $\rho_1 = 1\text{g/cm}^3$ ,  $A = 6.5$ , and  $Z = 3.5$  using a pressure of 50 Mbars (i.e.,  $5 \times 10^{13}$  dynes/cm<sup>2</sup>). The plastic behaves like a polytropic gas with  $\gamma \sim 4/3$ , so the shocked density is  $\rho_2 \sim 7\text{g/cm}^3$ . The shock velocity, from (4.18), is approximately 80 km/s ( $8 \times 10^6$  cm/s), so the shock will traverse a 100  $\mu\text{m}$  thick layer in about 1.2 ns. The postshock temperature, from (4.20), is approximately 25 eV. For comparison, consider the shock in the interstellar medium produced by a supernova remnant such as Tycho. The velocity is  $\geq 1,000$  km/s (faster in younger remnants) and  $\rho_1 \sim 1\text{amu/cm}^3 \sim 10^{-24}\text{g/cm}^3$ . Thus  $p_2 \sim 10^{-8}$  dynes/cm<sup>2</sup>. This seems small, but the postshock temperature is  $\geq 1$  keV, so the resulting plasma is quite hot. However, even for the velocities approximately 10 times higher that are present in very young supernova remnants, the temperature is not near relativistic values. Thus, except for the

cosmic rays produced at the shock, nonrelativistic theories and experiments can address the behavior of such systems.

#### 4.1.4 Entropy Changes Across Shocks

While mass, momentum, and energy are conserved across shock transitions, entropy is not. This should not come as a surprise to anyone who has studied statistical or thermal physics, as one is increasing the temperature of the fluid in a nonadiabatic transition. Here we determine and discuss the change in entropy across the shock transition.

Using (2.11) for the specific entropy and (4.12), one can find the difference in entropy across the shock wave to be

$$s_2 - s_1 = c_V \ln \left[ \frac{p_2}{p_1} \left( \frac{\rho_1}{\rho_2} \right)^\gamma \right]. \quad (4.24)$$

Thus, in an adiabatic transition that keeps  $p/\rho^\gamma$  constant, the entropy does not increase. Shocks increase entropy because they are irreversible, non-adiabatic transitions. For single strong shocks, the argument of the logarithm is dominated by  $p_2$ , as the density ratio varies only over a limited range. Taking the strong shock limit, we find

$$s_2 - s_1 = c_V \ln \left[ \frac{p_2}{p_1} \left( \frac{\gamma - 1}{\gamma + 1} \right)^\gamma \right] \sim c_V \left[ \ln \left( \frac{p_2}{p_1} \right) - 2.2 \right], \quad (4.25)$$

in which the final equality is obtained by setting  $\gamma = 5/3$  and evaluating  $p_2$  in Mbars and  $p_1$  relative to 30 kbars, with  $s$  and  $c_V$  in cgs units. Noting once again that in a fully ionized plasma both electrons and ions carry heat, one has

$$c_V = \frac{(Z + 1)k_B}{Am_p(\gamma - 1)} = \frac{(Z + 1)}{A(\gamma - 1)} 9.57 \times 10^7 \text{ J/(keV g)}, \quad (4.26)$$

expressed to accommodate temperature in keV. As we will see, the entropy in an inertial fusion capsule needs to be kept below about  $4 \times 10^8 \text{ J/(keV g)}$ . Since the ratio of pressures is  $>1,000$  in this case, compression for inertial fusion cannot occur by means of a single shock.

#### Homework 4.6

Evaluate the entropy variation of (4.24) as the Mach number approaches 1.

It may seem strange to the reader that entropy is generated by shock waves, when we have shown that shock waves are consistent with the fundamental equations for mass, momentum, and energy in the fluid, because these same equations produced (2.1.3), which we showed in Chap. 2 to express the conservation of entropy. In other words, if the conservation of entropy is consistent with and derived from the equations that include the presence of

shocks, why is entropy not conserved across shocks? The solution to this puzzle is that shocks are in fact dissipative structures whose details cannot be described by (4.1)–(4.3). From the point of view of these three equations, the shock transition can be taken to be a thin layer of zero thickness, because it does not alter the mass, momentum, or energy fluxes in the flow. However, the shock does convert kinetic energy to heat, and thus is an entropy source. If we were to describe the action of the shock in detail, we would need to add terms to the momentum and energy equations to account for the changes in the fluid. When we then used these equations to obtain an equation for the entropy, we would be left with an entropy source term, and this source term would not have a limit of zero as the thickness of the shock transition approached zero.

Now suppose that instead of using a single strong shock to achieve a desired value of  $p_2/p_1$ , which we write here as  $p_{\text{final}}/p_{\text{init}}$ , we use  $n$  shocks. Each of these will produce pressure ratio  $R_p = (p_{\text{final}}/p_{\text{init}})^{1/n}$  and a density ratio  $R_\rho$ . The density ratio produced by each shock may or may not approach  $(\gamma+1)/(\gamma-1)$ , but the final density ratio  $\rho_{\text{final}}/\rho_{\text{init}}$  will be much larger than  $(\gamma+1)/(\gamma-1)$  for  $n > 1$ . The temperature will be correspondingly smaller, as the pressure is the same yet the density is higher. Thus, the entropy increase will be smaller for multiple shocks than for one shock. Specifically,

$$s_2 - s_1 = n c_V \ln \left[ R_p \left( \frac{1}{R_\rho} \right)^\gamma \right] = c_V \ln \left[ \frac{p_{\text{final}}}{p_{\text{init}}} \left( \frac{1}{R_\rho} \right)^{n\gamma} \right]. \quad (4.27)$$

Figure 4.5 shows the resulting entropy increase, using (4.12) to compute  $R_\rho$ , for a value of  $p_{\text{final}}/p_{\text{init}} = 1,000$ . One can see that only a few shocks are needed to greatly reduce the total increase of entropy. The limit of a very large number of shocks with progressively smaller individual pressure jumps, as  $n \rightarrow \infty$ , is an adiabatic compression, which produces no entropy increase.

The use of multiple shocks to apply the available pressure is thus an important design tool. If one desires to achieve low entropy or high densities, as one does, for example, in ICF, then one should use several shocks. If one desires high temperature, as one does to produce fast ejecta, then one should use a single shock. If one desires to more carefully tailor the density, temperature, and entropy of a final state, then one can use multiple shocks chosen for that purpose.

#### 4.1.5 Oblique Shocks

We now return to the issue of *oblique shocks*, which we must analyze when there is a reason to choose a shock frame with an upstream velocity component transverse to the shock that is nonzero. This issue arose in the context of (4.5)–(4.7), which are valid only for zero transverse velocity. Note that an oblique shock may develop in more than one way. A shock can be launched as an oblique shock, by something as simple as a tilted piston. Also, a shock can

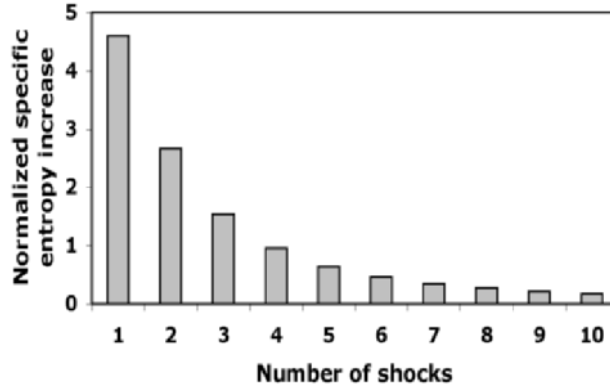


Fig. 4.5. The total increase in normalized specific entropy,  $(s_2 - s_1)/c_v$  is shown vs. the number of shocks, for an overall pressure increase of a factor of 1,000

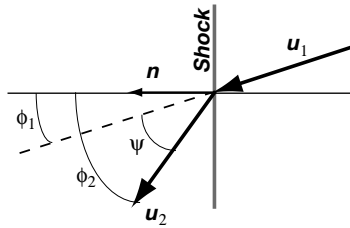


Fig. 4.6. Definitions for oblique shocks

become oblique by interacting with an interface. The geometric definitions we need for this problem are shown in Fig. 4.6.

To deal with nonzero transverse velocity, we define the shock normal, a unit vector  $\mathbf{n}$ , which is normal to the shock front and in the direction of the normal flow into the shock front (thus,  $\mathbf{n}$  points from right to left in our standard orientation). Then the component of  $\mathbf{u}$  in the normal direction is  $\mathbf{n}(\mathbf{u} \cdot \mathbf{n})$  while the transverse component of  $\mathbf{u}$  is  $\mathbf{u}_\perp = (\mathbf{n} \times \mathbf{u}) \times \mathbf{n}$ . Using the subscript  $n$  for the normal direction, (4.1)–(4.3) give us the following relations:

$$\rho_1 u_{n1} = \rho_2 u_{n2}, \quad (4.28)$$

$$\rho_1 u_{n1}^2 + p_1 = \rho_2 u_{n2}^2 + p_2, \quad (4.29)$$

$$u_{\perp 1} = u_{\perp 2}, \text{ and} \quad (4.30)$$

$$\rho_1 \epsilon_1 + p_1 + \rho_1 \frac{u_{n1}^2}{2} = \rho_2 \epsilon_2 + p_2 + \rho_2 \frac{u_{n2}^2}{2}. \quad (4.31)$$

**Homework 4.7**

Derive (4.28)–(4.31).

By comparison with the previous equations, one can see that we have gained one important new piece of information: the transverse velocity is conserved across the shock. Otherwise we have regained the previous equations (or an equivalent one in (4.31)) with  $u$  replaced by  $u_n$ . It is not surprising that the change in  $u_n$  is responsible for the changes in pressure and internal energy.

A clear consequence of these equations is that the fluid velocity is bent away from the normal as it crosses the shock. The normal velocity decreases but the transverse velocity does not, and this is the consequence. We can develop the mathematics for this situation as follows. Suppose that  $\mathbf{u}_1 \cdot \mathbf{n} = u \cos \phi_1$ , thus defining the angle of the incoming flow,  $\phi_1$ . Then the angle of the outgoing flow can be found from

$$\tan \phi_2 = \frac{u_\perp}{u_{n2}} = \frac{u_\perp}{u_{n1}} \frac{\rho_2}{\rho_1} = \tan \phi_1 \frac{\rho_2}{\rho_1}, \quad (4.32)$$

which for a polytropic gas becomes

$$\tan \phi_2 = \tan \phi_1 \frac{M_{nu}^2 (\gamma + 1)}{M_{nu}^2 (\gamma - 1) + 2}, \quad (4.33)$$

in which the upstream Mach number, calculated using  $u_{n1}$ , is  $M_{nu}$ .

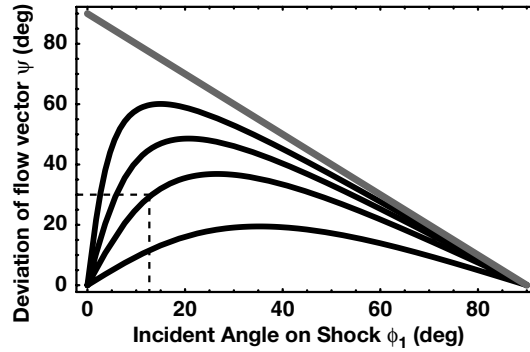
One may in some cases need to know how much the flow is (or can be) deflected by the shock, which is relevant, for example, to the supersonic movement of bodies through fluids. For this purpose we seek  $\psi$  defined by  $\mathbf{u}_1 \cdot \mathbf{u}_2 = u_1 u_2 \cos \psi$ . By expressing the velocities in terms of their normal and tangential components, then dividing by  $u_1 u_2$ , one can show

$$\cos \psi = \cos \phi_2 [\cos \phi_1 + \sin \phi_1 \tan \phi_2], \quad (4.34)$$

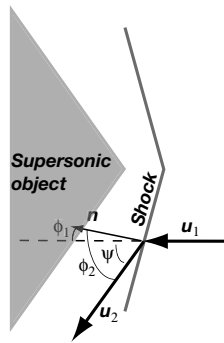
from which via (4.32) we have

$$\cos \psi = \frac{[\cos \phi_1 + \sin \phi_1 \tan \phi_1 (\rho_2 / \rho_1)]}{\sqrt{1 + \tan^2 \phi_1 (\rho_2 / \rho_1)^2}}. \quad (4.35)$$

An interesting implication of (4.35) is that there is a maximum possible angle of deflection that can be produced by a shock. Figure 4.7 shows the dependence of  $\psi$  on  $\phi_1$  for several specific density ratios. One can see that the maximum angle of deflection depends on the density ratio across the shock. If a supersonic object is too blunt, so that it attempts to deflect the incoming material by more than this angle, then a bow shock forms in front of the object, heating the material so that its flow around the object is subsonic. Figure 4.8 illustrates this case. These two figures are most useful for strong shocks with well-known density ratios. For weaker shocks in polytropic gases



**Fig. 4.7.** Flow deflection vs. incident angle for a density ratio from bottom to top of 2, 4, 7, and 14



**Fig. 4.8.** Schematic of object and resulting shocks

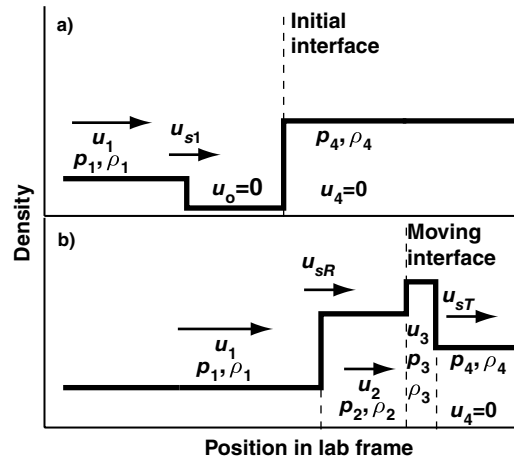
one could substitute from (4.33), noting that the upstream Mach number also depends on angle of incidence. This can be put into a standard form known as the *shock polar*, discussed in *Landau and Lifshitz* and many other texts on fluid dynamics. We will have an interest in the small-angle limit of (4.35) when we consider shock stability in Chap. 5. Assuming that  $\phi_1$  is small enough that  $-(\phi_1^2/2)(\rho_2/\rho_1)^2$  remains small, this is  $\psi = \phi_1(\rho_2/\rho_1 - 1)$ .

**Homework 4.8**

Derive (4.34) and (4.35).

**4.1.6 Shocks and Interfaces, Flyer Plates**

Understanding the basic properties of an isolated shock is an important fundamental building block. There are cases, such as the edge of a supernova remnant or the initial response of a target to laser ablation, in which the



**Fig. 4.9.** Schematic of steady shock incident on interface where density increases. (a) Before shock reaches interface. (b) After shock reaches interface

dynamics is essentially single-shock dynamics. There are many more cases, however, in which the dynamics is produced by the interaction of shock waves, interfaces, and other phenomena. This occurs, for example, when the shock wave in a supernova remnant encounters a molecular cloud or other dense obstacle, and also in experiments that use shock reverberation to compress and heat a material. The first of these that we discuss is the interaction of a shock wave (or in general, an incoming fluid with specific properties) with an interface at which the density increases. In general, this leads to a “reflected” shock in the initial fluid and a “transmitted” shock in the denser fluid. This is an example of the one possible response when a shock reaches an interface – two shocks are produced. Figure 4.9 shows a schematic of such an interaction. We discuss it further here. Other cases are discussed in Sect. 4.4.

The introduction of the second material and the interaction greatly complicates our bookkeeping. As is shown in Fig. 4.9, we designate the unshocked first fluid as state 0, the shocked fluid as state 1, the state of the reflected shock as state 2, the state of the transmitted shock as state 3, and the unshocked second fluid as state 4. We have little interest in the properties of state 0, except in whatever way they influence state 1. In addition, the material in state 4 would expand to the left unless this material is a solid (the typical case) or  $p_o = p_4$ . Our goal is to calculate the properties of the reflected and transmitted shocks, given the properties of the fluids and the initial shock. We can take advantage of two conditions in this calculation. These are  $p_2 = p_3$  and  $u_2 = u_3$ . That is, the pressure and velocity are both continuous across the density interface. If this were not the case, one would produce additional waves (or voids) at the interface in the postshock state, which is not consistent with our definition of the shocks as discontinuous,

localized transitions in fluid properties. To develop an analytic treatment of this behavior, we will designate the polytropic index of the first fluid by  $\gamma_1$  and that of the second fluid by  $\gamma_4$ .

Our approach here will be fairly general, which will give us results that can be applied to a number of specific cases. Some simpler cases, such as the reflection of a shock from a rigid wall or the behavior of an ideal flyer plate, are left as homework problems. In the case of an incoming shock as shown in Fig. 4.9a, one can find  $\rho_1$ ,  $u_1$ , and  $p_1$  from (4.5), (4.14), and (4.15) and the properties of state 0. Alternatively, Fig. 4.9b also may describe the interaction between an incoming block of solid material (a *flyer plate*) that strikes a second block of solid material. In this case, the properties of region 1 will not be determined by shock relations. As an extension of this analysis, one can use Fig. 4.9b to approximately describe the interaction of two colliding fluids with more general properties, if it makes sense after some initial transient to ignore the behavior of their leading edges. This may, for example, occur in experiments that produce a flyer plate that is in the plasma state. The net effect is that we start with the properties of state 1, in which the ratio of  $p_1$  to  $\rho_1 u_1^2$  will depend on the way in which the interaction develops. In any event we have the following equations:

$$\frac{\rho_1}{\rho_2} = \frac{M_{12}^2(\gamma_1 + 1)}{M_{12}^2(\gamma_1 - 1) + 2}, \quad (4.36)$$

$$\frac{p_1}{p_2} = \frac{2\gamma_1 M_{12}^2 - (\gamma_1 - 1)}{(\gamma_1 + 1)}, \quad (4.37)$$

$$\frac{\rho_3}{\rho_4} = \frac{M_{34}^2(\gamma_4 + 1)}{M_{34}^2(\gamma_4 - 1) + 2}, \quad (4.38)$$

$$\frac{p_3}{p_4} = \frac{2\gamma_4 M_{34}^2 - (\gamma_4 - 1)}{(\gamma_4 + 1)}, \quad (4.39)$$

$$u_{sT} - u_3 = u'_3 = u_{sT}\rho_4/\rho_3, \quad (4.40)$$

$$u_1 - u_{sR} = u_{s12}, \quad (4.41)$$

$$u_2 - u_{sR} = u'_2 = u_{s12}\rho_1/\rho_2, \quad (4.42)$$

$$p_2 = p_3, \quad (4.43)$$

$$u_2 = u_3, \quad (4.44)$$

in which  $M_{12} = u_{s12}/c_{s1}$  where  $c_{s1} = \sqrt{\gamma_1 p_1/\rho_1}$ ,  $M_{34} = u_{sT}/c_{s4}$  where  $c_{s4} = \sqrt{\gamma_4 p_4/\rho_4}$ , the upstream velocity in the reflected shock frame is  $u_{s12}$ , the postshock fluid velocity in the reflected shock frame is  $u'_2$ , the postshock fluid velocity in the transmitted shock frame is  $u'_3$ , and the reflected and transmitted shock velocities in the lab frame are  $u_{sR}$  and  $u_{sT}$ , respectively.

The known quantities in these nine equations are  $\rho_1, \rho_4, p_1, p_4, u_1, u_4, \gamma_1, \gamma_4$ . The unknowns are  $\rho_2, \rho_3, p_2, p_3, u_2, u_3, u_{s12}, u_{sR}$ , and  $u_{sT}$ , so the set of equations is closed if complex.



**Homework 4.9**

Derive (4.42). This requires thinking about which frame of reference one is working in, a key element in all such problems.

**Homework 4.10**

Determine the equations and derive the behavior of the simpler case in which a shock is incident on a stationary wall. Let state 0 be the state of the unshocked fluid, state 1 be that of the once-shocked fluid, and state 2 be the state of the reshocked fluid produced when the shock reflects from the wall.

One can, with some work, solve (4.36)–(4.44). One approach is to solve for the various quantities in terms of  $u_2$ , the postshock fluid velocity in both materials, and then to obtain an equation for  $u_2$  itself. This is productive since most aspects of the shock in one material do not depend on the properties of other material. This produces the following eight (4.45)–(4.52) in addition to (4.44):

$$0 = 16\gamma_1 p_1 + \left[ \sqrt{\frac{16c_{s1}^2 + (\gamma_1 + 1)^2(u_1 - u_2)^2}{(\gamma_1 + 1)^2(u_1 - u_2)^2}} - 1 \right] \left[ 4(\gamma_1 + 1)(p_1 - p_4) - (\gamma_1 + 1)(\gamma_4 + 1)\rho_4 u_2^2 \left( 1 + \sqrt{1 - \frac{16c_{s4}^2}{(\gamma_4 + 1)^2 u_2^2}} \right) \right], \quad (4.45)$$

$$\rho_2 = \frac{\rho_1 \left[ 4c_{s1}^2 + (u_1 - u_2) \left( (\gamma_1 + 1)(u_1 - u_2) + \sqrt{16c_{s1}^2 + (\gamma_1 + 1)^2(u_1 - u_2)^2} \right) \right]}{2 \left[ (\gamma_1 - 1)(u_1 - u_2)^2 + 2c_{s1}^2 \right]}, \quad (4.46)$$

$$p_2 = \frac{p_1}{\gamma_1 + 1} \left[ (-\gamma_1 + 1) + \frac{2\gamma_1 \rho_2^2 (u_1 - u_2)^2}{c_{s1}^2 (\rho_1 - \rho_2)^2} \right], \quad (4.47)$$

$$\rho_3 = \frac{(\gamma_4 + 1)\rho_4 \left( (\gamma_4 + 1)u_2 + \sqrt{16c_{s4}^2 + (\gamma_4 + 1)^2 u_2^2} \right)^2}{\left[ 32c_{s4}^2 + (\gamma_4 - 1) \left( (\gamma_4 + 1)u_2 + \sqrt{16c_{s4}^2 + (\gamma_4 + 1)^2 u_2^2} \right)^2 \right]}, \quad (4.48)$$

$$p_3 = \frac{p_4}{(\gamma_4 + 1)} \left[ (-\gamma_4 + 1) + \frac{\gamma_4}{8c_{s4}^2} \left( (\gamma_4 + 1)u_2 + \sqrt{16c_{s4}^2 + (\gamma_4 + 1)^2 u_2^2} \right)^2 \right], \quad (4.49)$$

$$u_{s12} = \frac{\rho_2(u_1 - u_2)}{\rho_2 - \rho_1}, \quad (4.50)$$

$$u_{sT} = \frac{1}{4} \left( (\gamma_4 + 1)u_2 + \sqrt{16c_{s4}^2 + (\gamma_4 + 1)^2 u_2^2} \right), \quad (4.51)$$

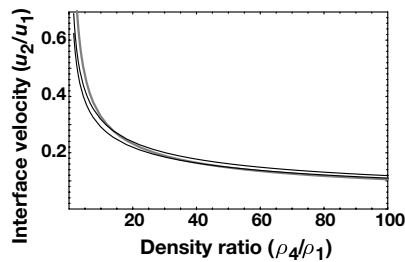
and

$$u_{sR} = \frac{\rho_2 u_2 - \rho_1 u_1}{\rho_2 - \rho_1}. \quad (4.52)$$

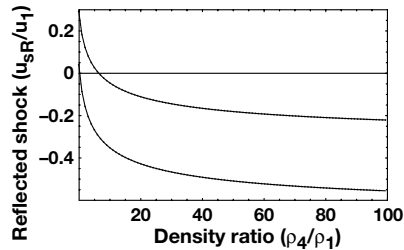
The first of these equations (4.45) is an implicit equation for  $u_2$  in terms of known quantities. It can be converted to a polynomial equation in  $u_2$ , whose order depends on the assumptions. The second (4.1.44) determines  $\rho_2$  based on  $u_2$  and known quantities. The remaining equations determine the other unknowns based on  $\rho_2$ ,  $u_2$ , and known quantities. A convenient and meaningful normalization is to divide velocities by  $u_1$ , densities by  $\rho_1$ , and pressures by  $\rho_1 u_1^2$ , since these quantities determine the dynamics. These equations may appear complicated, but this is mainly because of the numerous parameters that must be specified to define the system. We will show results for two examples.

The first example is that of reflected and transmitted shocks, produced when a first shock is incident on an interface where the density is greater than  $(\gamma_1 + 1)\rho_1/(\gamma_4 + 1) \sim \rho_1$ . Thus,  $\rho_1$ ,  $u_1$ , and  $p_1$  are produced by a shock, so that  $p_1 = (\gamma - 1)\rho_1 u_1^2/2$ . Figure 4.10 shows how the normalized interface velocity,  $u_2/u_1$ , varies with the density ratio  $\rho_4/\rho_1$ . One sees that the normalized velocity is quite close to  $1/\sqrt{\rho_4/\rho_1}$ . This is sensible. Once the density beyond the interface is a few times  $\rho_1$ , the incoming plasma is nearly stopped and the pressure on the interface is approximately constant and approximately equals  $\rho_1 u_1^2$ . For fixed pressure, the transmitted shock velocity scales inversely with the square root of the density. So this is what one sees.

The transmitted shock velocity is  $(\gamma_4 + 1)/2$  times  $u_2$ , which is shown in the figure. This is a simple relation because the upstream fluid for this shock is at rest in the lab frame. The reflected shock is more interesting. Its velocity is shown in Fig. 4.11. The velocity and direction of the reflected shock shows a strong dependence on  $\gamma$ . If the material being impacted is sufficiently compressible and low enough in density, then the momentum of the incoming fluid is sufficient to push the interface forward faster than the reflected shock retreats from it. In this case the reflected shock continues



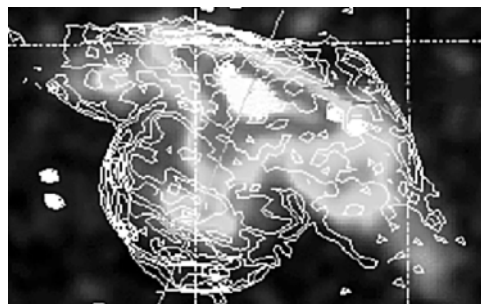
**Fig. 4.10.** The dependence of the normalized interface velocity on the density ratio when a shock encounters an interface. The two solid curves are for  $\gamma = 4/3$  (lower) and  $\gamma = 5/3$  (upper). The gray curve shows  $1/\sqrt{\rho_4/\rho_1}$



**Fig. 4.11.** The velocity of the reflected shock decreases as the density ratio increases. The lower curves is for  $\gamma = 5/3$  and the upper curve is for  $\gamma = 4/3$

to move forward in the lab frame. On the other hand, if the material being struck is sufficiently incompressible or sufficiently dense, then the reflected shock will recoil from the interface in the lab frame. The limiting reflected shock velocities are  $-u_1/3$  for  $\gamma = 4/3$  and  $-2u_1/3$  for  $\gamma = 5/3$ . This can be a useful limit, for example, to estimate how thick a wall one may need to contain a shocked material in an experiment. One can use shock reflections in measuring EOS. One measures the time it takes for the reflected shock to return to the downstream surface, known as the time for *shock reverberation*. This time is sensitive primarily to the compression produced by the initial shock.

In addition, this case is relevant to interaction of the forward shock in a supernova remnant with a molecular cloud. Figure 4.12 shows an image of such a collision. The spherical object is the remnant, and the figure shows both x-rays and radio emission produced by its interaction with the cloud. Leaving aside the clumpiness of the cloud, we can take  $\gamma$  to be  $5/3$  for both the interstellar medium and the cloud. Typical densities for the interstellar



**Fig. 4.12.** Interaction of a supernova remnant (the spherical object to the lower left) and a molecular cloud (the elongated object above). The figure shows a grayscale image of the x-ray emission, overlaid with a contour image of the radio emission. Credit: D. Burrows of Penn State and T. Landecker of the Dominion Radio Astronomy Observatory

medium and the cloud are  $2.5 \text{ cm}^{-3}$  and  $10^4 \text{ cm}^{-3}$ , respectively, so that the forward shock in the remnant has a density of  $10 \text{ cm}^{-3}$ . We will estimate its velocity as  $1000 \text{ km/s}$ ; it could be somewhat larger. Given these numbers, we would find the shock transmitted into the cloud to move at about  $40 \text{ km/s}$  while the reflected shock moves at  $330 \text{ km/s}$  back into the remnant. Given the comparable scale of these two objects in the image, it is clear that the entire remnant might be affected before the transmitted shock has traversed much of the cloud.

The second limit we will consider explicitly is the flyer-plate problem. The ideal flyer plate is a cold, planar material moving at a high velocity. An ideal flyer plate, by cleanly striking the surface of a target material, can create a very uniform and well-characterized shock in the target material. Flyer plates are used extensively to obtain equation of state data along the shock Hugoniot. Very often the flyer plate does not directly impact the sample to be studied, but instead impacts a fixed Aluminum layer, through which the resulting shock propagates to reach the sample. In this case  $\rho_4/\rho_1 = 1$ . As we mentioned above, the equations developed here can apply to the collision of a flyer plate with a target material if we choose  $p_1$  appropriately. Ideally  $p_1 \sim 0$ . Figure 4.13 shows the resulting dependence of  $u_2/u_1$  on the initial pressure in the flyer plate. As the pressure becomes small,  $u_2$  approaches  $0.5 u_1$ . This limited result is straightforward to obtain from the original equations, which is left as an exercise. In addition, so long as  $p_1$  is small, the interaction of the incident flyer plate with a material at any density produces reflected and transmitted shocks. (We discuss the behavior when  $p_1$  is not small when treating rarefactions at interfaces, below.) In this case one can simplify (4.45) to find that

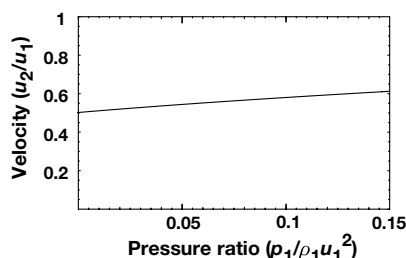
$$\frac{u_2}{u_1} = \frac{1}{1 + \sqrt{\frac{\rho_4(\gamma_4+1)}{\rho_1(\gamma_1+1)}}}, \quad (4.53)$$

#### Homework 4.11

For the simpler case in which  $p_1 = p_4 = 0$ ,  $\rho_1 = \rho_4$ , and  $\gamma_1 = \gamma_4 = \gamma$ , which is not a bad approximation for many flyer plate collisions, solve (4.44)–(4.1.50) to find the pressure and velocity of the shocked material.

## 4.2 Rarefaction Waves

A rarefaction is a decrease in density and pressure caused by expansion of a material. A rarefaction wave occurs when the onset of the expansion propagates through the material from one edge. Thus, for example, when the laser pulse that is creating pressure and plasma on the surface of an object ceases, the dense plasma expands outward. The corresponding decrease in pressure propagates into the object at the sound speed, and is accompanied by an



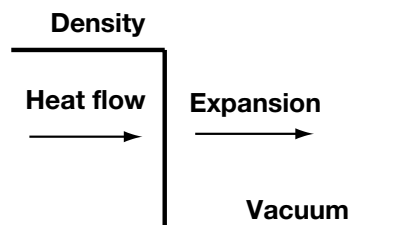
**Fig. 4.13.** The dependence of  $u_2/u_1$  on the normalized initial pressure of a flyer plate

outward flow of material and a corresponding density decrease. As another example, when a shock wave emerges from an object into a region of lower density, the high pressure produced by the shock wave causes material to accelerate forward from the object. As we will see, the flow of material outward into the rarefaction begins at a point that propagates inward into the material at the sound speed. In shock physics this behavior is known as the *release* of the shocked material. In high-energy-density physics or astrophysics it is often described as *shock breakout*.

In this section, we will first consider an idealized problem – the isothermal rarefaction – that is a useful model in many cases. This problem will also serve as our introduction to self-similar solutions, an important technique in many hydrodynamic problems.

#### 4.2.1 The Planar Isothermal Rarefaction and Self-Similar Analysis

There are cases in which a plasma expands from a planar surface at constant temperature. This requires a continuous supply of heat, to counteract expansion cooling, and so can happen only when the Peclet number (Chap. 2) is small or when the heating is rather magical. A very common example is the expansion of the low-density, laser-heated plasma from the irradiated surface of a laser target. But there are also other cases when the plasma does not cool too quickly and the very simple isothermal model can be used as a good first estimate. Figure 4.14 shows a sketch of the initial condition for this



**Fig. 4.14.** Sketch of initial condition for isothermal rarefaction

expansion. Our physical system at  $t = 0$  has warm, dense matter of uniform density  $\rho_o$  to the left of a boundary at which the density drops abruptly to zero, and heat flows into or through our system as necessary to keep the temperature constant. To analyze the expansion of this system, we begin by rearranging the continuity and momentum equations, (2.1) and (2.2), to obtain

$$\left( \frac{\partial \ln \rho}{\partial t} + \mathbf{u} \cdot \nabla \ln \rho \right) = -\nabla \cdot \mathbf{u}, \text{ and} \quad (4.54)$$

$$\left( \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) = -\frac{\partial p}{\partial \rho} \nabla \ln \rho, \quad (4.55)$$

in which the partial derivative is taken at constant temperature because the expansion is isothermal by definition. For polytropic fluids, we can recognize this derivative as the square of the sound speed,  $c_s$ , from Sect. 2.1, except that in this case it is an isothermal sound speed as opposed to an isentropic one. Heat flows outward in an isothermal rarefaction. Note the common structure of these two equations. They can be made identical by dividing the first by  $c_s$  and the second by  $c_s^2$ , and by considering a one-dimensional, planar expansion to obtain

$$\left( \frac{\partial \ln \rho}{c_s \partial t} + M \frac{\partial \ln \rho}{\partial x} \right) = -\frac{\partial M}{\partial x}, \text{ and} \quad (4.56)$$

$$\left( \frac{\partial M}{c_s \partial t} + M \frac{\partial M}{\partial x} \right) = -\frac{\partial \ln \rho}{\partial x}, \quad (4.57)$$

in which  $M = u/c_s$ . Note that here  $c_s$  is the isothermal sound speed, consistent with our assumptions. Thus  $c_s = \partial p / \partial \rho)_T$ , which is  $p/\rho$  for an ideal gas. The structure of the convective derivative suggests that some variable that couples space and time may be a key to describing the behavior of these equations. The solution we will discuss is a *similarity solution*. Many systems produce a time-invariant shape that can be described by some function  $f(\xi)$ , in which  $\xi$  is a similarity variable that in some way combines space and time. In this case, the functions  $M$  and  $\ln \rho$  are initially expressed in the variables  $x$  and  $t$ . We convert to a coordinate system, using  $\xi$  and  $t'$ , in which  $\xi = x/(c_s t)$  and  $t' = t$ . For some general function  $g(\xi, t')$ , we have by the chain rule

$$\frac{\partial g(\xi, t')}{\partial t} = \frac{\partial g}{\partial \xi} \frac{\partial \xi}{\partial t} + \frac{\partial g}{\partial t'} \frac{\partial t'}{\partial t} \quad (4.58)$$

for time derivatives and a similar expression for the spatial derivatives. Of course,  $\partial t' / \partial x = 0$  and  $\partial t' / \partial t = 1$ . However, the tricky point is that we are seeking a solution in which all the time dependence is included in the dependence on  $\xi$  so that  $\partial g / \partial t' = 0$ . Later in the chapter, we will see that one can take a more systematic approach to finding a similarity solution, and to determining whether or not one can be found. In this case we apply our definition of  $\xi$  to (4.56) and (4.57), to obtain

$$(M - \xi) \frac{\partial \ln \rho}{\partial \xi} = -\frac{\partial M}{\partial \xi}, \text{ and} \quad (4.59)$$

$$(M - \xi) \frac{\partial M}{\partial \xi} = -\frac{\partial \ln \rho}{\partial \xi}. \quad (4.60)$$

These equations have the fairly obvious solution

$$M = 1 + \xi = 1 + \frac{x}{c_s t} \text{ and} \quad (4.61)$$

$$\rho = \rho_o e^{-(1+\xi)} = \rho_o e^{-(1+x/(c_s t))}, \quad (4.62)$$

with the condition  $x \geq -c_s t$  imposed by the conservation of mass. This solution has several features worth mentioning. First, it has a linear velocity profile and an exponential density profile. Linear velocity profiles are common to many free expansions, which is not surprising as the distance any unforced parcel of fluid will travel is its speed times the time. The exponential density profile reflects the specifics of this case; we will see others soon. Second, profiles are often characterized by a scale length,  $L$ , typically defined as  $(d \ln \rho / dx)^{-1}$ , which is the distance over which an exponential profile decreases by a factor of  $e$  or a linear profile decreases to 0. A potentially independent definition of  $L$  is the distance over which the velocity changes by  $c_s$  (thus  $L = (dM/dx)^{-1}$ ). In this case by either definition one finds  $L = c_s t$ . Thus, the scale length is the distance an acoustic wave would travel. It is also the distance over which the initial material has begun to flow outward. Third, the density stays constant at the original interface as the expansion proceeds. The value of the density there is  $\rho_o e^{-1}$ . When an isolated dense block of material expands,  $\rho_o$  is the initial density of the material. In other expansions, such as those produced by laser heating, the electron density profile may tend to be exponential below the density at which the laser heating is strongest, but to have a different shape at higher density. In this case one would replace  $\rho_o e^{-1}$  in (4.62) by the density below which the profile is exponential.

#### Homework 4.12

Show that the conservation of mass in fact requires  $x \geq -c_s t$  in (4.61) and (4.62).

One can also find isothermal models of cylindrical or spherical self-similar expansions in the specialized literature. However, these do not produce simple solutions. They also have rather limited applicability, as diverging expansions cool much more strongly than planar ones, so that the isothermal assumption is more readily violated. So instead of pursuing them here, we turn to adiabatic expansions, in which there is no heating and no heat transport.

### 4.2.2 Riemann Invariants

Adiabatic expansions, or rarefactions, are common to many laboratory and astrophysical systems. Releases of energy, as when a shock wave emerges from a dense material layer or an exploding star, produce expansions. The flow of material, whether emerging from a channel in an experiment or emerging from a star to form a planetary nebula, produces an expansion. The cessation of pressure when a radiation source, whether a laser or a  $z$ -pinch or a star, becomes less powerful is followed by an expansion toward the source. Thus, expansions have broad relevance. To derive their behavior, one must first take a detour through some further fluid mechanics, which we begin here. It will turn out that there are two quantities, known as *Riemann invariants*, that (for isentropic flows) do not change along specific trajectories, known as *characteristics*. These invariants permit the calculation of adiabatic rarefactions, as we will see. Adiabatic processes, which may involve mechanical work but not the flow of heat, are also isentropic. Thus, they are fully described by the Euler equations. (But see also the discussion of the leading edge of expansions at the end of this section.)

A general disturbance in the properties of a moving fluid can affect the rest of fluid in two ways. It can generate sound waves, which move at the sound speed relative to the fluid flow, or it can generate local changes in properties (such as composition) that flow with the fluid at its velocity  $\mathbf{u}$ . Of course, a general disturbance produces sound waves that move in all possible directions. It requires very special conditions to produce sound waves moving in a restricted range of directions. The trajectories of such sonic or fluid disturbances are known as characteristics. The position vector of a characteristic,  $\mathbf{x}$ , changes for fluid disturbances, as

$$\frac{d\mathbf{x}}{dt} = \mathbf{u} \quad (4.63)$$

and for sonic disturbances as

$$\frac{d\mathbf{x}}{dt} = \mathbf{u} + c_s \hat{\mathbf{k}}, \quad (4.64)$$

in which  $\hat{\mathbf{k}}$  is a unit vector defining a direction of propagation and in which the fluid velocity  $\mathbf{u}$ , the sound speed  $c_s$ , and  $\hat{\mathbf{k}}$  depend, in general, on  $\mathbf{x}$  and  $t$ . The trajectory defined by (4.63) is known as the  $C_o$  characteristic. The trajectories defined by (4.64) when  $\hat{\mathbf{k}}$  is aligned with or opposed to the  $x$ -axis are known as  $C_+$  and  $C_-$ , respectively. As we will see, it helps visualize and understand planar rarefactions to plot the evolution of the fluid with position along the abscissa and time along the ordinate. Then a surface moving at constant velocity is a straight line.

We now develop the equations of motion into a form relevant to propagation along characteristics. First recall that in general the derivative of some function  $f(\mathbf{x}, t)$  along a specific trajectory defined by  $d\mathbf{x}/dt = \mathbf{w}$  is



$$\left(\frac{df}{dt}\right)_w = \frac{\partial f}{\partial t} + \mathbf{w} \cdot \nabla f. \quad (4.65)$$

Finding equations of motion that connect with characteristics turns out to be easiest working with pressure rather than density so we take

$$d\rho = \left(\frac{\partial \rho}{\partial p}\right)_S dp = \frac{dp}{c_s^2}, \quad (4.66)$$

where the derivative is taken at constant entropy. Then (2.1) and (2.2) become

$$\frac{1}{\rho c_s} \frac{\partial p}{\partial t} + \frac{1}{\rho c_s} \mathbf{u} \cdot \nabla p = -c_s \nabla \cdot \mathbf{u} \quad \text{and} \quad (4.67)$$

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{\nabla p}{\rho}. \quad (4.68)$$

To seek potential behavior along trajectories, we multiply (4.68) by a unit vector  $\hat{\mathbf{k}}$  (which could be a direction of sonic propagation), add the equations, and seek an equation in the form of (4.65) to obtain

$$\begin{aligned} \hat{\mathbf{k}} \cdot \left[ \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} + c_s \hat{\mathbf{k}}) \cdot \nabla \mathbf{u} \right] + \frac{1}{\rho c_s} \left[ \frac{\partial p}{\partial t} + (\mathbf{u} + c_s \hat{\mathbf{k}}) \cdot \nabla p \right] \\ = -c_s \left[ -\nabla \cdot \mathbf{u} + \hat{\mathbf{k}} \cdot (\hat{\mathbf{k}} \cdot \nabla) \mathbf{u} \right]. \end{aligned} \quad (4.69)$$

Here the quantity in the leftmost square brackets is the vector generalization of 4.2.12, with  $\nabla \mathbf{u}$  being the tensor with element  $(i,j)$  equal to  $(x_i)u_j$ , written in *dyadic notation*. We can recognize that the two square brackets on the left-hand side contain derivatives of the functions  $\mathbf{u}$  and  $p$  along the trajectory given by (4.64), if  $\hat{\mathbf{k}}$  is a direction of sonic propagation. In other words, we can write (4.69) as

$$\hat{\mathbf{k}} \cdot \left(\frac{d\mathbf{u}}{dt}\right)_{\mathbf{u}+c_s\hat{\mathbf{k}}} + \frac{1}{\rho c_s} \left(\frac{dp}{dt}\right)_{\mathbf{u}+c_s\hat{\mathbf{k}}} = -c_s \left[ -\nabla \cdot \mathbf{u} + \hat{\mathbf{k}} \cdot (\hat{\mathbf{k}} \cdot \nabla) \mathbf{u} \right]. \quad (4.70)$$

One might have hoped to find here general three-dimensional invariants of the flow with very broad applicability, which would have required that the right-hand side be identically zero. However, there are special cases for which one does find invariants. They include planar, one-dimensional flow, which is all we will consider from this point forward. In this case we have

$$du + \frac{dp}{\rho c_s} = 0 \text{ along the } C_+ \text{ trajectory } \frac{dx}{dt} = u + c_s, \quad (4.71)$$

and by taking the difference of equations 4.67 and 4.68 we also find

$$du - \frac{dp}{\rho c_s} = 0 \text{ along the } C_- \text{ trajectory } \frac{dx}{dt} = u - c_s \quad (4.72)$$

These two equations, upon integration, yield the Riemann invariants  $J_+$  and  $J_-$ , usually written as

$$J_+ = u + \int \frac{dp}{\rho c_s} \quad \text{and} \quad J_- = u - \int \frac{dp}{\rho c_s}. \quad (4.73)$$

Here the integral is the indefinite integral. In effect, it is evaluated only at a specific point of interest. By integrating either (4.2.18) or (4.2.19) from an arbitrary starting point to two distinct but arbitrary final points and then subtracting the two results, one can show that the value of  $J_+$  and  $J_-$  thus defined must be constant. The meaning of these equations and of the fact that these are invariants is that they remain constant along the trajectories, and can thus be used to help find the properties of the flow. Note that neither  $\rho$  nor  $c_s$  can be removed from the integral, as both in general are functions of the pressure. The important application of (4.73) is to polytropic gases. In this case one finds

$$J_+ = u + \frac{2c_s}{\gamma - 1} \quad \text{and} \quad J_- = u - \frac{2c_s}{\gamma - 1}. \quad (4.74)$$

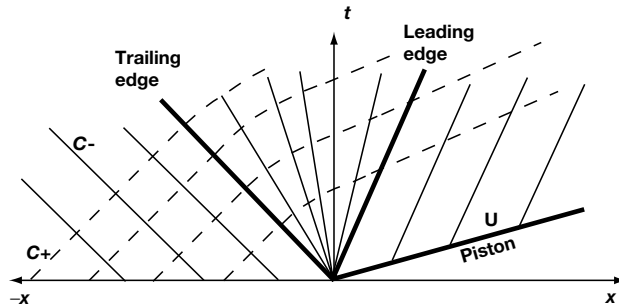
#### Homework 4.13

Obtain (4.74) from (4.73).

To obtain useful information from the Riemann invariants, there are certain properties one must understand. First of all, recalling that we are discussing only planar isentropic flows, once one has specified an initial state of the fluid, only  $u$  and one other quantity are needed to specify completely any other state of the fluid. The second quantity can be density, pressure, sound speed, or any combination of these, such as the  $\int p/(\rho c_s)$  in  $J_+$  or  $J_-$ . Because of this, it is also true that  $J_+$  and  $J_-$  also completely specify the state of the fluid. Thus, if you can follow  $C_+$  and  $C_-$  characteristics to specify  $J_+$  and  $J_-$  at their intersection, then you can infer the properties of the fluid there.

We generally plot the characteristics in a space of  $x$  along the horizontal axis and  $t$  along the vertical axis, as in Fig. 4.15. Consider (4.71) and (4.72) for the characteristics. When both  $u$  and  $c_s$  are constant, the characteristics are straight lines. Along  $C_+$ , for example, we know that  $J_+$  is constant, yet both  $J_+$  and  $J_-$  depend on similar variables. A change in slope of  $C_+$  requires that either  $u$  or  $c_s$  changes. Since  $J_+$  is constant, this means that  $J_-$  must change in order for the slope to change. This has a very useful implication: *If one Riemann invariant is constant over some region, then the characteristics for the other Riemann invariant as they cross this region are straight lines.* We will see how these two properties can be used in describing the planar adiabatic rarefaction.

Next suppose that either  $C_+$  or  $C_-$  starts where  $u = 0$ , and, for a polytropic gas,  $c_s = c_o$ . Then ask, what is the largest speed that the fluid can



**Fig. 4.15.** Trajectories of the characteristics in a planar adiabatic rarefaction.  $C_-$  characteristics are thin solid lines;  $C_+$  characteristics are dashed lines

flow? This will occur where the internal energy becomes zero. This is no surprise – the limiting speed of a fluid must be the speed it reaches when it has no internal energy so all its energy is kinetic energy. For a polytropic gas, one finds the maximum speed to be  $2c_o/(\gamma - 1)$ , which is  $3c_o$  for  $\gamma = 5/3$ . This is a very useful estimate. It is, for example, the limiting speed when one material expands against a second material of much lower density.

An example of this phenomenon is found in the design of experiments. It is common to create a shock wave or a blast wave that produces a rarefaction when it reaches a second material layer of very low density. As we discuss further below, the rarefaction drives a shock wave into the second material. Intuitively, one would expect the driven shock wave to become faster when the density of the second material is reduced. In practice, this often is not the case. If the density of the second material is low enough, then the leading edge of the rarefaction is moving close to the maximum possible speed. In this regime, changing the density of the low-density material has little impact on the speed of the shock driven into it.

#### Homework 4.14

Sketch the  $C_+$  and  $C_-$  characteristics in a fluid flowing uniformly with velocity  $u$ .

On the other hand, any high-energy-density fluid is a plasma, and this single-fluid description does not accurately describe the leading edge of a freely expanding plasma. The behavior of the leading edge of a freely expanding plasma is much closer to the behavior of an isothermal rarefaction, because as the density decreases the electrons are able to transport heat throughout the expanding plasma. The dynamical behavior is that the fast, light electrons try to rush out ahead of the ions, establishing an electric field that accelerates the ions. Thus, the electrons progressively give their energy to the electric field, which gives it to the ions. The electrons get more energy from heating by the ions and from the new electrons reaching low enough density to transport heat readily. The ions are accelerated by their own pressure

and by the electric field, and also lose some of their internal energy to electron heating. From the perspective of two-fluid theory as discussed in Chap. 2, the electric field is  $eE = k_B T_e d[\ln n_e]/dx$ . The limitation on the acceleration of the leading edge is a kinetic one. The electric field keeps nearly all of the electrons in the hot plasma. They reflect at some density and return to the denser region. (This behavior can be visualized as the electrons attempting to climb a long potential hill – most of them roll back down.) Only the most energetic electrons reach the leading edge of the expansion, where interactions with the surrounding gas can cool them and can help shield the potential of the plasma. An accurate analysis of this problem does not exist; it would require an entertaining foray into collisional kinetic theory. But one would roughly anticipate that the leading edge will form where  $n_e \sim n_{\text{gas}}$ , the gas density. Using a typical value of  $n_{\text{gas}} \sim 10^{10} \text{ cm}^{-3}$  and a maximum density  $n_e \sim 10^{23} \text{ cm}^{-3}$ , one finds that the *plasma potential* in electron volts is roughly  $30 k_B T_e$ , with  $k_B$  in  $J$  per unit temperature (for example,  $1.6 \times 10^{-19} \text{ J/eV}$ ). This is consistent with measurements.

### 4.2.3 Planar Adiabatic Rarefactions

We are now ready to analyze the planar adiabatic rarefaction. We imagine that there is an initial, semi-infinite, uniform fluid to the left of the origin, bounded on the right by a piston. At time  $t = 0$ , the piston instantaneously accelerates to a velocity  $U$ . (We could consider gradual accelerations of the piston, but this has little importance for our long-term applications of this conceptual model.) We desire to find the profiles of the fluid parameters that result in time and in space.

Figure 4.15 shows such a rarefaction, with position along the abscissa and time along the ordinate. In the figure, the thick line to the right shows the velocity  $U$  at which a piston is withdrawing from its initial position at  $x = 0$ . (The fluid is uniform to the left of the origin at  $t = 0$ .) The line labeled “Piston” is the boundary of the fluid; it is not a characteristic. Now consider the region to the left of the origin, which has uniform properties at  $t = 0$ . To the left of the thick line labeled “trailing edge,” the  $C_+$  and  $C_-$  characteristics that intersect at any point originate from initial points whose properties are identical. Thus, both  $J_+$  and  $J_-$  are the same as they were at  $t = 0$ . In this region, the properties of the fluid are unchanged.

Indeed, changes in the fluid properties can occur only through the arrival of other values of  $J_-$ , from points where  $x \geq 0$ . The earliest this can occur is along the  $C_-$  characteristic from the origin that is the line labeled “trailing edge.” This is the tail of the rarefaction wave. One often sees such a feature propagate through a region that has first been shocked or otherwise heated, when the edge of the region is allowed to expand. This wave propagates at the sound speed of the initial medium. The  $C_+$  characteristics that reach the edge of the rarefaction wave continue across the system toward the right, everywhere the fluid goes. As a result, because  $J_+$  is constant along  $x \leq 0$

at  $t = 0$ ,  $J_+$  is constant everywhere. This implies as well that all the  $C_-$  characteristics are straight lines. For the case of a polytropic gas, one has  $J_+ = 2c_o/(\gamma - 1)$  so at any location one has

$$J_+ = u + \frac{2c_s}{\gamma - 1} = \frac{2c_o}{\gamma - 1}, \text{ from which} \quad (4.75)$$

$$c_s = c_o - \frac{\gamma - 1}{2}u. \quad (4.76)$$

This tells us how the sound speed (and thus temperature) varies through the rarefaction. As  $u$  increases the medium cools – this is the anticipated expansion cooling. Next consider the fluid properties at the piston. Since  $u$  is fixed there, as  $U$ , determining either  $J_+$  or  $J_-$  determines the state of the fluid. The  $C_+$  characteristics propagate from the initial state to the piston, so  $J_+$  is known. Knowing  $J_+$  and  $U$ , one can then find  $J_-$  for the  $C_-$  characteristics leaving the piston. For the case of a polytropic gas, one has

$$J_- = U - \frac{2c_s}{\gamma - 1} = 2U - \frac{2c_o}{\gamma - 1}, \quad (4.77)$$

using (4.76). Because  $J_+$  is constant, the  $C_-$  characteristics emerging from the piston are straight lines with

$$\frac{dx}{dt} = U - \left( c_o - \frac{\gamma - 1}{2}U \right) = \frac{\gamma + 1}{2}U - c_o. \quad (4.78)$$

Along these characteristics,  $J_+$  and  $J_-$  are both constant, so the state of the fluid is constant. This portion of the fluid moves with the piston at velocity  $U$  and has sound speed

$$c_s = c_o - \frac{\gamma - 1}{2}U. \quad (4.79)$$

This region with constant fluid properties is bounded on the left by the thick line labeled “leading edge,” which is the front end of the region of expansion. What remains is to describe the region between the leading edge and the trailing edge. The  $C_-$  characteristics form a fan of straight lines emanating from the origin. This is sometimes known as a *rarefaction fan*. The rarefaction wave is *centered*, which refers to the fact that one set of characteristics emerges from a common point. Equation (4.76) gives  $c_s$  as a function of  $u$ . As a result, the equations for the  $C_-$  characteristics are

$$\frac{dx}{dt} = u - c_s = \frac{\gamma + 1}{2}u - c_o = \text{const.} \quad (4.80)$$

We can integrate this equation and rearrange it to find an equation for  $u$ :

$$u = \frac{2}{\gamma + 1} 2 \left( c_o + \frac{x}{t} \right). \quad (4.81)$$

For the polytropic gas, we now have a complete description of the fluid because

$$\frac{\rho}{\rho_o} = \left(\frac{c_s}{c_o}\right)^{2/(\gamma-1)} \quad \text{and} \quad \frac{p}{p_o} = \left(\frac{c_s}{c_o}\right)^{2\gamma/(\gamma-1)}, \quad \text{which become} \quad (4.82)$$

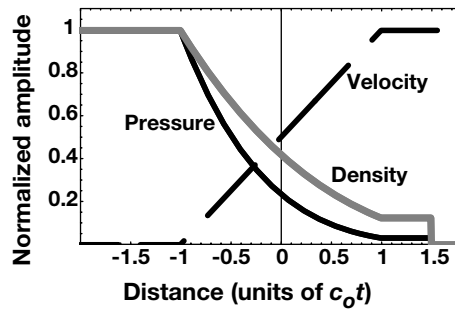
$$\frac{\rho}{\rho_o} = \left(1 - \frac{(\gamma-1)u}{2c_o}\right)^{2/(\gamma-1)} \quad \text{and} \quad \frac{p}{p_o} = \left(1 - \frac{(\gamma-1)u}{2c_o}\right)^{2\gamma/(\gamma-1)}, \quad \text{or} \quad (4.83)$$

$$\frac{\rho}{\rho_o} = \left(\frac{2}{\gamma+1} - \frac{\gamma-1}{\gamma+1} \frac{x}{c_o t}\right)^{2/(\gamma-1)} \quad \text{and} \quad (4.84)$$

$$\frac{p}{p_o} = \left(\frac{2}{\gamma+1} - \frac{\gamma-1}{\gamma+1} \frac{x}{c_o t}\right)^{2\gamma/(\gamma-1)}, \quad \text{for} \quad (4.85)$$

$$-c_o t \leq x \leq \frac{\gamma+1}{2} U t - c_o t \leq \frac{2}{\gamma-1} c_o t. \quad (4.86)$$

These profiles are illustrated in Fig. 4.16. One can observe that density and pressure reach their initial values at  $x = -c_o t$  and reach their steady values near the piston when  $u = U$ . (With the specific choice of parameters in Fig. 4.16, this occurs at  $x = c_o t$ . This is *not* a general result.) In addition, the density pressure and velocity are constant at  $x = 0$ . In (4.86), the limits on  $x$  are imposed by the trailing edge on the left and the leading edge on the right, with the upper limit reached when  $U$  reaches the maximum value the fluid can accommodate,  $2c_o/(\gamma-1)$ . At this point the leading edge equals the path of the piston. If the piston withdraws faster than that, it will pull away from the expanding fluid. This case is equivalent to a free adiabatic rarefaction, which is described by (4.2.22) and (4.2.27)–(4.2.32), with  $U = 2c_o/(\gamma-1)$  in the final equation.



**Fig. 4.16.** Density (gray), pressure (black), and velocity (dashed) profiles in a planar adiabatic rarefaction, normalized to the  $\rho_o$ ,  $p_o$ , and  $U$ , respectively. Here  $\gamma = 5/3$ ,  $c_o t = 1$ , and  $U = 1.5c_o$

**Homework 4.15**

Plot the minimum density and pressure in the rarefaction as a function of  $U$ . Discuss the meaning of the plots. Reasonable normalizations are recommended.

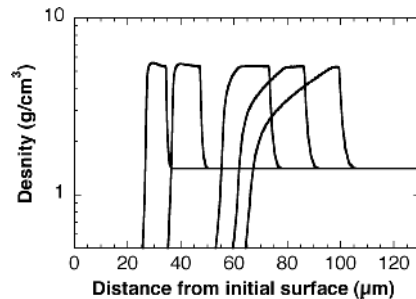
Now instead of a piston, suppose that there is a zero-pressure, lower-density material to the right of the initial interface, which has density  $\rho_r$  and polytropic index  $\gamma_r$ . A strong shock will be driven into this material, at some velocity  $u_s$ , with postshock velocity  $u_r$ , producing a pressure  $p_r = (\gamma_r - 1)\rho_r u_r^2/2$ . The motion of the interface then acts like the piston, so the rarefaction in the denser material will proceed as described above, for a value of  $u_r = U$  that is self-consistent, so that  $p_r$  is equal to the pressure at the piston in the description above.

Adiabatic rarefactions often are produced when shock waves reach an interface where the density drops. This occurs several times during the explosion of a star, with the added complication that the shock wave is a blast wave (discussed in the next section). It happens frequently in experiments as well. We will apply the above equations to this case when we consider the behavior of a shock at a density drop in Sect. 4.4.1.

**4.3 Blast Waves**

A uniform shock wave can come to an end in two ways. The previous section discussed one of them: the shock wave can reach a lower-density medium and a rarefaction wave can propagate backward into the shocked material. This section discusses the other one: the source of pressure can end, allowing a rarefaction wave to propagate forward and overtake the shock. This forms the structure illustrated in Fig. 4.17. In the figure, the first two curves show the nearly steady shock produced by the pressure of laser ablation (Chap. 8). The curves are not completely flat because, in the simulation and perhaps in a real system, the pressure produced by the laser does evolve with time. The later three curves, all after the end of the laser pulse, show the rarefaction wave developing on the left edge of the structure and soon overtaking the shock to form a *blast wave*. (A word on semantics is in order here. This definition of a blast wave as the structure formed when a rarefaction overtakes a shock is fairly common in the astrophysical literature, and is the one we will use. In the shock physics literature, the term *blast wave* is more often restricted to such structures produced by spherical expansions from a point explosion, while other cases would be described as “waves produced by impulsive loading,” or perhaps as “planar blast waves.”)

Blast waves are very common because releases of energy are of limited duration. On a small scale, processes such as solar flares release energy, causing blast waves to form in the solar wind. In some systems, hydrogen can



**Fig. 4.17.** Simulated evolution of a blast wave from a shock wave. Here laser irradiation from the left drives a nearly steady shock wave for 1 ns, after which the rarefaction from the front surface overtakes the shock wave to form a blast wave. The curves show the profile at 0.6, 0.8, 1.2, 1.4, and 1.6 ns after the start of the laser pulse

accumulate on the surface of a neutron star, leading to occasional nuclear explosions known as “astrophysical flashes.” Such flashes will drive blast waves through the surrounding material. Stellar explosions at first drive shocks, not blast waves. But eventually, when the interior pressure is much reduced and the accumulated interstellar material exceeds the mass of the star, supernova remnants develop a blast-wave structure which they retain for much of their evolution. One can think of many other cases, such as the interaction of jets with clouds, in which blast waves are produced. The most common blast wave in the Earth environment is produced by lightning, which briefly deposits energy within the lightning channel.

Planar blast waves are often useful in high-energy-density experiments. They can drive Rayleigh–Taylor instabilities at interfaces, as is discussed in Chap. 5. In addition, they can be used as timescale converters. One may, for example, have a laser that can provide power most effectively for 1 ns (or a  $Z$  pinch that can do so for 10 ns), but need to deliver energy over a longer timescale to some object. By forming a blast wave and then letting it propagate, one creates a store of energy. With time and distance, the blast wave carries more mass at lower velocity. If it is then allowed to release that energy, for example, by encountering a lower-density medium, it can drive further hydrodynamics for a much longer period than the duration of the initial energy source.

### 4.3.1 Energy Conservation in Blast Waves

Because most of the material in a blast wave is near the shock, it is useful and informative to see what one can infer from energy and momentum conservation. Consider again Fig. 4.17. We will discuss the spherical case. Spherical blast waves are often known as Sedov–Taylor (or Taylor–Sedov) blast waves as Sedov and Taylor, along with von Neumann, were the first to discuss their



behavior. There is an initial shock transition, followed by a rarefaction. The shock is nearly always a strong shock, and thus converts half the incoming energy (in the lab frame) into kinetic energy and half into thermal energy. However, the source of energy in the problem is the energy behind the shock. The total mass in the blast wave is the amount of mass that has been swept up by the shock. If the mass density  $\rho$  of the external medium is constant and the shock radius is  $R$ , then the total mass within a spherical blast wave is  $M = 4\pi\rho R^3/3$ . In cases where very little energy has been lost by radiation or heat conduction, the energy within the blast wave is approximately

$$E = M \left( \frac{2}{\gamma + 1} \right)^2 \dot{R}^2, \quad (4.87)$$

in which we assume the material is a polytropic gas to obtain the fluid velocity in terms of the shock velocity  $\dot{R}$ . The important result does not depend on this assumption. Equation (4.87) remains valid despite the conversion of thermal energy to kinetic energy in the rarefaction, but does assume that all the matter has been accelerated by the shock wave at its present velocity. In other words, it assumes that the change in shock velocity is slow. Conserving energy, we obtain

$$\frac{dM}{dt} \dot{R}^2 = -M 2\dot{R} \frac{d^2 R}{dt^2}, \quad (4.88)$$

in which  $dM/dt = 4\pi\rho R^2 \dot{R}$ , so

$$3\dot{R}^2 = -2R \frac{d^2 R}{dt^2}. \quad (4.89)$$

If one seeks a power-law solution of the form  $R = R_o t^\alpha$ , in which  $R_o$  and  $\alpha$  are constants, one finds  $\alpha = 2/5$ . The Sedov–Taylor blast wave thus has a radius that increases in proportion to  $t^{2/5}$ .

#### Homework 4.16

Show that this type of analysis produces  $\alpha = 1/2$  for cylindrical blast waves and  $\alpha = 2/3$  for planar blast waves.

If an energy-conserving blast wave is able to propagate far enough, which happens, for example, with blast waves from lightning, then the shock wave becomes a weak shock wave and eventually a disturbance that propagates at the local sound speed. This case is discussed in more detail in *Zeldovich and Raizer*. An estimate of the radius at which this will occur can be found by setting the explosion energy  $E_x$  per unit volume equal to the thermal energy. For a spherical explosion, this occurs when  $r \sim (3E_x/(8p_o))^{1/3}$ , where  $p_o$  is the initial pressure. Thus, for a typical supernova, which deposits about  $10^{51}$  ergs in exploding stellar material, and a typical interstellar pressure of 1.6 picodynes ( $1 \text{ cm}^{-3}$  and 1 eV), one finds a radius of  $6 \times 10^{20}$  cm,

or about 600 light years. For a cylindrical explosion, where  $E_x$  is now the energy per unit length, one finds  $r \sim \sqrt{E_x/(9p_o)}$ . For lightning, which has  $E_x \sim 10^5 \text{ J/m} = 10^{10} \text{ ergs/cm}$ , one finds  $r \sim 33 \text{ cm}$ . One can see that one could not experience the shock wave produced by lightning at a safe distance from the lightning itself.

To examine momentum conservation in a blast wave, it is tempting to take the same approach and to set the derivative of  $M\dot{R}$  equal to 0. However, this is not generally valid. While the conversion of thermal energy to kinetic energy in the rarefaction does not affect the overall energy balance, the production of momentum in the rarefaction does affect the momentum balance. While outward momentum is added to the newly shocked matter, inward momentum is generated by the rarefaction. In consequence, momentum conservation is more difficult to calculate and does not generate much insight into the global evolution of an energy-conserving blast wave.

However, momentum conservation is sometimes important because radiative energy losses are not always negligible. There are circumstances in which a blast wave enters a strongly radiating phase, so that it no longer conserves energy. All supernova remnants eventually enter this phase, when they become slow enough that the postshock material cools rapidly by radiation (see Chap. 7). The thermal energy produced by the shock is radiated away, so the energy remaining in the system steadily decreases. At the same time, there is little thermal energy to drive a rarefaction so the shocked material tends to become a dense shell moving with the shock. The pressure of this shell must equal the ram pressure of the incoming material, so as it loses energy by radiation it becomes quite dense and cool. At the same time, its velocity approaches the shock velocity in the frame of the unshocked matter. (In the shock frame, the very dense material that has been shocked and then cooled can move only very slowly away from the shock.) Such a system is often described as a momentum-conserving snowplow.

The time evolution of a momentum-conserving snowplow can be found by setting the derivative of  $M\dot{R}$  equal to 0. The approach to the solution is identical to that used for the conservation of energy. For the spherical case, one finds  $\alpha = 1/4$  for this case. Thus, spherical momentum-conserving snowplows expand with  $R \propto t^{1/4}$ .

#### Homework 4.17

Find the coefficients  $\alpha$  for cylindrical and planar momentum-conserving snowplows.

#### 4.3.2 A General Discussion of Self-Similar Motions

We have dealt previously in this chapter with two types of rarefaction that are self-similar. In the case of the isothermal rarefaction we identified a similarity variable, then we made an explicit though, simple transformation of

our equations. In the case of the adiabatic rarefaction, we solved for the behavior using Riemann invariants, though, if you look again at the solutions, they are self-similar with similarity variable  $x/(c_0 t)$ . In general, self-similar solutions have shapes in space that are independent of time, with a spatial scale,  $R(t)$ , that is a function of time. Thus, the generally useful similarity variable, which traces out the shape of the fluid parameters, is  $\xi = r/R$ .

We consider the possibility of self-similar motions in systems whose evolution is symmetric (planar, cylindrical, or spherical). These motions are described by the corresponding versions of (2.1), (2.2), and (2.14), which are

$$\frac{\partial \rho}{\partial t} + u \frac{\partial \rho}{\partial r} + \rho \left( \frac{\partial u}{\partial r} + \frac{su}{r} \right) = 0, \quad (4.90)$$

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial r} + \frac{1}{\rho} \frac{\partial p}{\partial r} = 0, \text{ and} \quad (4.91)$$

$$\frac{\partial p}{\partial t} + u \frac{\partial p}{\partial r} - c_s^2 \left( \frac{\partial \rho}{\partial t} + u \frac{\partial \rho}{\partial r} \right) = 0, \quad (4.92)$$

in which  $s = 0, 1,$  and  $2,$  respectively, for planar, cylindrical, and spherical symmetry. Observe that so long as the EOS that relates sound speed to density and pressure is a function of these quantities and numeric parameters (such as  $\gamma$ ), these equations contain only variables. We will use a polytropic equation of state, as is already assumed in (4.92). The parameters having numerical values with physical dimensions (such as a density of  $1 \text{ g/cm}^3$ ) enter through the boundary conditions and initial conditions that are necessary to solve these equations for some specific case. The general problem of *similarity transformations*, in which a scaling of some variables leads to identical equations using scaled variables, is discussed briefly in Chap. 10 and at length in Zeldovich and Raizer. Here we are concerned with finding *self-similar solutions*, in which there is a single variable  $\xi$  that describes the shape of the solution for all time and all space.

The key to finding a self-similar solution to these equations is to transform them to ordinary differential equations involving dimensionless functions of  $\xi$  that represent the shape of the fluid variables. We will work with density, velocity, and pressure. Alternatively, we could use the sound speed instead of the pressure. To reduce (4.90)–(4.92) to self-similar form, we must express each of  $u, \rho,$  and  $p$  as the product of a dimensionless function  $\xi,$  a function of  $R,$  and other necessary parameters. This turns out to require that we specify an initial condition for either density or pressure. Here we work with density, which is the most common case. We take

$$u = \dot{R}U(\xi), \quad \rho = \rho_o(r, t)\Omega(\xi), \quad \text{and} \quad p = \rho_o(r, t)\dot{R}^2 P(\xi), \quad (4.93)$$

in which the overdot represents a time derivative and the initial density  $\rho_o$  is in general a function of both space and time. Recognizing that

$$\frac{\partial h(\xi)}{\partial t} = -\xi \frac{\dot{R}}{R} h'(\xi) \quad \text{and} \quad \frac{\partial h(\xi)}{\partial r} = \frac{1}{R} h'(\xi), \quad (4.94)$$

in which the ' designates the derivative with respect to  $\xi$ , we can obtain after some algebra the following equations, in which  $\rho'_o$  is the spatial derivative of  $\rho_o$ ,

$$\begin{aligned} \frac{\dot{\rho}_o}{\rho_o} \frac{R}{\dot{R}} \Omega(\xi) + \frac{\rho'_o R}{\rho_o} U(\xi) \Omega(\xi) + [U(\xi) - \xi] \Omega'(\xi) \\ + \Omega(\xi) U'(\xi) + \frac{sU(\xi) \Omega(\xi)}{\xi} = 0, \end{aligned} \quad (4.95)$$

$$\frac{\dot{\rho}_o R}{\rho_o} P(\xi) + \frac{R \ddot{R}}{\dot{R}^2} U(\xi) \Omega(\xi) + [U(\xi) - \xi] U'(\xi) \Omega(\xi) + P'(\xi) = 0, \quad \text{and} \quad (4.96)$$

$$\begin{aligned} \frac{\dot{\rho}_o}{\rho_o} \frac{R}{\dot{R}} (1 - \gamma) P(\xi) + \frac{\rho'_o R}{\rho_o} U(\xi) (1 - \gamma) P(\xi) + 2 \frac{R \ddot{R}}{\dot{R}^2} P(\xi) \\ + (U(\xi) - \xi) \left( P'(\xi) - \gamma P(\xi) \frac{\Omega'(\xi)}{\Omega(\xi)} \right) = 0. \end{aligned} \quad (4.97)$$

#### Homework 4.18

Derive (4.95)–(4.97).

We note that here we have implicitly assumed that velocity, density, and pressure have no remaining dependence on time in our transformation from variables  $(r, t)$  to  $(\xi, t')$  (see Sect. 4.2.1). This in the end imposes a restriction on the number of constraints imposed by any other boundary or initial conditions.

In (4.95)–(4.97), we have obtained three ordinary differential equations, with explicit dependences on time and space in some terms. These dependences must also cancel out if the evolution is to be self-similar. Consider first (4.96). This equation depends only on time and on  $\xi$ , and the dependence on time cancels out if  $R$  is a power law in time or an exponential with an argument that is linear in time. Here we will emphasize power-law solutions, and so we will take

$$R = R_o t^\alpha, \quad (4.98)$$

in which  $R_o$  is the position at time  $t = 1$ . (In some practical applications, it can be useful to specify a starting time, and so to replace  $t$  with  $t/t_o$  in this equation.) This is sufficient to remove all the time dependences from (4.95)–(4.97) except those involving the density. If the density has a time dependence, one can see that self-similar behavior can follow only if the dependence of  $\rho_o$  and  $R$  is the same type of function. They can both be power-law dependences, for example, and any difference in the exponent will just produce a constant factor in the equations. If the time dependence of  $\rho_o$  does not have the same functional form as  $R$ , then the system will not exhibit a self-similar evolution. In other words, its shape will change with time.

In order for the spatial dependence to drop out of (4.95), the quantity  $\rho'_o R / \rho_o$  must have a constant value. This will occur if  $\rho_o$  is a power law function of position, so that

$$\rho_o(r) = \hat{\rho} r^\delta = \hat{\rho} \xi^\delta R^\delta \quad \text{and} \quad (4.99)$$

$$\frac{\rho'_o(r)R}{\rho_o} = \frac{\rho'_o(\xi)}{\rho_o R} R = \frac{\delta}{\xi}. \quad (4.100)$$

Thus, any power-law dependences of  $\rho_o$  on  $r$  and  $t$  are consistent with a self-similar evolution involving a scale  $R$  that is a power of  $t$ . However, the time dependence of  $R$  is part of the time dependence of  $\rho_o$ , so if  $\rho_o$  is  $\propto t^\beta$  overall, one must have  $\hat{\rho} \propto t^{(\beta-\alpha\delta)}$ . At times, self-similar solutions can be found in two adjacent regions, for example in which  $\rho_o$  may have two different dependences on space and time. What is required is that one be able to specify boundary conditions that connect the two regions. This is the case, for example, in the treatment of the structure of young supernova remnants by Chevalier (1982).

At this point we have identified the conditions under which a hydrodynamic system will exhibit self-similar evolution. While the notion of a self-similar evolution has obvious mathematical elegance, the practically oriented reader may wonder why we care, and whether such solutions are ever useful, and how to know when one may work. We do care, because there are in the end many problems that satisfy, or approximate, the conditions for self-similar evolution. This means that self-similar models are often very useful in the approximate description of how a system evolves. For example, good experiment design is very often based on simple physical reasoning, and self-similar models are an important tool for the experiment designer. They also provide a useful conceptual framework to discuss (and to estimate) how a system will evolve. This makes it worthwhile to know when they may work, which we discuss next.

To obtain (4.95)–(4.97) in a self-similar form, we must specify an initial density (or pressure) and are restricted to certain types of functional dependences in space and time, as just discussed. This amounts to having specified a single parameter with physical dimensions. If this is all that is specified, then one has a self-similar problem. An example is the propagation of a shock wave through a fluid whose density decreases as a power law in space and has a specified initial profile. This occurs, for example, when a shock wave emerges from a star. This specific case is treated by Zeldovich and Raizer, when they discuss this type of self-similar problem. In this case, one must solve the equations numerically to determine the value of the parameter  $\alpha$ . Curiously, the “impulsive loading” problem (the planar blast wave problem), discussed thoroughly there, is in this second category, even though it seems to have two parameters. The estimate of  $\alpha = 2/3$ , obtained in a homework problem in the previous section on the assumption that all the mass is near the shock, is only an upper limit because the mass is in fact distributed to minus infinity. (The lower limit is  $\alpha = 1/2$ , and for  $\gamma = 5/3$ ,  $\alpha = 0.611$ .) The

open boundary creates special problems, because beyond it the acceleration is large enough to cause a divergence of the energy integral in the self-similar solution. The solution to this problem is that there is always a small initial quantity of mass to which the self-similar solution does not apply, and that this mass contains only finite energy.

If the specification of the system includes a second parameter with physical dimensions, such as a total energy, then one has self-similar behavior of the type discussed by *Sedov*. The definition of  $\xi$  must be related to the known properties of the system, since these establish the relation between  $r$  and  $t$  in the actual physical system. For example, more energy will lead to faster motions, corresponding to larger values of  $\dot{R}$ , so if energy is specified then this must be included in the definition of  $R$  and hence  $\xi$ . One can define  $R$  to within a constant by creating a dimensionless combination of the known physical quantities,  $r$ , and  $t$ . This must be constant for  $r = R$ , and so can be solved to find the dependence of  $R$  on time (i.e.,  $\alpha$ ) and on the specified physical quantities. The constant, unknown coefficient can then be specified during the solution of the problem. We carry out this exercise for the spherical blast wave, in the following section.

If the specification of the system includes a third parameter with physical dimensions, such as the location of an interface in a blast-wave problem, then the evolution is not self-similar. The solution of (4.95)–(4.97) permits only one additional parameter to be defined, whether as a boundary condition, an initial condition, or an integral property of the system. Once this constraint is imposed on the solution, there remain no further undefined variables. Adding an additional constraint causes the self-similar problem to be overconstrained. The system may still evolve to have a fixed shape in space and time, but it will have distinct reference scales for  $r$  and for  $t$ . We discuss an example of this in Chap. 10.

This makes it fairly easy to tell whether a similarity solution may apply to a given problem. If the system is described by one or two dimensional parameters and these are simple power laws (or perhaps exponentials) in space and time, then a self-similar solution will exist. If there are more than this, then there will be no self-similar solution.

### 4.3.3 The Sedov–Taylor Spherical Blast Wave

We now turn specifically to the problem of finding the profiles of the fluid variables in a spherical blast wave. This is often known as the *point explosion problem*, as self-similar solutions require one to assume that the energy originated at an initial point (or line). Solutions found under this assumption will apply only when the blast wave is far enough from the source that this assumption becomes accurate. This problem can be solved analytically, as is discussed by Sedov. The first numerical solution was reported by G. I. Taylor, and von Neumann also contributed an early solution. Here we show how to develop a set of ordinary differential equations in terms of an appropriate

similarity variable. These equations can typically be integrated quite quickly using a computational mathematics program.

The point explosion problem has only constraints having physical dimensions. There are the explosion energy,  $E_x$ , and the initial density of the surrounding medium (assumed constant),  $\rho_o$ . The independent variables are space  $r$  and time  $t$ . This allows us to carry out the procedure described in the previous section. The ratio of  $E_x$  to  $\rho_o$  has units of length to the 5th power divided by time squared, so we can obtain a dimensionless parameter from these quantities. For a self-similar motion, we then have

$$\left(\frac{E_x t^2}{\rho_o r^5}\right) = \text{const}, \quad (4.101)$$

in the sense that the motion of any feature or point must keep this parameter constant. For this problem it is convenient to consider the position of the shock,  $R(t)$ , for which

$$R = \frac{1}{Q} \left(\frac{E_x}{\rho_o}\right)^{1/5} t^{2/5}, \quad (4.102)$$

where  $Q$  is a constant to be determined later. Then we can use  $\xi = r/R$  as the similarity variable and we see that  $\alpha = 2/5$ . Note that we have determined the scaling of the radius with time much more easily and exactly than we did in the energy argument of Sect. 4.3.1. This is rather amazing, since we needed only a little simple reasoning to do so. Note also that the shock velocity is  $(2/5)(R/t)$ . We could use the general transformation relations of the previous section, but for convenience we will use instead transformations that are specific to this system. This will be the basis of our transformation of the equations of motion, writing

$$u = \frac{2}{5} \frac{R}{t} U(\xi), \quad \rho = \rho_o \Omega(\xi), \quad \text{and} \quad p = \left(\frac{2}{5} \frac{R}{t}\right)^2 \rho_o P(\xi). \quad (4.103)$$

Here  $U$ ,  $\Omega$ , and  $P$  are dimensionless functions providing the shape of each of the fluid variables. We then have for any function  $h(\xi)$ ,

$$\frac{\partial h(\xi)}{\partial t} = h'(\xi) \frac{\partial \xi}{\partial t} = -\left(\frac{2}{5} \frac{\xi}{t}\right) h'(\xi), \quad \text{and} \quad (4.104)$$

$$\frac{\partial h(\xi)}{\partial r} = h'(\xi) \frac{\partial \xi}{\partial r} = \left(\frac{\xi}{r}\right) h'(\xi). \quad (4.105)$$

Here the prime designates the derivative of the function with respect to its argument. Using (4.103)–(4.105), the equations of motion become, after some manipulation, and with the assumption that the fluid is a polytropic gas,

$$[U(\xi) - \xi] \xi \Omega'(\xi) + [\xi U'(\xi) + sU(\xi)] \Omega(\xi) = 0, \quad (4.106)$$

$$-\frac{3}{2}\Omega(\xi)U(\xi) + [U(\xi) - \xi]\Omega(\xi)U'(\xi) + P'(\xi) = 0, \quad \text{and} \quad (4.107)$$

$$-3\Omega(\xi)P(\xi) + [U(\xi) - \xi][\Omega(\xi)P'(\xi) - \gamma P(\xi)\Omega'(\xi)] = 0. \quad (4.108)$$

Here  $s = 2$  for this spherical case. These are the ordinary differential equations we set out to obtain. Note that  $\gamma$  enters explicitly as a parameter in these equations. Thus, the solution is not independent of the equation of state. The numerical coefficients arise from the scaling of the dimensionless parameter, and thus are specific to this problem.

#### Homework 4.19

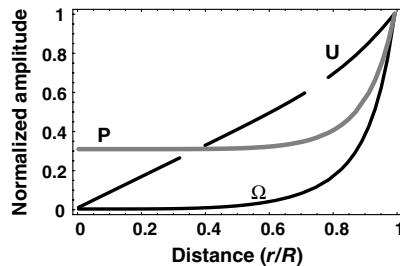
Derive (4.106)–(4.108).

The boundary conditions required to integrate the equations are obtained at the shock front, where

$$U(1) = \frac{2}{\gamma + 1}, \quad \Omega(1) = \frac{\gamma + 1}{\gamma - 1}, \quad \text{and} \quad P(1) = \frac{2}{\gamma + 1}. \quad (4.109)$$

By numerically integrating (4.106)–(4.109), one finds the profiles of the three dimensionless functions. These are shown in Fig. 4.18 for the spherical case ( $s = 2$ ) and for  $\gamma = 5/3$ . This value of  $\gamma$  is reasonable for nearly all astrophysical systems and for some laboratory experiments. For explosions in air, it would be better to take  $\gamma = 1.4$  and for many laboratory experiments  $\gamma$  could be as low as  $4/3$  or even less, as was discussed in Chap. 3. One sees that nearly all the mass is concentrated near the shock (even more than it first appears, when one realizes that the total mass per unit radius is proportional to  $r^2\Omega$ ).

The above equations are sufficient to give us the profile shapes, but not to determine the quantity  $Q$ , which sets the absolute value of  $R$  at a given time. Here is where we must make use of the second parameter having physical



**Fig. 4.18.** Dimensionless profiles for the Sedov–Taylor blast wave.  $U$  (dashed),  $\Omega$  (black), and  $P$  (gray) are shown as a function of  $r/R$ . The amplitudes are normalized to unity at the shock radius  $r = R$ .



dimensions,  $E_x$ . To find  $Q$ , one can evaluate the total energy in the self-similar profile, which must equal the explosion energy  $E_x$ . The integral for  $E_x$  is

$$E_x = \int_0^R \left( \frac{p}{\gamma - 1} + \frac{\rho u^2}{2} \right) 4\pi r^2 dr, \quad (4.110)$$

from which one can show that

$$Q^5 = \frac{16\pi}{25} \int_0^1 \left( \frac{P(\xi)}{\gamma - 1} + \frac{1}{2} \Omega(\xi) U^2(\xi) \right) \xi^2 d\xi. \quad (4.111)$$

Evaluating this integral for the profiles shown in Fig. 4.18, which assumes  $\gamma = 5/3$ , one finds  $Q = 0.868$ . This in turn lets us evaluate from (4.102) the radius for actual cases. For a supernova remnant formed by the release of  $10^{51}$  ergs into a medium with a density of  $10^{-23}$  g/cm<sup>3</sup>, one finds  $R = 1.2t^{2/5}$ , with  $R$  in light years and  $t$  in years. Obviously this does not apply within 1 year, and in fact hundreds to thousands of years are required to sweep up enough mass that a point-explosion model is appropriate. As another example, a laboratory blast wave experiment might release 100 J into a gas at a density of 10 mg/cm<sup>3</sup>. In this case  $R = 11.1t^{2/5}$ , with  $R$  in mm and  $t$  in  $\mu$ s.

#### Homework 4.20

Use a computational mathematics program to integrate these equations to find and plot the profiles, and to evaluate  $Q$ , for a cylindrical case. Apply this to find the behavior of a lightning channel produced by a deposited energy of  $10^{10}$  ergs/cm.

## 4.4 Phenomena at Interfaces

All of the discussion above, with the exception of the discussion of reflected shocks, relates only to the behavior of an isolated hydrodynamic phenomenon in an unbounded medium. This is a necessary start, but the features of interest in most physical systems arise from the interaction of hydrodynamic phenomena with structure in the medium or with each other. Understanding these effects is also needed to design clever experiments.

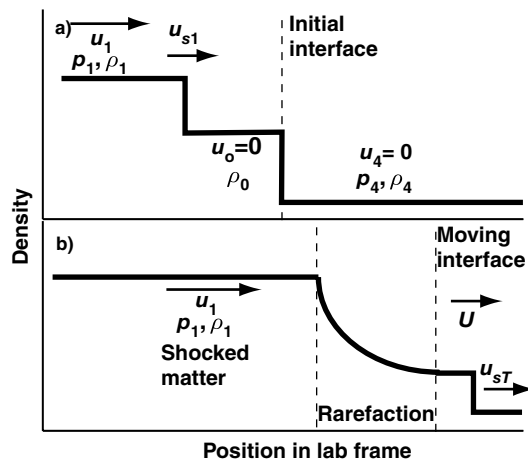
### 4.4.1 Shocks at Interfaces and Their Consequences

In Sect. 4.1.5, we discussed the generation of reflected and transmitted shocks when a shock wave approaches an interface where the density increases. We also discussed the flyer-plate case, described by the same mathematics, in which a cold, moving material collides with another material. In general, the material approaching (or creating) the interface has a velocity  $u_1$  and a

pressure  $p_1$ , and the pressure  $p_1$  can range from 0 to  $(\gamma - 1)\rho_1 u_1^2/2$ , which is the limit obtained when the pressure was produced by a strong shock (and is  $\rho_1 u_1^2/3$  for  $\gamma = 5/3$ ). Here we want to consider the more general case in which a shock wave (or in general a moving slab of material) approaches an interface beyond which there is material of arbitrary density, specified as  $\rho_4$  in the notation of Sect. 4.1.5, which we will use in this discussion.

This more general case is of interest in the laboratory and in astrophysical systems. In the laboratory, a radiation source can accelerate material in addition to shocking it, as is discussed in Chap. 9. This can be useful in an experiment if the goal is to make more energy available for the later evolution of the system. Such an object – a shocked and accelerated slab – might be described as a *plasma flyer plate*. In astrophysics, enduring radiation sources can shock and accelerate the objects they irradiate. This is the case, for example, in the star-forming region that includes the Eagle nebula, where the bright young stars have shocked and are now accelerating the nearby molecular clouds. When such an accelerated object encounters a stationary one, which might be a clump at higher density or a cloud at lower density, then on a global scale the kinds of interactions discussed here will ensue.

If  $\rho_4$  is small enough, we expect to see an adiabatic rarefaction when the shock reaches the interface. Figure 4.19 illustrates the situation in the lab frame. Let us apply the theory of Sect. 4.2.3 to this case. The new feature here is that it is best to do the mathematics in the *downstream frame*, in which the fluid downstream of the shock is at rest. We want to examine the system at the precise moment when the shock has reached an interface where the density decreases. In this frame, the expansion of the shocked matter



**Fig. 4.19.** A rarefaction may occur when a shock wave reaches an interface where the density drops, as illustrated here

follows identically (4.81)–(4.86). All the new aspects of this problem then have to do with correctly specifying the properties of the upstream medium

In the downstream frame, at the moment the shock reaches the interface, the new upstream material (of density  $\rho_4$ ) is moving toward the shocked material at a velocity  $u_1$ . The rarefaction then pushes shocked material forward at a velocity  $U$  in the downstream frame, which is  $U + u_1$  in the lab frame. Thus,  $U + u_1$  is the postshock fluid velocity of the upstream material in the lab frame. This implies that the transmitted shock velocity, in the lab frame, is  $(\gamma_4 + 1)(U + u_1)/2$ . We described the head of a centered rarefaction wave as moving backward in the material, which it does do in a Lagrangian sense. In the lab frame, however, the head of the rarefaction wave moves forward at  $u_{rw}$ , which is

$$u_{rw} = u_1 \left( 1 - \frac{\sqrt{\gamma_1(\gamma_1 - 1)}}{\sqrt{2}} \right). \quad (4.112)$$

Note that  $u_1$  can also be written  $2u_{s1}/(\gamma + 1)$ . The pressure in the shocked upstream medium,  $p_3$ , based on (4.18), is

$$p_3 = \frac{\gamma_4 + 1}{2} \rho_4 (U + u_1)^2. \quad (4.113)$$

Matching this pressure is the value at the interface,  $p_i$ . This can be found from (4.83), evaluated at the piston, which with current variable definitions is

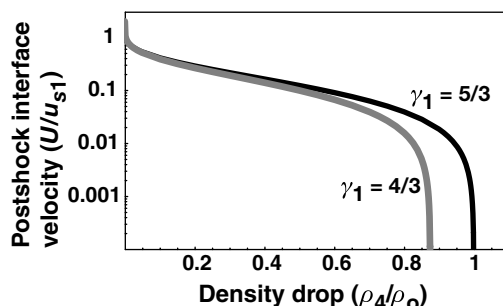
$$p_i = p_1 \left( 1 - \frac{\gamma_1 - 1}{2} \frac{U}{c_{s1}} \right)^{2\gamma/(\gamma-1)}, \quad (4.114)$$

in which  $c_{s1}$  is the sound speed in the initial shocked matter. One can find  $U$  by setting  $p_1 = p_3$ . For any specific choice of  $\gamma$ , the resulting equation can be converted to a polynomial equation for  $U$ . Figure 4.20 shows, for  $\gamma_4 = 5/3$  and for two values of  $\gamma_1$ , how the resulting value of  $U/u_{s1}$  depends on the density ratio  $\rho_4/\rho_o$ . Remember that this is in the downstream frame; in the lab frame,  $U$  is increased by  $2u_{s1}/(\gamma_1 + 1)$ .

#### Homework 4.21

Assuming that a strong shock reaches an interface beyond which the density ( $\rho_4$ ) is 0.1 times the density of the shocked material to the left of the interface ( $\rho_1$ ), solve for the profiles of the fluid parameters in the rarefaction that results.

One can see in Fig. 4.20 that there is a limiting value of  $\rho_4/\rho_o$  beyond which  $U$  disappears on the plot. In terms of the mathematics just described,  $U$  becomes negative, but this is not the physical solution. Instead, at this point the response of the system is to produce a reflected shock rather than a rarefaction. This can occur even if  $\rho_4 < \rho_1$ . The gray curve in Fig. 4.20 illustrates this case. To see how this occurs, consider that a reflected shock



**Fig. 4.20.** Profiles of the normalized interface velocity in the downstream frame when a shock encounters a density drop. The black curve shows  $\gamma_1 = 5/3$  and the gray curve shows  $\gamma_1 = 4/3$ , with  $\gamma_4 = 5/3$  in both cases

will form once the pressure in the shocked, low-density material exceeds  $p_1$ . Consider also that the postshock fluid velocity in the low-density material decreases as  $\rho_4$  increases, and has a limiting value of  $u_1$  at the transition from a rarefaction to a reflected shock. Thus, assuming the shock in the low-density material to be a strong shock (so (4.23) applies), we would expect this transition to occur when  $(\gamma_4 - 1)\rho_4 u_1^2/2 = p_1$ . In the specific case in which  $p_1$  is produced by a strong shock in a stationary material (which then approaches a stationary interface), one has  $p_1 = (\gamma_1 - 1)\rho_1 u_1^2/2$ , from which one can obtain the threshold density for a reflected shock as

$$\rho_4 = \frac{(\gamma_1 - 1)}{(\gamma_4 + 1)}\rho_1 = \frac{(\gamma_1 + 1)}{(\gamma_4 + 1)}\rho_o. \quad (4.115)$$

This is no surprise. It just says, for  $\gamma_1 = \gamma_4$ , that the transition occurs when the  $\rho_4/\rho_o$  at the interface exceeds 1, or in other words when the interface changes from a density decrease to an increase. But on the other hand if  $\gamma_4 > \gamma_1$ , then this transition will occur while there is still a density drop.

Consider also the more general case of a plasma flyer plate in which material that has been shocked or otherwise heated is accelerated to a higher velocity before it impacts a second material, creating an interface. In this case  $p_1 < (\gamma_1 - 1)\rho_1 u_1^2/2$ . If we express this as  $p_1 = f\rho_1 u_1^2$ , where as discussed above  $0 < f < 1/3$  for  $\gamma = 5/3$ , then the condition to produce a reflected shock becomes

$$\rho_4 = \frac{2f}{(\gamma_4 + 1)}\rho_1. \quad (4.116)$$

Here we see that the transition from a rarefaction to a reflected shock can occur at an arbitrarily low density, which depends on the properties and thus the history of the incoming matter.

**Homework 4.22**

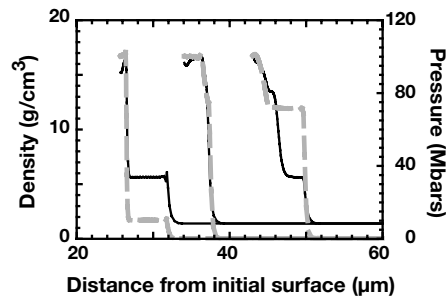
Assuming that  $\gamma_1 = \gamma_4$  (or not, if you wish), derive (4.116) from (4.44)–(4.52) by letting  $p_3$  approach  $p_1$  as the definition of the transition to a rarefaction. Hint: This one is not easy. Taking a limit will be necessary and the approach to the solution will matter.

**4.4.2 Overtaking Shocks**

It is not uncommon to find shock waves produced in succession, whether by a sequence of energy releases at the solar surface or by a sequence of irradiation pulses in an experiment. If a second shock is stronger than a first shock, then it moves more rapidly and will overtake the first one. The discussion so far in this chapter makes the qualitative behavior of such systems fairly obvious. We review it briefly here.

As the stronger second shock overtakes the first shock, there is a moment when they coalesce. In Fig. 4.21, the left set of curves show the two shocks, before the stronger one has overtaken the weaker one. The middle set of curves shows them as they coalesce. At this instant, the total density jump is the product of the density jump produced by each of the two shocks. However, unless the two shocks are fairly weak, this density jump is not consistent with a single shock. For example, if a shock with a density jump of 4-to-1 overtakes a shock with a density jump of 3-to-1, the resulting instantaneous density jump is 12-to-1, which few materials can sustain. So what does the system do?

One can see what happens by considering the moment of coalescence as an initial condition, in which the postshock material has a certain density, pressure, and velocity. What we have then is identical to the plasma-flyer-plate problem with a low value of the density beyond the interface created when the flyer plate reaches the object it collides with. The fact that the



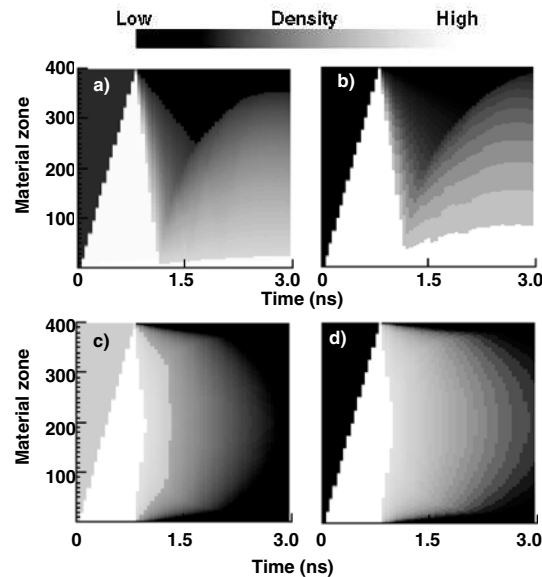
**Fig. 4.21.** Dynamics of overtaking shocks. Here a polyimide material of density  $1.4 \text{ g/cm}^3$  is driven at 10 Mbars for 1 ns and then is driven at 100 Mbars. The curves show 10.5 ns, 12 ns, and 13.5 ns. The dashed gray curves are pressure

density of the doubly shocked material,  $\rho_1$ , is produced by two shocks guarantees that the unshocked density,  $\rho_4$ , will be less than that given by (4.116). As a result, there will be a rarefaction in the shocked material, and a strong shock will be driven forward into the unshocked material. One can see this shock and rarefaction in the rightmost curves in Fig. 4.21. Soon after the rarefaction forms, the head of the rarefaction wave will return to the driven surface where the pressure that drove the shocks was applied. We take up next what happens then.

### 4.4.3 Reshocks in Rarefactions

The longer-term behavior of shocked layers of material is often very relevant to systems of interest. As shock waves or rarefactions traverse a system, they encounter interfaces or other waves and interact. This can be a complicating factor in any system for which the study of a later interface is of interest. Once more than one wave reaches the interface, its behavior becomes more complex. It is tempting to form the conclusion that every wave always begets a next wave, but this is not correct. In particular, when a rarefaction wave crosses a system, it may or may not produce a subsequent wave. Suppose specifically that a pressure source creates a shock wave in a layer of material, as described in Sect. 4.1. Suppose further that when the shock wave reaches the end of the layer, a rarefaction forms as described in Sect. 4.2 and 4.4.1. The head of the rarefaction moves back through the shocked material, and eventually reaches the initial surface. What happens then is illustrated in Fig. 4.22.

In these simulations whose results are shown in the figure, a pressure of 30 Mbars drives a shock through  $C_1H_1$  of density  $1 \text{ g/cm}^3$ . This produces the shock wave that moves up and to the right across the system. Once the shock wave reaches zone 400 (the end of the system), a rarefaction wave returns toward the driven surface. Consider first parts (a) and (b) of the figure. In the simulation producing these results, the driving pressure is always present. One can clearly see the rarefaction structure as a variation in density (shade of gray) between the rarefaction wave and the top of the image. When the rarefaction wave reaches the driven surface, a new shock is launched back into the plasma. This is easy to understand as follows. The rarefaction wave decreases the pressure of the shocked material. Even so, by assumption the driving pressure does not change. Once the pressure in the material begins to decrease at the driven interface, the interface will accelerate in response. This will launch either a sound wave or a shock wave back into the material. One cannot tell which from simple reasoning, but it turns out in this particular case that in response to the first rarefaction a second shock wave is launched. In this specific system, there is no next wave, as the second shock just travels down the density gradient to the end of the expanding plasma where it disappears. If there were further layers of material, then the second



**Fig. 4.22.** A grayscale display vs. Lagrangian zone and time can be an effective way to see the waves in a hydrodynamic system. In the first row are (a) density and (b) pressure for a reshock in a rarefaction, created by the continuous application of a 30 Mbar pressure to a CH material at 1 g/cc. In the second row are (c) density and (d) pressure for a system in which the driving pressure ends before the rarefaction returns to the driving surface, so no reshock is produced. The jagged boundaries are caused by the finite zone size and finite number of time outputs in the simulation

shock might produce further shock or rarefaction waves that would traverse the plasma. When this happens, it is known as *reverberation*.

#### Homework 4.24

An entertaining aspect of this specific problem is that it is one case where the traditional model in which shocks are driven by moving pistons does not produce correct qualitative behavior. Consider a rarefaction as it approaches a piston that is moving forward at a constant velocity. What will happen?

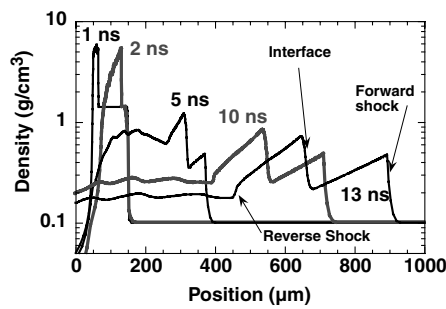
Consider now parts (c) and (d) of Fig. 4.22. In the simulation producing these results, the driving pressure ends after 0.8 ns. One can see the shock and the beginning of the rarefaction wave returning toward the driven surface. However, there is a second rarefaction wave moving upward from the driven surface, due to the expansion of this surface now that it is no longer driven. When rarefaction waves meet, they do not produce further waves. There is no source of pressure to produce a reshock in either rarefaction. Thus, if a rarefaction approaches a surface that was driven but is no longer driven, for example, because the radiation source has turned off, then there will be no

further wave. The result in Fig. 4.22 is that the density and pressure of the plasma decrease smoothly in space and time as the plasma expands and cools.

#### 4.4.4 Blast Waves at Interfaces

We discussed above how common blast waves are in astrophysics, because the originating event that produces a shock wave very often is short lived compared to the lifetime of the shock. As a result, the rarefaction from the source overtakes the shock and produces a blast wave. The blast wave then may encounter interfaces where the density changes, in response to which these interfaces will evolve. A very important application of this lies within a Type II supernova, in which the blast wave generated near the core of the star encounters density drops at the boundary between the C–O layer and the He layer, and again at the boundary between the He layer and the H layer. The density also decreases within each layer as radius increases, but this turns out not to be essential to the behavior at the interface.

Figure 4.23 illustrates the behavior. When the blast wave reaches the interface, the density drop at the interface cannot be sustained by a single shock. Just as in the cases of the adiabatic rarefaction and of overtaking shocks, a forward shock is driven into the low-density material. The surprising development is that a shock develops in the high-density material, despite the fact that (4.115) is not satisfied. We can explain what this is and why it is called a *reverse shock* by contrasting this case with the adiabatic rarefaction. In the adiabatic rarefaction, there is an indefinite supply of density and pressure behind the interface. In the adiabatic rarefaction, both the density and pressure increase with distance behind the interface, until they eventually reach their initial values. In contrast, in the blast-wave case, a rarefaction



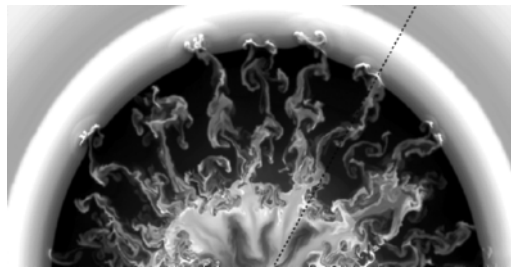
**Fig. 4.23.** Development of a reverse shock when a blast wave passes through an interface. This case is produced in a simulation when a laser with an irradiance of  $4.2 \times 10^{14}$  W/cm<sup>2</sup> strikes a 150- $\mu$ m-thick layer of polyimide, followed by a low-density carbon layer



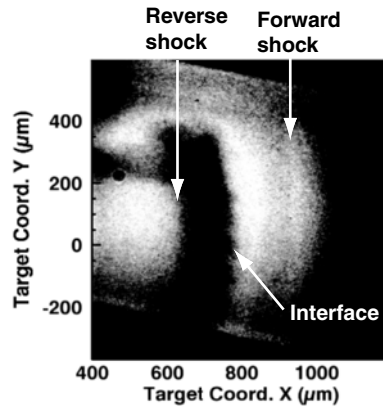
does propagate backwards, causing the plasma behind the interface to accelerate, but the pressure in the blast wave soon drops below the pressure in the shocked low-density material in front of the interface. One then has an expanding and accelerating flow of material that encounters the slower, higher-pressure material near the original interface. A shock develops at this transition. Thus, a *reverse shock* is a shock formed when a freely expanding plasma encounters an obstacle. This is distinct from the reflected shock formed when a shock wave crosses an interface. In the blast-wave case, the blast wave creates its own obstacle through its interaction with the interface.

One can also see in Fig. 4.23 that the density decreases behind the forward shock and behind the interface. This reflects the gradual expansion of the region between the shocks in response to the pressure gradient that develops as the system slows. This has the additional consequence that the forward shock soon develops a blast-wave structure itself. However, the shape of this structure may differ significantly from the shape of the initial blast wave. It also may not soon be self-similar, as the distance from the interface introduces an additional physical scale into the problem.

Systems in which blast waves encounter interfaces have been an important area of activity in the early years of experimental astrophysics. This has been motivated by instabilities in Type II supernovae, which we discuss further in the next chapter, and by the question whether errors in calculations of their nonlinear evolution might explain some discrepancies with data. Figure 4.24 shows results from a calculation of the explosion of a Type-II supernova (1987A). The forward and reverse shocks are clearly evident. Figure 4.25 shows data from an experiment in which these two features were also produced. The experiment was a well-scaled reproduction of the supernova explosion for reasons discussed in Chap. 10. This particular experiment was done to confirm that the correct one-dimensional behavior was achieved. It lacked the initial perturbations that would have produced unstable structures.



**Fig. 4.24.** Result of a simulation of the explosion of SN 1987A. The forward and reverse shocks are evident near the outer edges. The dramatic structures are discussed in Chap. 5. Credit: Kifonidis et al. (2003)



**Fig. 4.25.** Data from experiment sending a blast wave through an interface. Laser irradiation produced a shock in a 150- $\mu\text{m}$ -thick layer of dense plastic ( $1.41 \text{ g/cm}^3$ ), then ended allowing a blast wave to form. At the time of the image, the interface between the plastic and  $50 \text{ mg/cm}^3$  foam has moved 650- $\mu\text{m}$  and 2D effects, producing curvature and rollups at the edges, are becoming important

#### 4.4.5 Rarefactions at Interfaces

Rarefactions never proceed unimpeded forever. Whenever a rarefaction develops in astrophysics, whether at the edge of a supernova when the blast wave emerges from the star, at the edge of a molecular cloud when a shock wave emerges from it, or somewhere else, the rarefaction encounters at minimum the interstellar medium. In addition, it may encounter other objects as it propagates. This produces a situation in which a flowing, expanding, cool plasma produces an interface through its interaction with something. After reading the prior section, it will come as no surprise that the interaction produces a forward shock and a reverse shock.

Experiments can produce similar phenomena, by creating rarefactions that encounter a layer of material. In general, this can be a way to produce a high-Mach-number flow and then to let it interact. It also can have the effect of converting a brief source of energy into a lower-pressure source of much longer duration. Among recent applications of this technique have been equation of state studies (see Chap. 3), experiments related to supernova remnants (see below), experiments to produce jets, and designs of experiments to study the long-term interactions of shocks and clumps.

The classic example of a rarefaction that encounters an interface is the young supernova remnant. We can observe the explosion of a star for at most a few years. In contrast, we can observe nearby supernova remnants for centuries if not millennia. Supernova remnants are the observable structures that form through the interaction of the ejecta from a stellar explosion with the surrounding (circumstellar) environment. They are widely believed

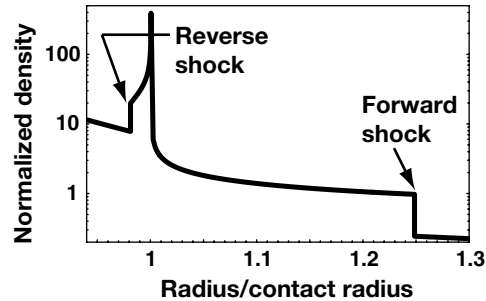
to produce most of the Cosmic Rays that irradiate the Earth. Despite our ability to observe a number of supernova remnants in considerable detail, the structure and the evolution of supernova remnants pose many challenges to our understanding.

The energy that creates the supernova remnant is the kinetic energy of the exploding star, typically about  $10^{51}$  ergs. An interesting feature is that the “interface” that leads to the structure has neither a decrease nor an increase of the density, but rather has an abrupt decrease in the density gradient. The material emerging from most stellar explosions can be argued to be self-similar (see Zeldovich and Raizer) and to have an inverse-power-law dependence on radius and time. The profile is quite steep, with an exponent of 8 or 9. The stellar ejecta undergo a homologous expansion, with velocity,  $v$ , radial distance,  $r$ , and time,  $t$ , related by  $v = r/t$ . Expansion cooling reduces the temperature of this material to a low value early on, so that nearly all the energy of the ejecta is kinetic energy.

In contrast, the circumstellar density falls off much more slowly, as  $1/r^2$  if it is due to a prior stellar wind or perhaps more slowly if the star has been an inactive white dwarf, as in the case of Type Ia explosions. When the rapidly expanding ejecta from the star interact with the nearly stationary circumstellar matter, forward and reverse shocks develop. This initially velocity of the forward shock is of order 10,000 km/s. This first phase of supernova-remnant evolution is the free-expansion (or “young-remnant”) phase. Ignoring clumps and instabilities, the entire structure between the two shocks moves at a velocity that is determined by the properties of the ejecta and the circumstellar material.

The system involves initial densities that are power laws, and the expansion velocity, which is  $x/t$ , introduces no additional scales, which suggests that the evolution might be self-similar. In 1982, R.A. Chevalier showed that it can be analyzed as two self-similar regions that are matched across the contact surface. One can find three coupled equations for the self-similar evolution of the density, the velocity, and the sound speed between the reverse shock and the interface, and between the forward shock and the interface, just as we discussed in Sect. 4.3.2. Here again, modern computational mathematics programs make the integration of these equations straightforward. Figure 4.26 shows the density profile for parameters relevant to SN 1987A, with the ejecta density scaling as  $r^{-9}$  and the circumstellar density scaling as  $r^{-2}$ .

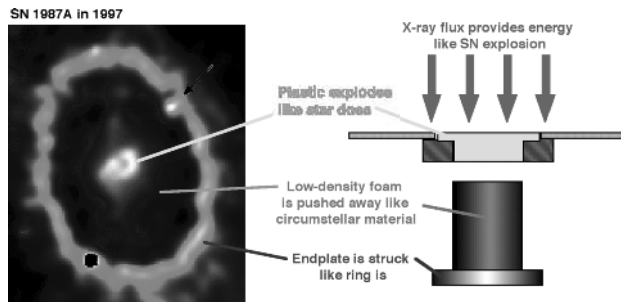
It may seem strange to treat the supernova remnant as a hydrodynamic object, because the average density of the circumstellar medium may be of order 1 or 10 particles per cubic centimeters. The feature that permits a hydrodynamic treatment is the presence of a magnetic field that is small enough that it does not affect the dynamics yet large enough and structured enough to confine the particles to a very small volume on the scale of the entire supernova remnant. This turns out to be very much the case. The



**Fig. 4.26.** Self-similar profile of density in a young supernova remnant, showing forward and reverse shocks. The supernova ejecta come in from the left

primary uncertainty in the hydrodynamic treatment is the potential effect on the hydrodynamics of a developing population of cosmic rays.

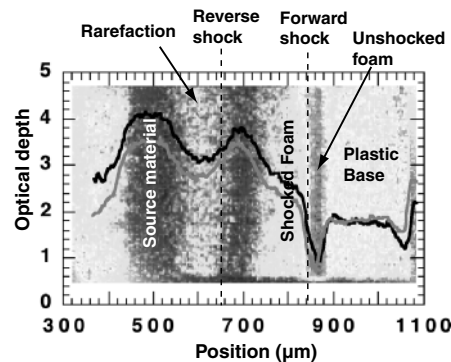
One example of an extremely young supernova remnant is the remnant from SN 1987A (reviewed by Chevalier in 1993), shown in Figure 4.27. At only 150,000 light years, this object is far closer and thus far more diagnosable than any other supernova of the modern era. The ring shown, and two larger rings as well, are of unknown origin and provide an added element of excitement. During the years after the explosion, the development of radio and x-ray emission from this object were followed by the advent of visible emission at “hot spots” as the stellar ejecta began to collide with the innermost ring.



**Fig. 4.27.** The supernova remnant from SN 1987A and a related experiment. (On left) SN 1987A in 1997. The arrow on the image of SN1987A shows the hot spot where interaction of the shocked matter and the ring had begun. The image of SN 1987A is from the Hubble Space Telescope. It was created with support to the Space Telescope Science Institute, operated by the Association of Universities for Research in Astronomy, Inc., from NASA contract NAS5-26555, and is reproduced with permission from AURA/STScI. (On right) Schematic of the experiment. The thickness of the plastic layer is 200  $\mu\text{m}$ . The diameter of the foam cylinder is 700  $\mu\text{m}$

Laboratory experiments can help improve our understanding of some of the mechanisms present in supernova remnants, and can help test the computational models we build to interpret their behavior. The design of the first such experiment, by the author and colleagues (Drake et al., 2000) is also illustrated in Fig. 4.27. The arrows and labels in the figure identify the correspondence between features in the experiment and those in SN 1987A. These experiments were in a planar geometry, intended to simulate a small segment of the overall supernova remnant expansion. The experiment began when an intense x-ray flux, produced by laser heating of a gold hohlraum (see Sect. 8.2), irradiated a 200- $\mu\text{m}$ -thick layer of plastic. The x-rays ablated the plastic, launching a strong shock wave through it, at a pressure of  $5 \times 10^{13}$  dynes/cm<sup>2</sup> (50 Mbars). This was the analog of the initial blast wave produced by the SN explosion. This shock wave compressed, accelerated, and heated the plastic. When the shock broke out of the plastic, the ejecta from its rear expanded, accelerated, cooled, and decompressed across a 150- $\mu\text{m}$ -wide gap. In an actual supernova remnant, spherical expansion provides the decompression (see McKee, 1974). Here the gap served an analogous function. The ejecta then launched a forward shock into the ambient matter, in this case a foam whose density was less than 1% of the density of the compressed plastic layer. The ejecta stagnated against the (moving) contact surface with the foam, which launched a reverse shock into the ejecta, just as occurs in a supernova remnant.

Figure 4.28 shows the measured profile, obtained by x-ray radiography. The forward shock and reverse shock are clearly established. This system is a well-scaled model of the basic hydrodynamic structure of a young supernova



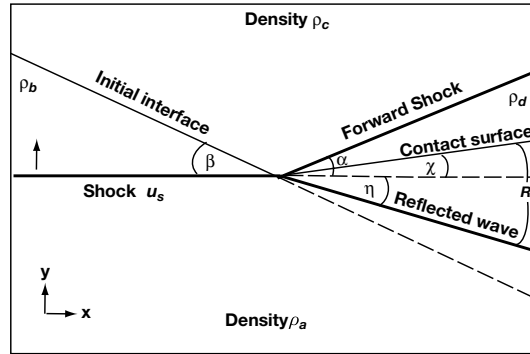
**Fig. 4.28.** Data from an ejecta-driven shock experiment. The image shows the x-ray transmission through the experimental system. One can clearly see the reverse shock and forward shock. The curves show evaluations of the optical depth of the system as a function of position from an initial surface. The spatial resolution is limited, so the transitions in optical depth are smoothed out somewhat. From Drake et al. (2000)

remnant (see Chap. 10 regarding how to scale such systems). Specifically, with reference to quantities defined in Chap. 2, in the supernova remnant and the experiment, respectively,  $Re = 6 \times 10^8$  and  $7 \times 10^6$ ,  $Pe = 10^7$  and  $10^4$ . Radiative losses are unimportant in both systems. This basic experiment design has subsequently been used to address instabilities in such systems and their interaction with other structures.

The later evolution of the supernova remnant connects with other topics in this book. Eventually, the mass of the accumulated circumstellar matter exceeds the mass of the stellar ejecta. This is generally taken to mark the (gradual) transition to the “Sedov–Taylor” phase. As this phase begins, the reverse shock runs in to the center of the supernova remnant and dissipates, after which the supernova remnant is believed to develop the characteristic structure of the Sedov–Taylor blast wave discussed above. Throughout the development of the supernova remnant, the shocked matter also radiates energy. Radiative losses are in some cases important during the young-supernova-remnant phase, but were not in SN 1987A. They are never important during the Sedov–Taylor phase, but eventually the remnant slows down and cools enough that they become important (they pass through the minimum of the “cooling function” discussed in Sect. 6.2.2). Once enough cooling has occurred, the remnant becomes a momentum-driven snowplow (Sect. 4.3.1) and the shell structure (now much thinner) may become unstable to thin-shell instabilities. The above is the one-dimensional story, but the extent to which three-dimensional effects such as instabilities or interactions with clumps may distort this picture is not entirely known. Not only supernova remnants but also other objects such as molecular clouds are observed to be clumpy in general.

#### 4.4.6 Oblique Shocks at Interfaces

To prepare for the discussion in the next chapter, we also need to consider the behavior that develops when an oblique shock wave arrives at an interface, where the density increases or decreases. One can see from the discussion earlier in this chapter that in general the result will involve a transmitted shock beyond the interface and a reflected wave propagating backwards (relative to the interface). The postshock contact surface will be between these, with the sign of the angles ( $\alpha$  and  $\chi$  in Fig. 4.29) depending on the type of reflected wave. The properties of the interface will determine whether the reflected wave is a shock wave or a rarefaction wave. So long as the EOS is the same on the two sides of the interface, the reflected wave will be a shock when the density increases across the interface and a rarefaction when the density drops. Figure 4.29 is a schematic of the essential geometry, assuming the system to be uniform in the direction out of the page. We will label the shocked or unshocked regions  $a, b, c$ , and  $d$ , as indicated on the subscripts on the density  $\rho$ , and will use the subscript  $R$  for the region between the contact surface and the reflected wave. So long as the pressure source driving



**Fig. 4.29.** Sketch of behavior occurring when a shock approaches an oblique interface. The angles shown are for a reflected rarefaction wave. For convenience,  $\eta$  and  $\beta$  as shown are taken to be positive angles

the shock remains constant, and so long as the edges of the system have no effect, the various waves will each have a constant velocity. As a result, they will radiate in straight lines from the point where the shock and the interface meet, and the fluid velocity will be independent of the distance from this point. The entropy will be constant across each region of this system, changing only at the shocks.

Following Section 109 of Landau and Lifshitz, we can observe that the sensible way to analyze this system is in a cylindrical coordinate system centered at the point where the shock meets the interface. We can make this coordinate system be stationary by working in a frame in which  $\mathbf{u} = \mathbf{u}_{lab} + \mathbf{u}_I$ , where  $\mathbf{u}_I$  is given by  $(u_s / \tan \beta) \hat{x} - u_s \hat{y}$ , with the  $x$  and  $y$  directions defined as shown in Fig. 4.29. We define the azimuthal angle  $\phi$  relative to the  $x$ -axis, as usual. In this frame of reference, a point on the shock wave or the interface moves radially inward with time, while a point on the transmitted shock or the reflected wave moves radially outward. Assuming that  $u_z$  is constant everywhere, and based on the assumptions of the previous paragraph, the derivatives in  $r$  and  $z$  are zero in this coordinate system while  $u_r$  and  $u_\phi$  may vary with  $\phi$ . With these assumptions the momentum equation implies

$$\frac{\partial u_r}{\partial \phi} - u_\phi = 0 \text{ and} \tag{4.117}$$

$$u_\phi \frac{\partial u_\phi}{\partial \phi} + u_r u_\phi = -\frac{1}{\rho} \frac{\partial p}{\partial \phi} = -\frac{\partial w}{\partial \phi}, \tag{4.118}$$

where  $dw = dp/\rho$  is the differential enthalpy at constant entropy, while the continuity equation implies

$$\left( u_r + \frac{\partial u_\phi}{\partial \phi} \right) + \frac{u_\phi}{\rho} \frac{\partial \rho}{\partial \phi} = 0. \tag{4.119}$$

**Homework 4.25**

To obtain these results, one must evaluate the equations in cylindrical polar coordinates. Beginning with the first two Euler equations, carry out this evaluation.

By combining these equations, we can obtain some insight into the behavior. Equations (4.117) and (4.118) imply that

$$w + \frac{1}{2}(u_r^2 + u_\phi^2) = \text{constant}, \quad (4.120)$$

while these equations in combination with (4.119) imply that

$$\left(u_r + \frac{\partial u_\phi}{\partial \phi}\right) \left(1 - \frac{u_\phi^2}{c_s^2}\right) = 0. \quad (4.121)$$

Equation (4.120) connects the changes in velocity with changes in enthalpy as one moves across the system. We will return to it. Equation (4.121) evidently has three solutions. One of these solutions occurs when the argument of the left parentheses is zero. This corresponds to uniform flow.

**Homework 4.26**

Thus, a property of uniform flow is that  $u_r = -\partial u_\phi / \partial \phi$  in any cylindrical polar coordinate system. Landau and Lifshitz use a geometric argument to demonstrate this. Instead, demonstrate this using a vectorial argument. (Hint: Begin by taking dot products of unit vectors along  $r$  and  $\phi$  with an arbitrary velocity vector.)

The flow is uniform in regions  $a, b, c$ , and  $d$ , with discontinuities at the boundaries between regions. The equations describing these parts of the flow do not depend on the nature of the reflected wave. To simplify the mathematics, we will assume the initial pressure in regions  $b$  and  $c$  to be negligible and the entire system to have constant  $\gamma$ . Then in regions  $b$  and  $c$  the velocity equals the velocity of our moving frame of reference,

$$\mathbf{u}_b = \mathbf{u}_c = \mathbf{u}_I = \frac{u_s}{\sin\beta} [\cos\beta\hat{x} - \sin\beta\hat{y}], \quad (4.122)$$

while in region  $a$  we have  $\rho_a = \rho_b(\gamma + 1)/(\gamma - 1)$ , and we use the results of Sect. 4.1.5 for oblique shocks to find the postshock velocity, which is

$$\mathbf{u}_a = \frac{u_s}{\sin\beta} \left[ \cos\beta\hat{x} - \left(\frac{\gamma + 1}{\gamma - 1}\right) \sin\beta\hat{y} \right], \quad \text{and} \quad (4.123)$$

$$p_a = \frac{\rho_b u_s^2}{\gamma + 1}. \quad (4.124)$$



In region  $d$  we have  $\rho_d = \rho_c(\gamma + 1)/(\gamma - 1)$ , and the shock is also oblique in the moving frame. One can show that

$$\mathbf{u}_d = \frac{u_s}{\sin\beta} \left[ \frac{\gamma\cos\beta + \cos(2\alpha + \beta)}{\gamma + 1} \right] \hat{x} - \frac{u_s}{\sin\beta} \left[ \frac{\gamma\sin\beta - \sin(2\alpha + \beta)}{\gamma + 1} \right] \hat{y} \quad \text{and} \quad (4.125)$$

$$p_d = \frac{\rho_c u_s^2}{\gamma + 1} \frac{\sin^2(\alpha + \beta)}{\sin^2\beta}. \quad (4.126)$$

The ratio of the component of  $\mathbf{u}_d$  normal to the surface of the forward shock to the component parallel to this surface gives  $\tan(\alpha - \chi)$ , as it is the radial flow away from the forward shock that establishes the downstream boundary of region  $d$ . This gives

$$\left( \frac{\gamma - 1}{\gamma + 1} \right) \tan(\alpha + \beta) = \tan(\alpha - \chi). \quad (4.127)$$

All of the above applies whether there is a reflected shock or a rarefaction. We consider these possibilities in turn. In the case of the reflected shock, the flow within the reflected shock is also uniform. The difference with the planar case of Sect. 4.4.1 is that the transmitted and reflected shocks are both oblique. The angles  $\alpha$  and  $\chi$  in Fig. 4.29 are both negative in this case, but (4.125)–(4.127) still apply. For region  $R$  we have

$$\mathbf{u}_R = \frac{u_s}{(1 + \gamma)^2 \sin\beta} ([\gamma(\gamma + 1)\cos\beta + \gamma\cos(\beta - 2\eta) + \cos(\beta + 2\eta)] \hat{x} + [-\gamma(\gamma - 1)\sin\beta + \gamma\sin(\beta - 2\eta) + \sin(\beta + 2\eta)] \hat{y}), \quad (4.128)$$

$$p_R = \frac{\rho_b u_s^2}{\gamma - 1} \frac{[\sin(\beta + \eta) - \gamma\sin(\beta - \eta)]^2}{(\gamma + 1)^2 \sin^2\beta}, \quad \text{and} \quad (4.129)$$

$$\left( \frac{\gamma - 1}{\gamma + 1} \right) \frac{\sin(\beta + \eta) - \gamma\sin(\beta - \eta)}{\cos(\beta + \eta) + \gamma\cos(\beta - \eta)} = \tan(\eta + \chi). \quad (4.130)$$

Given the known parameters, one can solve numerically for  $\alpha$  and  $\eta$  by setting  $p_d = p_R$  and setting equal the solutions of (4.127) and (4.130) for  $\chi$ .

In the case of the reflected rarefaction wave, the flow is not uniform in the region between the rarefaction wave and the contact surface. This region, which we can call the rarefaction, is isentropic so  $p/c_s^{2\gamma/(\gamma-1)}$  and  $\rho/c_s^{2/(\gamma-1)}$  are both constant. In the rarefaction one must replace (4.128) through (4.130) with a description of the rarefaction. Within this region,  $u_r$  is clearly positive and the relevant solution of (4.121) has  $u_\phi = c_s$ . If we designate the properties at the head of the rarefaction wave as  $w_a$ ,  $u_{ra}$ , and  $c_a$ , then we can write from (4.120)

$$u_r^2 = u_{ra}^2 + c_a^2 - c_s^2 + 2(w_a - w). \quad (4.131)$$

Substituting these two results into (4.119), rearranging, and integrating gives

$$\phi + \eta = \int_{\rho_a c_a}^{\rho c_s} \frac{d(\rho c_s)}{\rho \sqrt{u_{ra}^2 + c_a^2 - c_s^2 + 2(w_a - w)}}. \quad (4.132)$$

Here  $\eta$  is as shown in Fig. 4.29, and would correspond to  $-\phi_o$  if we designated the angle of the rarefaction wave as  $\phi_o$ . For a polytropic gas, this can be reduced to

$$\phi + \eta = \sqrt{\frac{\gamma+1}{\gamma-1}} \int_{c_s/c_a}^1 \frac{d\xi}{\sqrt{Q^2 - \xi^2}}, \quad \text{in which} \quad (4.133)$$

$$Q^2 = 1 + \frac{\gamma-1}{\gamma+1} \frac{u_{ra}^2}{c_a^2} > 1 \quad (4.134)$$

Note that  $Q$  depends on  $\eta$ , because

$$\frac{u_{ra}^2}{c_a^2} = \frac{(\gamma+1)^2 \cos^2(\beta - \eta)}{\gamma(\gamma-1) \sin^2 \beta} > 1 \quad (4.135)$$

We can find the angle  $\eta$  by considering the behavior of the rarefaction wave. In a planar system, the rarefaction wave flows away from the shocked interface into the downstream system at the downstream sound speed. One way to think of this is to realize that sound waves are launched in all directions from the disturbed interface, and that in the planar geometry the leading edge of their phase fronts initiates the rarefaction wave. Taking this same point of view, we can say that sound waves propagate from any point on the interface in the system of Fig. 4.29, beginning at the moment the shock reaches the interface (taken as  $t = 0$ ). The vector  $\mathbf{f}$  describing their location is then

$$\mathbf{f} = (\mathbf{u}_I + c_a \hat{\mathbf{k}}) t, \quad (4.136)$$

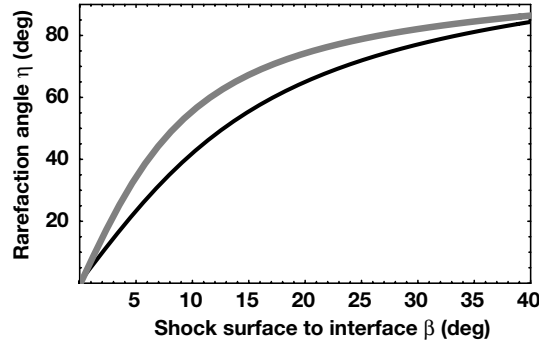
in which  $\hat{\mathbf{k}}$  is a unit vector in an arbitrary direction. Defining the angle of this vector as  $\phi_s$ , and expressing both  $c_a$  and  $\mathbf{u}_I$  in terms of  $u_s$ , we can find the following equation for angle of the rarefaction,  $\eta$ :

$$-\tan \eta = \text{Min} \left[ \frac{-\frac{\gamma+1}{\gamma-1} \sin \beta + \sqrt{\frac{\gamma}{\gamma+1}} \sin \beta \sin \phi_s}{\cos \beta + \sqrt{\frac{\gamma}{\gamma+1}} \sin \beta \cos \phi_s} \right], \quad (4.137)$$

where the minimum is found by varying  $\phi_s$ . One typically finds  $\phi_s \sim 180$  to 250 degrees. Figure 4.30 shows  $\eta$  vs.  $\beta$ .

Knowing  $\eta$ , (4.76) allows one to evaluate  $c_s$  (and hence other quantities) as a function of the angle  $\phi$  in between the rarefaction wave and the contact surface as

$$c_s = c_o Q \sin \left[ \sin^{-1} \left( \frac{1}{Q} \right) - \sqrt{\frac{\gamma-1}{\gamma+1}} (\phi + \eta) \right]. \quad (4.138)$$



**Fig. 4.30.** The angle of rarefaction produced by oblique interface turns out to depend only on  $\gamma$  and  $\beta$ . Solid is  $\gamma = 5/3$ ; gray is  $\gamma = 4/3$

At the contact surface, this gives an equation for  $\chi$ , in terms of the sound speed  $c_{\text{end}}$  there:

$$\chi + \eta = \sqrt{\frac{\gamma + 1}{\gamma - 1}} \left[ \sin^{-1} \left( \frac{1}{Q} \right) - \sin^{-1} \left( \frac{c_{\text{end}}/c_a}{Q} \right) \right], \quad (4.139)$$

where the fact that the pressure in the rarefaction at the contact surface equals  $p_d$  gives

$$\frac{c_{\text{end}}}{c_a} = \left( \frac{p_d}{p_a} \right)^{(\gamma-1)/(2\gamma)} = \left( \frac{\rho_c \sin^2(\beta + \alpha)}{\rho_b \sin^2 \beta} \right)^{(\gamma-1)/(2\gamma)}. \quad (4.140)$$

Thus, (4.139) gives  $\chi$  as a function of  $\eta$  (already known as described above) and  $\alpha$ . Equation (4.127) still gives  $\chi$  as a function of  $\alpha$ . This allows one to obtain numerically a solution for  $\alpha$  and  $\chi$ .

This completes our description of the shock at an oblique interface. When we consider rippled interfaces in Chap. 5, we will work with the small-angle limits of the above equations.

## 5 Hydrodynamic Instabilities

Our discussion of the previous chapter focused on one-dimensional phenomena, in which a physical system was structured as a function of linear or radial distance, but by assumption was not structured in the other two dimensions. Common experience tells us that this will rarely be a good assumption. We see turbulent clouds and whirlwinds in the air and complex eddies in water. We know of three-dimensional turbulent motions within the Earth and within the Sun. For that matter, if we focus our attention, we can see amazing hydrodynamic phenomena every day in the bathroom sink. We know that one can save fuel in cars and in airplanes by careful design that reduces the energy delivered to turbulence in the air. In fact, our experiences and common knowledge would lead us to conclude that hydrodynamic fluids are more often than not unstable and turbulent in some sense.

We might suppose, however, that all such effects are well understood, because they have been studied in depth for more than a century. Surely a series of brilliant humans, armed with modern mathematics and, more recently, with powerful computers, working with phenomena that can very readily be observed in nature or produced in laboratories, will have come to understand this subject thoroughly. As it happens, this is not true. One could nominate hydrodynamic instabilities and turbulence for an award in the category of area of physics in which the least fundamental progress has been made during the last century. There are, of course, positive outcomes of the effort in this area. Much has been learned, much is being learned, and what has been learned has often had real practical importance far beyond any direct impact of the quest for the next quark. But one still must wonder why this has been so difficult.

Much of the answer to this question can be found by contemplating the first Euler equation – the continuity equation. This equation contains the divergence of the product  $\rho u$ , which makes it a nonlinear equation. It often proves feasible in physics to deal with nonlinear terms in physical equations by assuming that one of the variables is constant or by linearizing both variables. Indeed, these approaches will provide some insight into the fundamental hydrodynamic instabilities as this chapter develops. In some physical systems, the nonlinear terms drive waves that saturate themselves in ways that do not affect the global dynamics of the system. In hydrodynamic systems, the

variations in both density and velocity often become large and structured in comparison with the initial values. The resulting dynamics are not tractable. Even computer simulations cannot follow all of the behavior, as the finite size of computer memory and run time imposes significant limits on the resolution with which one can examine the dynamics.

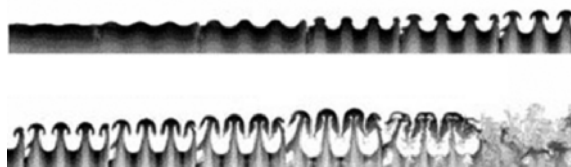
Even so, gaining some understanding of hydrodynamic instabilities is feasible. In particular, one can identify various circumstances that produce unstable behavior; these give us the instabilities with well-known names. One can use linearized theory to evaluate the initial growth rate of the instabilities when their amplitudes are small. We will pursue this for some instabilities that are important to high-energy-density physics. The cases we will examine all involve modulations of a system in only two dimensions. This is where one finds the strong effects that tend to initiate the growth of unstable structures in real fluids. As the modulations grow, they proceed to develop structure in the third dimension, which sometimes sets the stage for secondary instabilities that involve modulations of a system in three dimensions. In contrast to the two-dimensional instabilities, which are few and ubiquitous, the three-dimensional instabilities tend to be specific to a given detailed geometry. We will leave their details to the specialized literature, but we will discuss their consequence, which is a state of the fluid known as hydrodynamic turbulence.

We are about to begin, but first we have to consider some sad news. The fluids of interest to us are compressible. They respond to pressure gradients in part by compressing. So our strong preference would be to analyze potential instabilities for compressible fluids. Unfortunately, the mathematics involving compression rapidly becomes too complicated, making analytic theories of limited use. Most practical studies of instabilities in compressible systems involve hydrodynamic simulations, which can handle the compression and perhaps even the nonlinear evolution of the instability. In applications, this nonlinear evolution is often of most interest. In much of this chapter we consider first the incompressible theory to develop some intuition for the processes and to get a first estimate of their growth rates. Fortunately, the incompressible theory has worked out better than one might have feared. The instabilities in compressible systems behave very much like the incompressible theory would suggest. One possible reason for this is that unstable fluctuations that do not compress the fluid often grow faster than those that do, perhaps because they spend no energy on compression. A caution for applications of the incompressible results is that rarefactions in compressible systems can stretch the unstable structures, creating the illusion of larger unstable growth.

## 5.1 Introduction to the Rayleigh–Taylor Instability

The *Rayleigh–Taylor instability* causes the interpenetration of fluid regions having different density. Figure 5.1 shows an example. These fluid regions

may be two different materials, divided by an interface, or the same material at two different average densities, with a density gradient between the regions. The instability, which we will designate as the RT instability, is often said to occur whenever a less-dense fluid supports a more-dense fluid against gravity. A first generalization of this condition is to say that the RT instability occurs whenever fluid regions that differ in density experience a pressure gradient that opposes the density gradient. A second and broader generalization of this condition is to say that an *entropy mode* occurs whenever the entropy gradient is parallel to the pressure gradient (see Sect. 5.3).



**Fig. 5.1.** Growth and saturation of the Rayleigh–Taylor instability, observed by acceleration of two fluids in a test facility. Credit: University of Arizona, Jeff Jacobs.

### 5.1.1 Buoyancy as a Driving Force

We will focus here primarily on the condition of opposed pressure and density gradients. This condition is perhaps too general to be immediately clear, but this is necessary to cover most cases of interest in high-energy-density physics. When this condition for instability is satisfied, the system can reduce its potential energy through the interpenetration of the two fluid regions. Our approach to this instability will be to begin with some simple analysis for the sake of improving our intuition about both buoyancy and acceleration. Then we will proceed in the next section with a formal derivation of the unstable behavior.

We begin with bubbles. We know that bubbles of air rise in water, and can define analogies in other fluids. We can understand the upward force on the bubble, as Archimedes did, by thinking about the force required to insert the bubble into the water, thereby lifting the water surface. But this does not give us any understanding of what really goes on in the bubble. So it is natural to ask why, in detail, a bubble experiences a net upward force. We could begin to think about this by considering the fluid momentum equation in a fluid without viscosity,

$$\rho \frac{\partial}{\partial t} \mathbf{u} + \rho \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p - \nabla \Psi, \quad (5.1)$$

in which  $\Psi$  is the potential, equal to  $\int \rho g dz$  for gravity. (We ignore surface tension here, although that would not be justified for air bubbles in water.) In steady state, the gradients of pressure and gravitational potential balance one another, and indeed we often determine the pressure at some point by thinking about the weight of the fluid above it. If we think only in the vertical direction, it would seem that any distribution of matter could come to an equilibrium, with as much pressure as is needed to balance gravity. Then there would be no buoyancy (and no Rayleigh–Taylor instability either).

In a structured fluid under gravity, the weight of the matter above various points will vary. If this determined the local pressure, then there would be lateral pressure gradients in the fluid. But the fluid cannot sustain such gradients. In a compressible fluid they would be relaxed through compression and sound waves. In the limit that the fluid becomes incompressible, the pressure is instantaneously constant along surfaces of constant potential energy (i.e., at the same height in a gravitational potential). This implies that the pressure is determined by the average of the weight of the matter above any such surface. In mathematical terms, if  $z$  defines an axis parallel to the direction of the potential gradient, then

$$p = \frac{1}{A} \int_A \int_z^{z_{\max}} -\frac{\partial \Psi}{\partial z} dz dA = \frac{-1}{A} \int_A \int_z^{z_{\max}} \rho(z, x, y) g dz dA, \quad (5.2)$$

in which the area of the fluid over the  $(x, y)$  plane is  $A$  and  $z_{\max}$  is a height beyond which there is no influence on the location of interest, and the right-most expression is specific to mass in a gravitational field. The corresponding term in (5.1) is

$$-\frac{dp}{dz} = \frac{1}{A} \int_A \frac{\partial \Psi}{\partial z} dA = \frac{1}{A} \int_A \rho(z, x, y) g dA = \bar{\rho} g, \quad (5.3)$$

which defines the average density as  $\bar{\rho}$ . For the simple gravitational case, we thus have

$$\rho \frac{\partial}{\partial t} \mathbf{u} + \rho \mathbf{u} \cdot \nabla \mathbf{u} = (\bar{\rho} - \rho) g. \quad (5.4)$$

Now we can see that the bubble, having  $\rho < \bar{\rho}$ , experiences an upward force. Also we see that the remaining fluid experiences a downward force. This corresponds to the settling that must occur as the bubble moves higher. For a small bubble in a fluid of density  $\rho_2$ ,  $\bar{\rho} \approx \rho_2$ , and by integrating over the bubble volume  $V$  one finds the standard result that the upward force is  $(\rho_2 - \rho)V$ . At this point we understand in detail why the bubble rises. To understand the dynamics of the bubble as an entity, we would also have to consider the work involved in displacing the fluid above the bubble and any other forces such as surface tension. But (5.4) takes us far enough to address RT instabilities, so we stop here for now.

The standard simple example of an RT instability is the evolution of a system in which a denser fluid, such as water, is initially oriented above a

less dense fluid, such as oil, in a gravitational field (whose direction defines “above”). A standard demonstration uses a jar filled with oil and water, which is quickly yet smoothly inverted. One can also find toys or desktop knick-knacks that display the resulting dynamics. (For such demonstrations it does matter that the motion which inverts the two fluids does not cause a change of state of the fluid. Thus, one may observe that dark beer is less dense than amber beer, but the attempt to invert a glass containing these two unmixed fluids is likely to have comic consequences unrelated to RT.) Returning to our reference situation, first note that the pressure increases toward the bottom of this structure, because of the weight of the fluid above it. Thus, the pressure gradient is downward. Second, by assumption the density gradient is upward. Within any ripple at the interface between the two fluids, the less dense fluid will feel an upward force while the more dense fluid will feel a downward force. The result is that small ripples of the interface, always present because of thermal noise, will grow. The comparison with the case of the bubble makes it seem natural to describe the region of less-dense, rising fluid as the *bubble*. This is standard jargon in discussions of RT. The denser material that penetrates into the less dense fluid is known as the *spike*.

### Homework 5.1

Consider a system with water above oil as just described. Suppose there is a small, sinusoidal ripple on the surface. Find the vertical profile of the force density between the lower and upper boundaries of the ripple for a region of denser fluid and for a region of less-dense fluid. Discuss the comparison of the two fluids and the shape of the force density profile.

The cases that arise in high-energy-density physics rarely involve gravity as such. Instead, they tend to involve a low-density fluid that is pushing against a higher density fluid, causing the acceleration or deceleration of the higher density fluid. One example is that of hot air beginning to rise against cooler, higher density air. An analogous case is that of a pulsar wind (a very high temperature yet low-density fluid of positrons and electrons) accelerating the denser matter surrounding the pulsar ahead of it. A second analogous case is that of the low-density, laser-heated corona surrounding a laser-fusion capsule pushing the denser capsule and accelerating it inward. In all these cases the pressure is higher in the low-density fluid, so there is a pressure gradient that opposes the density gradient. One way to see intuitively how the instability works in this case is to realize that the inertia of any fluid will cause it to resist being accelerated. The denser fluid has more inertia and the interpenetration of the two fluids allows some of it to lag behind.

There is a counterintuitive aspect to such systems, though. The potential energy is reduced when the denser fluid “falls” up the pressure gradient. In the context of fluid dynamics, one tends to think that pressure gradients cause the acceleration of material down the pressure gradient. This is true in isolation, but here it is the behavior of each individual fluid parcel relative to



the average behavior that matters. If one thinks of the position and velocity of a fluid parcel relative to the position of an accelerating interface, one can see that more energy must be invested to move ahead of the interface rather than to stay with the interface, but less energy must be invested to lag behind. Thus, in the context of a system with a steady, imposed acceleration of an interface, the potential energy is proportional to the distance material has moved ahead of the interface. This defines the analog of “up” in this system. The less dense fluid will feel a force, relative to the interface position, that is in the direction of the acceleration, while the denser fluid will feel a force in the opposite direction, causing it to lag behind.

A mathematically identical situation develops at a decelerating interface when the densities are reversed. This occurs when denser matter has an initial velocity, so it is moving into a region of less-dense matter. The compression and heating of the less-dense matter, perhaps by a shock, establishes a pressure gradient that acts to decelerate the denser matter, thus opposing the density gradient. Examples occur in the laboratory when a blast wave exits dense matter into less dense matter, as for example, at the interior surface of an inertial-fusion capsule. Astrophysical cases abound, for example, at the head of some astrophysical jets, where the denser material in the jet is decelerated by the less-dense material in front of it. In these cases with a decelerating interface, less energy is again invested to lag “behind” the interface, though now this is accomplished by slowing down less than the interface does. Here the interpenetration of the fluids reduces the potential energy of the matter relative to that generated by the imposed deceleration of the interface.

Another example of a decelerating interface is found in supernovae. Figure 4.25 showed the results of a simulation of one particular supernova explosion (SN1987A). During supernova explosions the blast wave from the explosion crosses the material interfaces in the star, where the density decreases more rapidly. The interface then decelerates, as the blast wave moves outward and the velocity of the interface decreases (see Sect. 4.4.4). In the process, a pressure gradient is established (again this is just part of the blast wave) that points outward, opposing the inward density gradient at the interface. In consequence RT develops at each interface, and in the nonlinear phase spikes of dense material flow outward through the star. The regions between the spikes, where less-dense material moves inward are the bubbles.

We can summarize the above mathematically by noting that the natural frame of reference in which to examine RT growth is that of the interface. This frame is typically accelerating relative to the frame of the laboratory. In the frame of the laboratory, the acceleration of the interface might be written as  $\mathbf{a} = -\nabla p/\rho$ . In the frame of the interface, any modulations see an average “gravitational” acceleration  $\mathbf{g}$  in the opposite direction. Thus  $\mathbf{g} = \nabla p_o/\rho_o$ , in which the subscript  $o$  designates the averaged values. This completes our introductory contemplation. We now proceed to develop a mathematical treatment of the linear phase of the RT instability.

### 5.1.2 Fundamentals of the Fluid-Dynamics Description

In the present section, we take up the fluid-dynamics description of the RT instability, in which we consider the behavior of the entire fluid in two dimensions. An alternative approach is the interface-dynamics description, in which one analyzes only the behavior of an interface, ignoring the dynamics elsewhere. Both are found in the literature. The interface dynamics description, being more limited, can be easier to formulate mathematically. However, it is a dead-end approach. It does not permit the inclusion of other effects such as viscosity or density gradients, and it does not generalize to allow for changing properties of the plasma in the third dimension.

So we proceed with a fluid-dynamics description. We will end up finding solutions for the RT modulations as *surface waves*, which are waves whose influence on the medium decays as one moves away from the surface. We can analyze the dynamics, using the fluid continuity and momentum equations. For momentum, we will use (2.27). We take the radiation pressure, the electromagnetic forces, and the other forces to be negligible, but we will keep the viscous force to explore the effects of viscosity. For comparison with other literature, note that surface tension, treated for example in *Chandrasekhar* would be one of the other forces in (2.27). We ignore this force because it has no relevance to high-energy-density systems, which are too hot to allow the molecular interactions that create it.

We define our fluid such that the initial unperturbed interface is in the  $x$ - $y$  plane. Our approach will be to linearize the fluid equations, so we take the unperturbed pressure and density to be  $p = p(z)$  and  $\rho = \rho(z)$ , respectively, and we take the first-order perturbations in the same quantities to be  $\delta p(x, y, z)$  and  $\delta \rho(x, y, z)$ . We work in a frame in which the interface is at rest, so the zeroth-order velocity is 0 and we can take the first-order velocity to be  $\mathbf{u}(x, y, z)$ . We assume the fluctuations to be incompressible, noting as discussed above that this may not require that the fluid itself is incompressible. This assumption is expressed as  $\nabla \cdot \mathbf{u} = 0$ , so the continuity equation becomes convective:

$$\frac{\partial \delta \rho}{\partial t} + \mathbf{u} \cdot \nabla \rho = 0, \quad (5.5)$$

and with our assumptions the linearized momentum equation becomes

$$\rho \frac{\partial \mathbf{u}}{\partial t} = -\nabla \delta p + \nabla \cdot \underline{\boldsymbol{\sigma}}_\nu - g \delta \rho \hat{\mathbf{z}}, \quad (5.6)$$

in which  $\hat{\mathbf{z}}$  is a unit vector in the  $z$  direction and the effective gravitational acceleration, in a (noninertial) frame of reference in which the interface is at rest, is  $g$ . This can be tricky to apply, as one may be inclined to assume that  $g$  is in the direction of the acceleration in the frame of reference of the laboratory. However, in the frame of reference of the interface,  $g$  points toward the region of higher pressure, for reasons discussed at the end of the previous subsection. Here as in Chap. 2 the viscosity tensor is given by  $\underline{\boldsymbol{\sigma}}_\nu$ , with elements

$$\sigma_{\nu ij} = \rho\nu \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3}\delta_{ij} \frac{\partial u_k}{\partial x_k} \right) + \zeta\delta_{ij} \frac{\partial u_k}{\partial x_k}, \quad (5.7)$$

in which  $y$  and  $z$  are  $x_2$  and  $x_3$ , respectively, one sums over repeated indices, and  $\delta_{ij}$  is the Kronecker delta function. Here the kinematic viscosity is  $\nu$  and the second coefficient of viscosity, which is not important here, is  $\delta$ . The term involving the viscosity tensor simplifies considerably (and the term involving  $\zeta$  vanishes), because  $\partial u_i/\partial x_i = 0$  from incompressibility. Also, it is consistent with our assumptions that the only nonzero derivative of  $\nu$  is  $d\nu/dz (= d\nu/dx_3)$ . With these observations and assumptions, one has for the  $k$  component of  $\nabla \cdot \underline{\sigma}_\nu$ ,

$$\nabla \cdot \underline{\sigma}_\nu)_k = \rho\nu \nabla^2 u_k + \frac{\partial(\rho\nu)}{\partial x_3} \left( \frac{\partial u_k}{\partial x_3} + \frac{\partial u_3}{\partial x_k} \right). \quad (5.8)$$

There are three very distinct directions in this problem, which are the direction of gravity, the direction of the wavevector of a surface modulation, and the direction perpendicular to these two. We will assume throughout that the mean surface is perpendicular to the pressure gradient. We further assume the surface modulations to be plane waves, expecting to express any actual surface modulation as a sum over all the possible plane waves, which form a complete basis set. Our goal is to find the evolution of an arbitrary plane wave, assuming that it grows in time from a very small initial amplitude. We define  $\mathbf{u}$  as  $\mathbf{u} = (u, v, w)$ , which allows us to write the components of (5.6), the incompressibility condition, and (5.5) as

$$\rho \frac{\partial u}{\partial t} = -\frac{\partial}{\partial x} \delta p + \rho\nu \nabla^2 u + \frac{\partial(\rho\nu)}{\partial z} \left( \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right), \quad (5.9)$$

$$\rho \frac{\partial v}{\partial t} = -\frac{\partial}{\partial y} \delta p + \rho\nu \nabla^2 v + \frac{\partial(\rho\nu)}{\partial z} \left( \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right), \quad (5.10)$$

$$\rho \frac{\partial w}{\partial t} = -\frac{\partial}{\partial z} \delta p + \rho\nu \nabla^2 w + \frac{\partial(\rho\nu)}{\partial z} \left( 2\frac{\partial w}{\partial z} \right) - g\delta\rho, \quad (5.11)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \text{ and} \quad (5.12)$$

$$\frac{\partial}{\partial t} \delta\rho = -w \frac{\partial\rho}{\partial z}. \quad (5.13)$$

This is the set of equations that describes the linear phase of the RT instability, including effects of viscosity or density gradients. We look for waves that represent growing modulations of the surface, and thus in general will have amplitudes with an unknown variation in  $z$  but proportional to  $\exp(ik_x x + ik_y y + nt)$  in  $x, y$ , and time  $t$ . Here  $k_x$  and  $k_y$  are the  $x$  and  $y$  components of the wavevector (which we could have chosen to lie along one of these axes) and  $n$  is the exponential growth rate. With these substitutions, we get a new equation set:

$$\rho n u = -ik_x \delta p + \rho \nu \left( \frac{\partial^2}{\partial z^2} - k^2 \right) u + \frac{\partial(\rho \nu)}{\partial z} \left( \frac{\partial u}{\partial z} + ik_x w \right), \quad (5.14)$$

$$\rho n v = -ik_y \delta p + \rho \nu \left( \frac{\partial^2}{\partial z^2} - k^2 \right) v + \frac{\partial(\rho \nu)}{\partial z} \left( \frac{\partial v}{\partial z} + ik_y w \right), \quad (5.15)$$

$$\rho n w = -\frac{\partial}{\partial z} \delta p + \rho \nu \left( \frac{\partial^2}{\partial z^2} - k^2 \right) w + \frac{\partial(\rho \nu)}{\partial z} \left( 2 \frac{\partial w}{\partial z} \right) - g \delta \rho, \quad (5.16)$$

$$ik_x u + ik_y v = -\frac{\partial w}{\partial z}, \quad \text{and} \quad (5.17)$$

$$n \delta \rho = -w \frac{\partial \rho}{\partial z}. \quad (5.18)$$

in which  $k^2 = k_x^2 + k_y^2$ .

We can reduce these equations from five to two through the following steps. Multiply the first by  $-ik_x$  and the second by  $-ik_y$ , then add the first two equations and use the fourth equation to simplify them. Also use the fifth equation to eliminate  $\delta \rho$  from the third equation. This gives us

$$\rho n \frac{\partial w}{\partial z} = -k^2 \delta p + \rho \nu \left( \frac{\partial^2}{\partial z^2} - k^2 \right) \frac{\partial w}{\partial z} + \frac{\partial(\rho \nu)}{\partial z} \left( \frac{\partial^2}{\partial z^2} + k^2 \right) w \quad \text{and} \quad (5.19)$$

$$\rho n w = -\frac{\partial}{\partial z} \delta p + \rho \nu \left( \frac{\partial^2}{\partial z^2} - k^2 \right) w + \frac{\partial(\rho \nu)}{\partial z} \left( 2 \frac{\partial w}{\partial z} \right) + w \frac{g}{n} \frac{\partial \rho}{\partial z}. \quad (5.20)$$

Recalling that  $\nu(z)$  and  $\rho(z)$  are given as properties of the unperturbed system, we can see that we need only to eliminate  $\delta p$  to have an equation for  $w$  in terms of known parameters. Doing this, we obtain

$$\begin{aligned} & \frac{\partial}{\partial z} \left[ -\rho n \frac{\partial w}{\partial z} + \rho \nu \left( \frac{\partial^2}{\partial z^2} - k^2 \right) \frac{\partial w}{\partial z} + \frac{\partial(\rho \nu)}{\partial z} \left( \frac{\partial^2}{\partial z^2} + k^2 \right) w \right] \\ & = k^2 \left[ -\rho n w + \rho \nu \left( \frac{\partial^2}{\partial z^2} - k^2 \right) w + \frac{\partial(\rho \nu)}{\partial z} \left( 2 \frac{\partial w}{\partial z} \right) + w \frac{g}{n} \frac{\partial \rho}{\partial z} \right]. \end{aligned} \quad (5.21)$$

This equation, along with the previous ones and boundary conditions, provides the tools we need to investigate RT growth rates in the linear regime. They apply within the fluid on each side of an interface.

The boundary conditions play an essential role in these instability calculations, justifying some effort to develop them. (In contrast, in a linear theory we are not particularly concerned with initial conditions.) A first obvious condition is that the fluids must remain in contact. This requires that  $w$  be continuous across the interface. We can integrate or subtract the fundamental equations across a boundary to find additional conditions. The integration is similar to the analysis we did in Chap. 4 when considering shock transitions, although the functions involved are more complex. It is worth pursuing now, as we will need to use the results.

Consider the interface to be at  $z = 0$ , with regions designated by the subscript 2 at  $z > 0$  and by the subscript 1 at  $z < 0$ . Suppose that  $q(z)$  is an arbitrary function that is continuous and differentiable everywhere. Suppose  $f(z)$  and  $h(z)$  are arbitrary functions that are continuous and differentiable everywhere except at the interface, so we can write, for example,  $f(z) = f_1(z)H(-z) + f_2(z)H(z)$ , in which  $H(z)$  is a Heavyside step function, equal to zero for  $z < 0$  and to 1 for  $z > 0$ , and  $f_1$  and  $f_2$  are continuous, differentiable functions. The derivative of  $f(z)$ ,  $g(z) = df/dz$ , can be written  $g(z) = g_1(z)H(-z) + g_2(z)H(z) + \Delta f \delta(z)$ , in which  $\delta(z)$  is the Dirac delta function,  $g_1$  and  $g_2$  are continuous, differentiable functions, and  $\Delta f = f_2(0) - f_1(0)$ . We then take the limit of the integral over a small region about the interface, as the width of the region goes to zero. Evidently this will give zero unless the argument of the integrand includes a delta function. Specifically

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} \int_{-\epsilon}^{\epsilon} f(z) dz &= \lim_{\epsilon \rightarrow 0} [\epsilon f_2(\epsilon/2) - \epsilon f_1(\epsilon/2)] = 0, \\ \lim_{\epsilon \rightarrow 0} \int_{-\epsilon}^{\epsilon} (\partial f(z)/\partial z) dz &= f_2(0) - f_1(0), \\ \lim_{\epsilon \rightarrow 0} \int_{-\epsilon}^{\epsilon} (\partial^2 q(z)/\partial z^2) dz &= \lim_{\epsilon \rightarrow 0} \left[ \frac{\partial q_2}{\partial z} - \frac{\partial q_1}{\partial z} \right] = \left[ \frac{\partial q_2}{\partial z} - \frac{\partial q_1}{\partial z} \right]_{z=0}, \\ \lim_{\epsilon \rightarrow 0} \int_{-\epsilon}^{\epsilon} q(z) \frac{\partial f(z)}{\partial z} dz &= q_s (f_2 - f_1)_{z=0} + \lim_{\epsilon \rightarrow 0} \int_{-\epsilon}^{\epsilon} \epsilon \frac{\partial q}{\partial z} \frac{\partial f}{\partial z} dz \\ &= q_s (f_2 - f_1)_{z=0}, \\ \lim_{\epsilon \rightarrow 0} \int_{-\epsilon}^{\epsilon} f(z) \frac{\partial q(z)}{\partial z} dz &= q_s (f_2 - f_1)_{z=0} - \lim_{\epsilon \rightarrow 0} \int_{-\epsilon}^{\epsilon} q(z) \frac{\partial f}{\partial z} dz = 0, \text{ and} \\ \lim_{\epsilon \rightarrow 0} \int_{-\epsilon}^{\epsilon} \frac{\partial f(z)}{\partial z} h(z) dz &= \lim_{\epsilon \rightarrow 0} \int_{-\epsilon}^{\epsilon} \Delta f \delta(z) (h_1(z)H(-z) + h_2(z)H(z)) dz \\ &= (f_2(0) - f_1(0)) \left( \frac{h_1(0) + h_2(0)}{2} \right). \end{aligned} \tag{5.22}$$

Here the subscript  $z = 0$  indicates that the quantities should be evaluated as the interface is approached. Just as in the case of shock waves, the interface may be treated as a mathematical discontinuity, although in microscopic reality all physical quantities and their derivatives vary continuously across the interface.

**Homework 5.2**

The final relation in (5.22) is significant for our specific application, in which one needs to integrate, across an interface, equations that contain discontinuous quantities along with derivatives of discontinuous quantities. By treating the delta function and the step function as limits of appropriate functions (see a mathematical methods book), prove this relation.

Applying the relations of (5.22) to (5.19) and (5.20), realizing that  $w$  is continuous across the interface and that all derivatives of  $w$  in  $z$  are continuous and bounded as one approaches the interface, we find

$$0 = \left[ \rho_2 \nu_2 \left( \frac{\partial^2}{\partial z^2} + k^2 \right) w_2 - \rho_1 \nu_1 \left( \frac{\partial^2}{\partial z^2} + k^2 \right) w_1 \right]_{z=0} \quad \text{and} \quad (5.23)$$

$$0 = -(\delta p_2 - \delta p_1)_{z=0} + \left[ 2(\rho_2 \nu_2 - \rho_1 \nu_1) \frac{\partial w_1}{\partial z} \right]_{z=0} + w_o \frac{g}{n} (\rho_2 - \rho_1)_{z=0}, \quad (5.24)$$

in which we have used  $w_o$  to designate the common value of  $w$  at the interface. The second of these equations in particular is essential to the analysis of RT, but this requires that we eliminate the pressure difference to obtain an equation involving only one unknown function. We can obtain another condition for the pressure, by subtracting (5.19) from itself across the boundary to find

$$\begin{aligned} & k^2(\delta p_2 - \delta p_1) \\ &= \rho_2 \left[ -n + \nu_2 \left( \frac{\partial^2}{\partial z^2} - k^2 \right) \right] \frac{\partial w_2}{\partial z} + \rho_1 \left[ n - \nu_1 \left( \frac{\partial^2}{\partial z^2} - k^2 \right) \right] \frac{\partial w_1}{\partial z} \\ & \quad + \frac{\partial(\rho_2 \nu_2)}{\partial z} \left( \frac{\partial^2}{\partial z^2} + k^2 \right) w_2 - \frac{\partial(\rho_1 \nu_1)}{\partial z} \left( \frac{\partial^2}{\partial z^2} + k^2 \right) w_1, \end{aligned} \quad (5.25)$$

and then combining this with (5.24) to eliminate the pressure. (We henceforth drop the notation “ $z = 0$ ”, realizing that in such boundary conditions all quantities are evaluated as  $z$  approaches the boundary from within the fluid designated by the subscript.) After using (5.23) to eliminate two terms, this gives a usable, if complex, boundary condition,

$$\begin{aligned} & w_o k^2 \frac{g}{n} (\rho_2 - \rho_1) + k^2 \left[ 2(\rho_2 \nu_2 - \rho_1 \nu_1) \frac{\partial w_1}{\partial z} \right] \\ &= \rho_2 \left[ -n + \nu_2 \left( \frac{\partial^2}{\partial z^2} - k^2 \right) \right] \frac{\partial w_2}{\partial z} - \rho_1 \left[ -n + \nu_1 \left( \frac{\partial^2}{\partial z^2} - k^2 \right) \right] \frac{\partial w_1}{\partial z}. \end{aligned} \quad (5.26)$$

For problems in which we include finite viscosity, a final boundary condition is that the first derivative of  $w$ ,  $\partial w / \partial z$ , must be continuous across the interface, as must the other derivatives of  $\mathbf{u}$ . The microscopic interactions of the particles that create viscosity assure this. (In contrast, when we neglect

viscosity in a fluid model, we allow locally discontinuous solutions like those we found for shocks.)

In many hydrodynamic circumstances the pressure is continuous across an interface. This is certainly the case across interfaces in steady motion, but it is not necessarily the case in unstable, accelerating systems, as (5.24) shows. However, in this case the  $xz$  and  $yz$  components of the viscous stress tensor are continuous. One has

$$\begin{aligned}\sigma_{xz} &= \rho\nu \left( \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right) = \rho\nu \left( \frac{\partial u}{\partial z} + ik_x w \right) \\ \sigma_{yz} &= \rho\nu \left( \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right) = \rho\nu \left( \frac{\partial v}{\partial z} + ik_y w \right), \text{ which implies}\end{aligned}\tag{5.27}$$

$$ik_x \sigma_{xz} + ik_y \sigma_{yz} = \rho\nu \left[ \frac{\partial}{\partial z} (ik_x u + ik_y v) - k^2 w \right] = -\rho\nu \left( \frac{\partial^2}{\partial z^2} + k^2 \right) w.\tag{5.28}$$

This quantity is continuous across the interface, as one can also see from (5.23).

Thus, the boundary conditions we have to work with are (5.23), (5.26), the continuity of  $w$  across the boundary, any additional constraints imposed by the geometry of the problem, and for viscous flows the continuity of  $\partial w/\partial z$  across the boundary.

## 5.2 Applications of the Linear Theory of the Rayleigh–Taylor Instability

At this point we have the tools we need to address various cases. We now proceed to consider three basic applications of the theory developed in the previous section. We begin with the simple case of an interface separating two fluids of different density. We then discuss the effects of viscosity on the instability. This is important for example in the atmosphere. It seems that it might not matter for high-energy-density systems, which have large Reynolds number (see Chap. 2). However, viscosity can play a role in such systems in altering the growth of small-scale structures. After that, we turn to the impact of density gradients, which are important in many applications.

### 5.2.1 Rayleigh–Taylor Instability with Two Uniform Fluids

The simplest case is that of two uniform fluids with a boundary at  $z = 0$  and with no viscosity. Equation (5.21) then becomes

$$\frac{\partial}{\partial z} \left[ -\rho n \frac{\partial w}{\partial z} \right] = k^2 \left[ -\rho n w + w \frac{g}{n} \frac{\partial \rho}{\partial z} \right],\tag{5.29}$$

in which  $\partial \rho/\partial z$  also equals zero for the uniform fluids. This then simplifies for uniform density in each region to

$$\frac{\partial^2 w}{\partial z^2} = k^2 w. \quad (5.30)$$

Since the fluid must be undisturbed at sufficiently large distances, so as  $w \rightarrow \pm\infty$ , the solutions are

$$\begin{aligned} w_1 &= w_o e^{kz} & \text{for } z < 0 \\ w_2 &= w_o e^{-kz} & \text{for } z > 0, \end{aligned} \quad (5.31)$$

where  $w_o$  is the same in both solutions because  $w$  must be continuous at the interface (to avoid the creating of voids or the accumulation of matter). Here we have defined fluid 1 as the region below the interface and fluid 2 as the region above it, by using subscripts on  $w$ . Our primary differential equation has thus given us the profiles but not the growth rate. To find this, we use the boundary condition (5.26) to find

$$w_o \frac{g}{n} (\rho_2 - \rho_1) = \frac{n}{k^2} (\rho_2 + \rho_1) k w_o, \text{ from which} \quad (5.32)$$

$$n_o = \sqrt{\frac{\rho_2 - \rho_1}{\rho_2 + \rho_1} k g} = \sqrt{A_n k g} \quad (5.33)$$

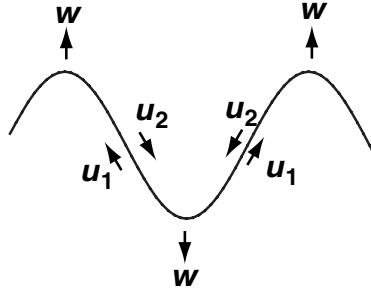
in which we have labeled the growth rate for this case as  $n_o$  and defined the Atwood number,  $A_n = (\rho_2 - \rho_1)/(\rho_2 + \rho_1)$ , which varies from  $-1$  to  $1$  and measures how strong the density jump is at an interface. When  $A_n$  is negative, meaning that the denser fluid is already “below“ the less dense one, then in the simple limit of (5.33)  $n_o$  is purely imaginary and the modulations oscillate but do not grow. (If we included finite viscosity such modulations would damp, as is discussed at length in Chandrasekhar.) Equation (5.33) gives the simplest result for the RT growth rate, and for this reason is often referred to as the “classical“ RT growth rate. (Thus continuing the flagrant abuse of the term “classical“ throughout physics.) This growth rate  $n_o$  provides a reference for the growth in more-complicated systems. Adding complications tends to reduce this growth rate below  $n_o$ .

### Homework 5.3

Find the solution for the velocity profiles and the growth rate for the RT instability for two uniform, constant density fluids that are confined by two planar surfaces each a distance  $d$  from the interface, which is accelerated at constant  $g$ .

Before considering complications, it is worthwhile to point out how RT inherently provides circumstances that may lead to further instabilities. Suppose that the wavevector points in the  $x$  direction, so that  $k = k_x$ . Then in the light the solution given by (5.31), (5.17) implies  $iu_1 = w_o$  and  $u_2 = -u_1$ . The first of these relations implies that  $u$  and  $w$  are out of phase spatially. The second implies that there is shear flow across the interface. Figure 5.2





**Fig. 5.2.** Shear flow induced by Rayleigh-Taylor. The arrows show the location and direction of the maxima in the velocity perpendicular to the interface and along the interface. The amplitude shown is nonlinearly large. In the linear limit,  $\mathbf{u}$  is horizontal to first order

illustrates this. Material must flow along the interface to provide the mass that penetrates across the original interface. Correspondingly, the material must flow in opposite directions on the two sides of the interface. This shear flow provides the potential for growth of the Kelvin-Helmholtz instability, discussed later in the chapter.

### 5.2.2 Effects of Viscosity on the Rayleigh-Taylor Instability

As a first example of a complication that reduces the RT growth rate, consider the effects of viscosity. (We leave to the specialized literature the effects of mass diffusion due to binary collisions, which complement viscosity and further reduce the RT growth rate.) As a preliminary exploration, let us assume that the viscosities on the two sides of the interface are nonzero, and that the densities on the two sides of the interface are different in magnitude, but that the densities and viscosities are both otherwise uniform in space. In this case, (5.21) becomes

$$\left[ -n + \nu \left( \frac{\partial^2}{\partial z^2} - k^2 \right) \right] \frac{\partial^2 w}{\partial z^2} = \left[ -nw + \nu \left( \frac{\partial^2}{\partial z^2} - k^2 \right) w \right] k^2. \quad (5.34)$$

This equation has a general solution,

$$w(z) = C_1 e^{s_1 z} + C_2 e^{-s_2 z} + C_3 e^{kz} + C_4 e^{-kz} \quad (5.35)$$

in which  $s_i = k\sqrt{1 + n/(k^2\nu_i)}$  with  $i$  being 1 or 2. To assure that  $w$  vanishes at  $\pm\infty$ , it is clear that  $C_1$  and  $C_3$  are zero for  $z > 0$  and  $C_2$  and  $C_4$  are zero for  $z < 0$ . Note that this choice regarding how to satisfy the specific boundary conditions used here implies that the real parts of  $k$ ,  $s_1$ , and  $s_2$  are all positive. This will be an important constraint as we develop our solution. We have four boundary conditions. These are the continuity of  $w(z)$ , the

continuity of  $\partial w/\partial z$ , (5.23) and (5.26). These with the definition of  $s_i$  give us

$$C_1 - C_2 + C_3 - C_4 = 0, \quad (5.36)$$

$$s_1 C_1 + s_2 C_2 + k C_3 + k C_4 = 0, \quad (5.37)$$

$$n(\rho_2 C_2 - \rho_1 C_1) + \rho_2 \nu_2 k^2 (2C_2 + 2C_4) - \rho_1 \nu_1 k^2 (2C_1 + 2C_3) = 0, \text{ and } (5.38)$$

$$0 = \frac{2A_n g k^2}{n} (C_1 + C_3) - k n (C_3(1 - A_n) + C_4(1 + A_n)) - 2k^2 (k C_3 + s_1 C_1)(\nu_1 - \nu_2) + 2A_n k^2 (k C_3 + s_1 C_1)(\nu_1 + \nu_2). \quad (5.39)$$

Since we have four equations that are linear in the four amplitudes, we can write (5.36) to (5.39) as an equation in which a matrix  $\mathbf{M}$  multiplies the vector  $(C_1, C_2, C_3, C_4)$ . Then the determinant of  $\mathbf{M}$  must be zero, which gives us the following general dispersion relation for this case:

$$0 = n^2 [2A_n^2 k - (s_2 + s_1) + A_n(s_2 - s_1)] + 2k^2 n [(s_2 - s_1) + A_n(2k - (s_2 + s_1))] [\nu_2(1 + A_n) - \nu_1(1 - A_n)] + 2 [k^5 - k^4(s_2 + s_1) + k^3 s_1 s_2] [\nu_2(1 + A_n) - \nu_1(1 - A_n)]^2 + A_n g k [s_2(1 - A_n) + s_1(1 + A_n) - 2k]. \quad (5.40)$$

For further discussion here, we will specialize to the case  $\nu_1 = \nu_2$  and thus  $s_1 = s_2$ , obtaining

$$0 = 2n^2 (A_n^2 k - s) + 8n A_n^2 k^2 \nu (k - s) + 8A_n^2 k^3 \nu^2 (k - s)^2 - 2A_n g k (k - s). \quad (5.41)$$

This equation is deceptively simple, as  $s$  depends on  $n$ . We could solve for either variable, but it is most useful to solve for  $s$  because  $s$  is constrained to have a positive real part. Substituting for  $n$  in (5.41), we obtain a fifth-order polynomial for  $s$ ,

$$0 = -s^5 \nu^2 + s^4 k \nu^2 A_n^2 + 2s^3 k^2 \nu^2 (1 - 2A_n^2) + s^2 k^3 \nu^2 (6A_n^2) + s(A_n g k - k^4 \nu^2 (1 + 4A_n^2)) - A_n g k^2 + A_n^2 k^5 \nu^2. \quad (5.42)$$

By taking the limit as the viscosity vanishes, one can see from (5.42) that the growth rate goes to  $\sqrt{A_n g k}$ , as it should. While one can solve (5.42) straightforwardly with a computational mathematics program, it is more useful to first cast it in a nondimensional form. If one compares the terms in the coefficient of  $s$ , it is clear that  $g$  corresponds to  $k^3 \nu^2$ , suggesting that one uses a normalized wavenumber  $\tilde{k} = k/(g/\nu^2)^{1/3}$ . It is worth noting that we can write this as  $\tilde{k} = [(k^2 \nu)/\sqrt{gk}]^{2/3}$ , so one sees that  $\tilde{k}$  depends on the competition between diffusion and growth. Specifically, for a spatial scale of  $1/k$ ,  $\tilde{k}$  is the  $2/3$  power of the ratio of the rate of viscous diffusion to the fundamental RT growth rate  $\sqrt{k g}$ . Comparing the third term on the right-hand side with the sixth term, one sees that  $s^3 \nu^2$  here corresponds to  $g$ , suggesting

that one uses a normalized value  $\tilde{s} = s/(g/\nu^2)^{1/3}$ . The corresponding normalization of the growth rate is  $\tilde{n} = n/(g^2/\nu)^{1/3}$ . This too can be written more intuitively, as  $\tilde{n} = (n/\sqrt{k}g)(k^2\nu/\sqrt{k}g)^{1/3}$ , which depends on the growth rate per unit fundamental RT growth rate times  $\sqrt{k}$ . With these normalizations, the zero-viscosity growth rate is  $\tilde{n} = \sqrt{A_n\tilde{k}}$ , and the dispersion relation becomes

$$0 = -\tilde{s}^5 + \tilde{s}^4 A_n^2 \tilde{k} + 2\tilde{s}^3 \tilde{k}^2 (1 - 2A_n^2) + 6\tilde{s}^2 A_n^2 \tilde{k}^3 + \tilde{s} \tilde{k} [A_n - \tilde{k}^3 (1 + 4A_n^2)] - A_n \tilde{k}^2 + A_n^2 \tilde{k}^5. \quad (5.43)$$

This equation provides a universal relation between the normalized growth rate and the normalized wavenumber, depending only on the value of the Atwood number. Any root of this equation (for  $\tilde{s}$ ), whose real part is positive, corresponds to a physical mode, but this mode is only exponentially growing if  $\Re(s) > k$ . Otherwise the mode is damped. Any roots with nonzero imaginary parts would correspond to oscillating modes, which might in principle be growing or damped.

Figure 5.3 shows the non-trivial solutions of (5.43) for  $\tilde{s}$ . One of the roots has a positive real part that always exceeds one. This is the exponentially growing mode. Two of the roots always have negative real parts, and so never correspond to solutions of this problem. These two roots also have imaginary parts; the other two are purely real. The final root is negative at small  $\tilde{k}$  (small viscosity) but becomes a damped mode as  $\tilde{k}$  increases.

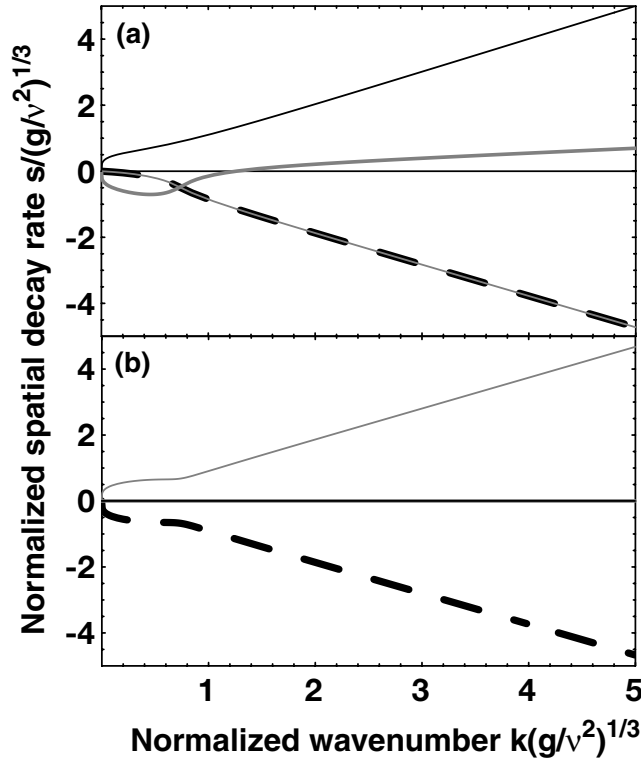
Figure 5.4 shows the corresponding values of the normalized growth rate  $\tilde{n}$ , for the root corresponding to an exponentially growing mode, for two values of  $A_n$ . The roots shown in Fig. 5.4 are readily obtained from a computational mathematics program, but are not algebraically simple. However, it turns out that a simpler equation captures much of the behavior with high accuracy except at very small  $A_n$ . The physical basis for this is that the growth will approach the zero-viscosity value when viscosity is small and that viscous effects will dominate at high viscosity, so that a solution that joins these two regimes may work well even through the transition between them. To obtain such a solution, one can replace  $s$  in (5.41) with the value from an expansion for high viscosity,  $s \sim k[1 + n/(2k^2\nu)]$ . Making this substitution and solving the resulting equation produces the much simpler dispersion relation

$$n = \sqrt{A_n k g + k^4 \nu^2} - k^2 \nu, \quad (5.44)$$

which in our dimensionless units becomes

$$\tilde{n} = \sqrt{A_n \tilde{k} + \tilde{k}^4} - \tilde{k}^2. \quad (5.45)$$

This solution, originally developed by Hurt, Harlow, and Duff, is shown as the dashed gray curves in Fig. 5.4.



**Fig. 5.3.** Solutions for the spatial decay rate  $s$  for the Rayleigh–Taylor instability with viscosity, for  $A_n = 0.5$ . The real and imaginary part of each root are shown using the same curve type in both (a) and (b). (a) Real parts. Modes with positive real parts are physical solutions to the problem considered here. (b) Imaginary parts

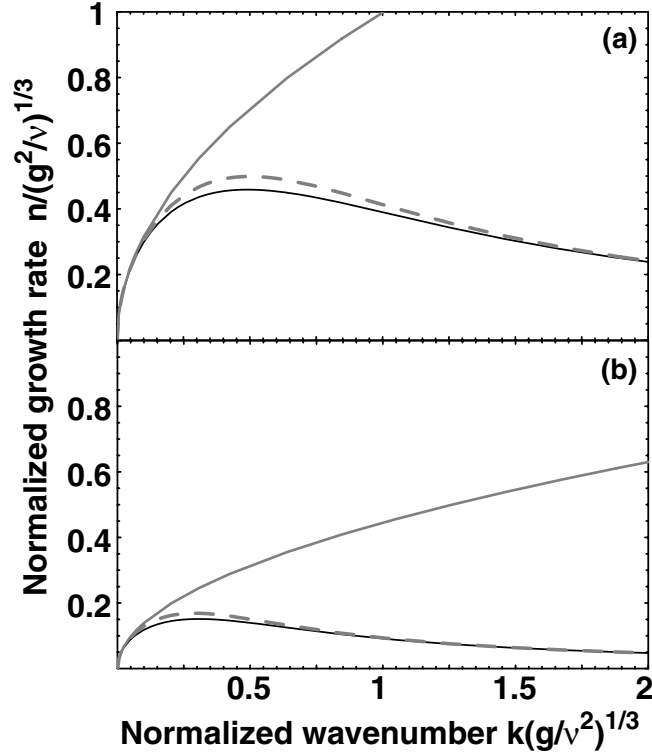
**Homework 5.4**

The discussion above (5.43) shows that  $\tilde{n} = (n/\sqrt{kg})\sqrt{\tilde{k}}$ . This would suggest that it might make more sense to separate the meaning of the axes more cleanly by using  $\tilde{\delta} = (n/\sqrt{kg})$  and  $\tilde{k} = [(k^2\nu)/\sqrt{gk}]^{2/3}$  as the two variables. Recast this equation in terms of these new variables, solve it, and plot the real roots from  $\tilde{k} = 0$  to 2. Discuss the results and compare them to  $n = \sqrt{A_n g k}$ .

**Homework 5.5**

Derive (5.44) and (5.45) from (5.41). Comment on the nature of the terms that have been dropped.

It is no surprise that the wavenumber of the mode with the highest growth rate has a normalized wavenumber that is some fraction of unity. The largest growth occurs at wavenumbers just smaller than those for which viscosity



**Fig. 5.4.** Rayleigh-Taylor instability with viscosity. The black curves show the actual growth rates. The solid gray curves show the zero-viscosity result,  $n = \sqrt{A_n k g}$ . The solid gray curves shows the approximation of (5.45). The panels show (a)  $A_n = 1$  and (b)  $A_n = 0.2$

begins to substantially reduce the growth. The wavenumber of maximum growth is approximately that given by (5.44) and (5.45), from which  $\tilde{k} = A_n^{1/3}/2$  or

$$k = \frac{1}{2} \left( \frac{A_n g}{\nu^2} \right)^{1/3} \quad (5.46)$$

at the maximum. The magnitude of the growth rate at this wavenumber is

$$n = \frac{3}{4} \sqrt{\frac{(A_n g)^{4/3}}{\nu^{2/3}}} - \frac{(A_n g)^{2/3}}{4\nu^{1/3}}. \quad (5.47)$$

Another observation from (5.44) and (5.45) is that, although the effect of viscosity is to reduce the growth rate, viscosity alone can never reduce it to zero. This makes physical sense, because while the viscosity can resist the flow of fluid and turn some kinetic energy into heat, the system will still seek its minimum-potential-energy state. Turning to real numbers,

the viscosity is given in (2.40), and in a very rough estimate has a typical value  $\sim 0.01 \text{ cm}^2/\text{s}$  in high-energy-density experiments. A characteristic value for  $g$  might be  $(100 \text{ km/s})/(10 \text{ ns}) = 10^{15} \text{ cm}^2/\text{s}$ . With these assumptions, the wavenumber of maximum growth for an RT mode, already well below  $\sqrt{A_n k g}$ , from (5.46) with  $A_n = 0.5$ , is of order  $10^6 \text{ cm}^{-1}$ , so the wavelength is of order  $0.1 \text{ }\mu\text{m}$ . In some experiments this wavelength can be larger, of order  $1 \text{ }\mu\text{m}$ . Wavelengths shorter than this will experience greatly reduced RT growth. This will definitely limit the ability of RT and related mechanisms to produce short-wavelength turbulence. In an astrophysical context, one might have  $\nu \sim 10^{20} \text{ cm}^2/\text{s}$  and  $g$  might be  $(100 \text{ km/s})/(100 \text{ years}) \sim 0.003 \text{ cm}^2/\text{s}$ . In this case, the maximum growth occurs for a wavenumber of  $10^{-15} \text{ cm}^{-1}$ , or a wavelength of order  $10^{16} \text{ cm} \sim 0.01 \text{ light years}$ . Wavelengths much shorter than this would be in the high-viscosity regime and experience reduced growth.

### 5.2.3 Rayleigh–Taylor with Density Gradients and the Global Mode

Some interfaces are abrupt, and one can design experiments to create abrupt interfaces at least initially. However, the RT instability occurs in many situations that have a gradual interface. Indeed, sometimes there is no “interface” as such but merely an extended density gradient that opposes a pressure gradient. This is the case, for example, in supernovae. So it is worthwhile to explore the effects of a density gradient on this instability. Given that the effects of viscosity are typically small, it is sensible to set  $\nu = 0$  for this calculation. In this case our basic differential equation (5.21) becomes

$$k^2 \left( \rho n - \frac{g}{n} \frac{\partial \rho}{\partial z} \right) w - \frac{\partial}{\partial z} \left( n \rho \frac{\partial w}{\partial z} \right) = 0. \quad (5.48)$$

Before proceeding to specific cases, it is worthwhile to observe that this equation has the solution  $w = w_0 e^{-kz}$  for an arbitrary density profile, corresponding to a growth rate of  $n = \sqrt{kg}$ . This mode is known as the *global Rayleigh–Taylor mode*, and this growth rate is the largest RT growth rate that exists (see Bychkov et al.). One might think that this is the end of the story for RT in density profiles. However, this mode does not always exist because it may not satisfy the boundary conditions. On the one hand, whenever a high-pressure region of negligible density is either accelerating or decelerating a fluid layer of some thickness  $L$ , the fluid layer will be unstable to the global RT mode for modes with  $kL \gg 1$ . The maximum amplitude of these modes will be at the free surface where the high pressure is located. This mode can also be viewed as a generalization of the mode we found in Sect. 5.2.2 to an arbitrary density profile and to  $A_n = 1$ . On the other hand, if there is a nonnegligible density on both sides of the interface, then the boundary conditions do not allow the global mode. We consider next such a case

in which the instability develops somewhere on an extended and continuous density profile.

We will assume, as a sensible general case, that the density is exponentially distributed, so that  $\rho(z) = \rho_0 e^{z/L}$ . Thus the density increases with “height“, defined as the direction opposite the acceleration  $g$  in the frame of reference of the interface. Thus  $\rho'(z) = \rho_0/L$ . Substituting for  $\rho = \rho(z)$ , (5.48) has the solution

$$w = C_1 \exp \left[ \left( \sqrt{1 + 4k^2 L^2 - 4gk^2 L/n^2} - 1 \right) \frac{z}{2L} \right] + C_2 \exp \left[ - \left( \sqrt{1 + 4k^2 L^2 - 4gk^2 L/n^2} + 1 \right) \frac{z}{2L} \right], \quad (5.49)$$

with two constants  $C_1$  and  $C_2$ . Here again these constants respond to the boundary conditions. If, for example, the unstable zone is confined between two boundaries, as can happen in the Earth’s atmosphere, then one would need the amplitude to be zero at these boundaries (though for a linear theory this would be relevant only to wavelengths of order the distance between boundaries). Such close boundaries are less common in the systems of interest to us, so we will take  $C_1 = 0$  for  $z < 0$  and  $C_2 = 0$  for  $z > 0$ , in order to assure that the perturbation dies out with distance. Then the remaining constants are of equal magnitude to keep  $w$  continuous at the interface.

It is worth focusing on the fact that the notion of an interface is somewhat artificial in a continuous density profile. The instability might develop at any location in the profile. The largest fluctuations in the profile, wherever they may be, will produce large modulations first. Figure 5.5 illustrates the impact of a single mode in such an environment. The mode is strongest at some location (about halfway up the figure) and results in the flow of material both laterally and vertically. Matter flows into the downward moving spikes and upward moving bubbles. In reality, instabilities are likely to be seeded throughout the profile, and the entire unstable region is likely to become very clumpy.

Indeed, the localized modes with  $\mathbf{k} \perp \nabla p$  that we consider here are a subset of the possible modes. In a continuous profile the direction of  $\mathbf{k}$  (and thus the “surface“ considered) are not restricted to lie in the plane perpendicular to  $\nabla p$ , but for a given magnitude of  $k$  this direction will correspond to the direction of largest growth. In addition, one can find plane-wave solutions to (5.48) in which the growth rate has both real and complex parts. These correspond to modes that grow while they propagate. Their growth rate is somewhat smaller than that of the global RT mode.

### Homework 5.6

Find the plane-wave solutions in  $x, y$  and  $z$  to (5.48) and discuss their behavior.



**Fig. 5.5.** A Rayleigh-Taylor mode on a density gradient. This grayscale representation shows a vertically exponential density profile in which there is a density perturbation due to a single mode

Returning to a localized RT mode in a continuous density profile, we find the RT growth rate by applying the boundary condition (5.26), noting that at the chosen interface  $\rho_1 = \rho_2 = \rho_o$ , so the term involving  $g$  in this equation drops out. Some simple algebra then gives an equation for the growth rate,

$$0 = n^2(1 + 4k^2L^2) - 4gk^2L, \quad (5.50)$$

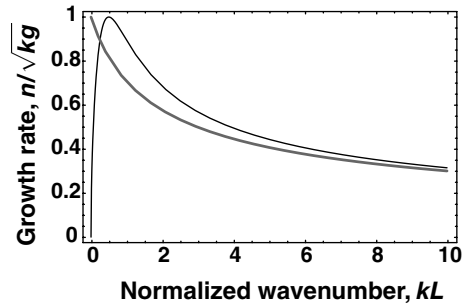
with the obvious solution for the growing mode

$$n = \sqrt{gk} \sqrt{\frac{4kL}{1 + 4k^2L^2}}. \quad (5.51)$$

The normalized growth rate,  $n/\sqrt{kg}$ , is plotted against  $kL$  in Fig. 5.6. The growth rate reaches the value for the global mode at  $kL = 1/2$ , corresponding to a wavelength about ten times the density scale length. (For comparison, Fig. 5.5 shows a mode whose wavelength is about 30% of the density scale length, so  $kL \sim 20$ ). The normalized growth rate is finite but rapidly decreasing at small  $kL$ , becoming proportional to  $\sqrt{2kL}$ . If one thinks about a steadily increasing wavelength in Fig. 5.5, one can see that more mass has to flow over longer distances as the wavelength increases. On the other hand, at large enough  $kL$  the growth rate goes to  $\sqrt{g/L}$ , losing all dependence on  $k$ . This is thought to be the relevant limit for many cases in astrophysics. If we write  $g = |\nabla p|/\rho = p/(\rho L_p)$ , where  $L_p$  is the scale length of the pressure profile, then the growth rate takes a form familiar to astrophysics, becoming  $(c_s^2/\gamma)/\sqrt{LL_p}$ . (There is also a contribution to growth from the pressure gradient alone in this limit, which we discuss in Sect. 5.3.)

There are cases in which a density gradient exists and may even be exponential, but only over a limited range of densities. A prime example is found in inertial fusion, at the inner surface of the fusion capsule. Unstable wavelengths that are small compared to the density scale length are affected by the density profile, while wavelengths that are long compared to the density





**Fig. 5.6.** Impact of density gradient on Rayleigh-Taylor growth rate. The black curve shows the result from (5.51). The gray curve shows the approximate relation,  $\sqrt{1/(1+kL)}$ , discussed in the text

scale length tend to respond to the densities at the upper and lower boundaries, behaving as though the interface is abrupt. The gray curve in Fig. 5.6, showing the function  $\sqrt{1/(1+kL)}$ , is a reasonable compromise to approximate the behavior under such conditions. It is widely used, as is the similar approximation  $\sqrt{A_n/(1+A_n kL)}$ .

### Homework 5.7

Consider an exponential density profile that decreases in the direction of the acceleration,  $g$ , as  $\rho = \rho_0 e^{-z/L}$ , and thus is the opposite of the case analyzed above. Apply the RT instability analysis to find  $n$  for this case. Discuss the results.

## 5.3 The Convective Instability or the Entropy Mode

The RT instability is in fact a special case, although it is a very important one. Consider, for example, the behavior of the Earth's atmosphere near the surface. The density gradient is negligible in comparison to the temperature gradient. On a hot day, when the air is hottest and the pressure is greatest near the surface, a parcel of air that rises slightly will expand to equalize its pressure. This in turn reduces the density of the parcel and makes it buoyant. The reverse happens when a parcel of air drops slightly. In short, the air is unstable to convective motions that will have the net effect of bringing cooler air down and hotter air up. Cumulus cloud formation is often a diagnostic of this. This instability is naturally called the *convective instability*.

The general instability of which the convective instability and the RT instability are special cases is the *entropy mode*. The instability occurs when  $\nabla s \cdot \nabla p > 0$ , where once again  $s$  is the specific entropy. This condition is known as the Schwarzschild stability criterion in the Western literature. When  $\nabla s \cdot \nabla p < 0$  the fluid supports stable, oscillating waves. The Rayleigh-Taylor

instability is all that remains in the fully incompressible limit. The condition  $\nabla s \cdot \nabla p > 0$  can be reduced to  $\nabla \rho \cdot \nabla p < 0$  by recalling that the specific entropy  $s$  can be expressed as  $s_o + c_V \ln(p/\rho^\gamma)$  and that in the incompressible limit  $\gamma \rightarrow \infty$ . The more general condition allows for the possibility described above that a fluid parcel may expand or contract adiabatically as it crosses the interface, because of the overall pressure gradient. In that case, interpenetration of the fluids leads to a reduction in potential energy if  $\nabla s \cdot \nabla p > 0$  is satisfied. Landau and Lifshitz anticipate this instability in their section entitled “internal waves in an incompressible fluid.” We can develop a linear theory of this mode as follows.

We use the same conventions as in Sect. 5.1.2, with  $s(z)$  being the initial entropy profile and  $\delta s$  being the first-order deviation. We also assume that only the first derivatives of  $p, \rho$ , and  $s$  are nonzero in the initial state. The linearized conservation of entropy can be written as

$$\frac{\partial}{\partial t} \delta s = -\mathbf{u} \cdot \nabla s. \quad (5.52)$$

Although we will take the medium to be compressible, we look for fluctuations that involve no compression to simplify the mathematics. There is some chance that these will be the fastest growing modes, as they invest no energy in longitudinal compression. Thus

$$\nabla \cdot \mathbf{u} = 0. \quad (5.53)$$

As in the case of Rayleigh–Taylor, we want to consider the motion in the plane of the interface, so the momentum equation in this accelerating frame becomes

$$\frac{\partial}{\partial t} \mathbf{u} = -\frac{1}{\rho} \nabla \delta p + \frac{\nabla p}{\rho^2} \delta \rho. \quad (5.54)$$

Here we have explicitly written the force introduced by the accelerating frame in terms of the pressure gradient. Equivalently we could write  $\mathbf{g} = \nabla p/\rho$ .

To solve these equations, we begin by using pressure and entropy as the thermodynamic variables. Thus

$$\frac{\delta \rho}{\rho} = \frac{1}{\rho} \left( \frac{\partial \rho}{\partial s} \right)_p \delta s + \frac{1}{\rho} \left( \frac{\partial \rho}{\partial p} \right)_s \delta p. \quad (5.55)$$

The second term in this equation is negligible, as  $\delta p \ll \rho c_s^2$ , which is necessary for a linear theory to be valid. This term also introduces no new dependences in the solution. Thus the momentum equation becomes

$$\frac{\partial}{\partial t} \mathbf{u} = -\frac{1}{\rho} \nabla \delta p + \left( \frac{\nabla p}{\rho^2} \frac{\partial \rho}{\partial s} \right)_p \delta s. \quad (5.56)$$

We look for solutions to this equation of the form  $\exp[nt + ik_x x]$ , allowing for exponential growth and for propagation in some direction perpendicular

to  $z$ , which we designate as  $x$ . The  $y$  component of  $\mathbf{u}$  is not affected by this dynamics, as the right-hand side of (5.56) has no curl. We seek a wave equation for the fluctuating velocity  $w$  as in Sect. 5.1.2. We begin by taking the dot product of (5.56) with  $\nabla s$ , to obtain

$$-\frac{\partial^2}{\partial t^2} \delta s = -\frac{1}{\rho} \frac{\partial s}{\partial z} \frac{\partial}{\partial z} \delta p + \left( \frac{1}{\rho^2} \frac{\partial p}{\partial z} \frac{\partial s}{\partial z} \frac{\partial \rho}{\partial s} \right)_p \delta s. \quad (5.57)$$

The time derivative of this equation, with (5.52) and dividing out the common factor, gives

$$n^2 w = -\frac{n}{\rho} \frac{\partial}{\partial z} \delta p - \left( \frac{1}{\rho^2} \frac{\partial p}{\partial z} \frac{\partial s}{\partial z} \frac{\partial \rho}{\partial s} \right)_p w. \quad (5.58)$$

Next we use the  $x$  component of (5.56) and (5.53) to eliminate  $\delta p$ , finding

$$\delta p = -\frac{n\rho}{k_x^2} \frac{\partial w}{\partial z}. \quad (5.59)$$

Noting that both  $w$  and  $\rho$  in this equation have finite derivatives in  $z$ , (5.58) becomes

$$n^2 w = \frac{n^2}{k_x^2} \frac{\partial^2 w}{\partial z^2} + \frac{n^2}{\rho k_x^2} \frac{\partial \rho}{\partial z} \frac{\partial w}{\partial z} - \left( \frac{1}{\rho^2} \frac{\partial p}{\partial z} \frac{\partial s}{\partial z} \frac{\partial \rho}{\partial s} \right)_p w, \quad (5.60)$$

which is a wave equation

$$k_x^2 \left( 1 + \frac{\omega_s^2}{n^2} \right) w - \frac{1}{L} \frac{\partial w}{\partial z} - \frac{\partial^2 w}{\partial z^2} = 0, \quad (5.61)$$

where for convenience we have defined  $1/L = \partial \ln \rho / \partial z$  and

$$\omega_s^2 = \left( \frac{1}{\rho^2} \frac{\partial p}{\partial z} \frac{\partial s}{\partial z} \frac{\partial \rho}{\partial s} \right)_p. \quad (5.62)$$

Equation (5.61), in combination with boundary conditions developed as described above, covers a wide variety of limiting cases. For example, in the limit as  $L \rightarrow \infty$ , one obtains plane-wave solutions having  $n^2 = -\omega_s^2 \sin^2 \theta$ , where  $\theta$  is the angle between the  $z$  axis and  $\mathbf{k}$ . This is the solution found in the fluid mechanics text of Landau & Lifshitz, which also covers stable gravity waves. We will develop these applications further shortly, but first it is useful to return to the use of  $p$  and  $\rho$  as the thermodynamic variables, in which case, using the thermodynamic relation

$$\left( \frac{\partial \rho}{\partial s} \right)_p \left( \frac{\partial s}{\partial p} \right)_\rho = - \left( \frac{\partial \rho}{\partial p} \right)_s. \quad (5.63)$$

This allows us to connect several useful forms of  $\omega_s$  as

$$\omega_s^2 = -\frac{1}{\rho} \frac{\partial p}{\partial z} \left[ \frac{1}{\rho c_s^2} \frac{\partial p}{\partial z} - \frac{1}{\rho} \frac{\partial \rho}{\partial z} \right] = -\frac{c_s^2}{\gamma} \left[ \frac{1}{\gamma L_p^2} - \frac{1}{L_p L} \right] = -gk_p, \quad (5.64)$$

in which the third term assumes a polytropic gas,  $p/L_p = \partial p/\partial z$ , and  $k_p = |\nabla p|/(\rho c_s^2) - (1/L)$ . Note that both  $L$  and  $L_p$  can be positive or negative. One sees that a pressure gradient is always destabilizing, which is sensible from the discussion at the beginning of this section, and that a density gradient must oppose the pressure gradient to be destabilizing. The frequency  $\omega_s$ , when real, is called the Brunt-Väisälä buoyancy frequency (see, for example, Tritton).

If we seek a general solution to (5.61), we find

$$w = C_1 \exp \left[ \left( \frac{z}{2L} \right) \left( \sqrt{1 + 4k_x^2 L^2 (1 - gk_p/n^2)} - 1 \right) \right] + C_2 \exp \left[ \left( \frac{-z}{2L} \right) \left( \sqrt{1 + 4k_x^2 L^2 (1 - gk_p/n^2)} + 1 \right) \right], \quad (5.65)$$

in which  $C_1$  and  $C_2$  are constants. For  $L \rightarrow \infty$  and no pressure gradient, the two terms here are proportional to  $\exp[\pm k_x z]$ , as they should be. Just as in the above, one needs the boundary condition to find the growth rate. If one finds the growth rate for the simple case that  $w \rightarrow 0$  at  $\pm\infty$ , with finite density and pressure gradients near an interface where the Atwood number is  $A_n$ , one finds

$$n^2 = gk_x \left( \frac{2k_x L}{1 + 4k_x^2 L^2 - A_n^2} \right) \left[ k_p L - A_n^2 + \sqrt{k_p^2 L^2 + A_n^2 (1 + 4k_x^2 L^2 - 2k_p L)} \right]. \quad (5.66)$$

This equation includes most of the cases one may encounter in the laboratory or in astrophysics, with the exception of a density gradient that extends for a finite distance between two layers of constant density.

### Homework 5.8

Carry out this calculation and find (5.66). Then find the limits when (a)  $k_p \rightarrow 0$  and  $k_x L \gg 1$  and (b) when  $A_n = 0$  and  $L_p = 0$ . Compare these with previous results in the chapter.

### Homework 5.9

Work out the linear theory to find an expression for the growth rate for the case of a density gradient that extends for a finite distance between two layers of constant density.

## 5.4 Buoyancy-Drag Models of the Nonlinear Rayleigh–Taylor State

Once the amplitude of a single-mode RT instability reaches about 10% of the wavelength of the initial perturbation, nonlinear effects begin to alter the

rate of growth. For a purely sinusoidal initial condition, the first development is that harmonics of the imposed wavelength begin to appear as the shape of the perturbation becomes distorted. This development has been studied in experiments and may have some relevance to specific applications. Even so, like the linear phase, the phase when harmonics are important is only transitory. Eventually the instability develops very elongated bubbles and spikes. In addition, the impact of Kelvin–Helmholtz processes at the tips of the bubbles and spikes is to broaden the tips until the interface, expressed as a function, becomes double-valued. This evolution can be seen in Fig. 5.1. This phase of the evolution, with elongated bubbles and spikes growing in time and having broad tips, may last for a significant time. The evolution during this phase can be thought of as the buoyancy-driven rising of the bubbles, limited by the drag on their tips. Models that describe this behavior are known as buoyancy-drag models. We discuss an example here. At present, more details can be found only in the literature, for example in Oron et al., in Dimonte, and in the references these contain.

A buoyancy-drag model describes the velocity of the interface,  $u_i$ , with the equation

$$(\rho_1 + \rho_2) \frac{du_i}{dt} = (\rho_2 - \rho_1) g - \frac{C_d}{\lambda} \rho_2 u_i^2, \quad (5.67)$$

for densities  $\rho_1 < \rho_2$  and with  $\lambda$  a “wavelength” corresponding to the width of a bubble. This is not quite a simple Newtonian force equation. Here we focus on the evolution of the bubbles. Similar considerations apply to the spikes. The contribution of  $\rho_2$  on the left-hand side represents the fact that as the bubbles rise the denser mass must be displaced sideways. This might not necessarily contribute with a factor of 1 as assumed here. The first term on the right-hand side gives the buoyancy force causing the bubble to rise. The second term on the right-hand side gives the drag force that resists the rise of the bubble. For three-dimensional bubbles,  $C_d = 2\pi$ . The factor of  $1/\lambda$  in this term is not genuinely an inverse wavelength. Physically it represents the ratio of the bubble volume, which contributes to the other two terms, to the bubble area, which produces the drag.

When (5.67) applies, the bubble will accelerate until the two terms on the right-hand side balance. This defines the asymptotic bubble velocity,

$$u_i = \sqrt{\frac{A_n g \lambda}{\pi(1 + A_n)}}, \quad (5.68)$$

in which we have employed the Atwood number. Defining the bubble height  $h$  as the displacement of the bubble from the mean position of the original interface, one has  $u_i = dh/dt$ . In addition, it is reasonable to suppose that the bubbles have a characteristic shape (see below). One can express this as a ratio  $b = h/\lambda$ . This gives an equation one can solve for  $h$ , finding

$$h = \alpha_B A_n g t^2, \quad (5.69)$$

in which a parameter  $\alpha_B$  or “alpha bubble” has been introduced, being equal in this model to  $[2\pi b(1 + A_n)]^{-1}$ . One can observe the growth of unstable structures in experiments or simulations to find a value of  $\alpha_B$ . Typical values of  $\alpha_B$  are within a factor of 2 of 0.05. There is much physics in the details that can be summarized by a certain value of  $\alpha_B$ , but these are not our concern here.

Interfaces that are not prepared with a specific initial mode typically have a broad spectrum of initial modes. Our discussion of RT makes some features of such systems evident. The initial growth of the unstable modes will be most rapid for the short-wavelength modes, whose exponential growth rate is proportional to  $\sqrt{A_n k g}$ . These modes grow faster and also have a smaller asymptotic velocity from (5.68). Thus, they reach their final velocity first. As time progresses, larger bubbles reach their final asymptotic velocity, overtaking and absorbing the smaller bubbles. This process is known as bubble competition. In what is known as the self-similar regime, the net effect of bubble competition is that the characteristic shape of the bubbles does remain constant as they grow in amplitude. Detailed calculations have shown that such a self-similar regime is reached, for an initial broadband spectrum and for constant acceleration. Experiments or other physical systems, however, may not have a broadband initial spectrum, may remain for a very long time in a state in which the initial conditions impact the structure, and may not have truly constant acceleration. As a result, the bubble-competition viewpoint and (5.69) are a useful model but not one that can be assumed to always apply.

## 5.5 Mode Coupling

Thus far we have considered the RT modes to be independent of one another, even in the nonlinear regime. This is consistent with our treatment of the equations in Sect. 5.1.2. Indeed, it is implied by these equations, because we began by linearizing them. Linearization amounts to assuming that the modes do not affect one another, because it is the nonlinear terms that would permit such effects. From the point of view that the nonlinear terms are small during the initial phases of RT growth, this is perfectly acceptable. However, as the modes grow (or if their initial amplitude is not very small), the existing modes do in fact couple to one another. This produces source terms for modes whose wave vector is the sum or difference of the wave vectors of existing modes. The production (or enhancement) of some modes by this process is known as *mode coupling*. We explore this here.

The general notion that the beating of two waves can produce sum and difference modes is a familiar one from general physics, often discussed in the context of music. In such discussions, it is only sometimes emphasized that it is the nonlinear terms in the underlying equations that create such possibilities. In general, such mode coupling can develop in two ways. On the

one hand, the coupling can occur throughout a volume in which waves are present. In this case, the equations of continuity and of momentum are key to describing the interaction of the waves throughout space. Harmonic generation in music, laser scattering, wave coupling in the ionosphere, and the interaction of fluctuations in the solar wind all are examples of such volumetric mode coupling. This type of coupling, however, is of limited importance in RT. On the other hand, the coupling can occur at a surface, where the requirement that the surface move self-consistently and the other boundary conditions may include nonlinear terms and introduce mode coupling. Such a surface is often, but not always, an interface between two regions with distinct properties.

Let us begin by considering the behavior of such a surface, and return to the volumetric behavior later. The velocity of a point on a continuous surface is determined by the combination of its local time variation and motion that propagates to that point from adjacent regions. The requirement for continuity of the surface can be written

$$\frac{\partial \mathbf{x}_s}{\partial t} + \mathbf{u}(\mathbf{x}_s) \cdot \nabla \mathbf{x}_s = \mathbf{u}_s, \quad (5.70)$$

in which  $\mathbf{x}_s$  is the location of a point on the surface,  $\mathbf{u}(\mathbf{x}_s)$  is the fluid velocity at that point, and  $\mathbf{u}_s$  is the velocity of the point on the surface at  $\mathbf{x}_s$ . Note, for example by reference to Fig. 5.2, that the fluid velocity may differ substantially from the velocity of the point on the surface.

The important point about (5.70) is that the second term in this equation is nonlinear. The fluid velocity  $\mathbf{u}$  includes motion due to all the modes that are present at the surface, as does the location of the surface  $\mathbf{x}_s$ . This inherently produces coupling of any two modes present at the surface to drive other modes. This is known as *second-order mode coupling*. In more detail, the fact that  $\mathbf{u}$  is evaluated at  $\mathbf{x}_s$  and not at an unperturbed, flat, initial interface, creates finite though weaker coupling at all higher orders. We will not discuss this aspect here.

To see more clearly what happens in second-order mode coupling at surfaces, we can explore mathematically a simple case that an experiment might attempt. Suppose we have an interface, separating two uniform fluids of different density and negligible viscosity, initially perturbed by some number of modes of small amplitude. The amplitude of each of these modes is made to be much larger than that of the other modes. All these other modes have finite initial amplitudes, at minimum corresponding to variations in the surface location on the atomic scale, but we take these to be negligible. We then apply and maintain a constant acceleration  $g$  to the system, beginning at some time  $t = 0$ .

We know from the differential equation (5.1.25) found in Sect. 5.2.1 that all the modes decay exponentially in  $\pm z$ . We can specify the perturbed velocity as a sum over surface fluctuations involving the possible wavevectors in the  $x - y$  plane,  $\mathbf{k}_m$ . As we are considering mode coupling, we cannot

use the usual complex notation without thought, but instead must represent the physical variables as real quantities. Taking all this into account, we can write the  $z$ -component of the velocity as

$$w = \sum_m w_m(t) e^{-sk_m(z-z_s)} \cosh(i\mathbf{k}_m \cdot \mathbf{x} - i\phi_m), \quad (5.71)$$

in which  $\phi_m$  is the phase of mode  $m$ ,  $s$  is  $-1$  for  $z > 0$  and  $+1$  for  $z < 0$ ,  $w_m(t)$  is its time-dependent amplitude, and  $z_s$ , also a function of time, is the location of the surface. It would be mathematically simpler but less intuitive to absorb the term involving  $z_s$  into the time-dependent function  $w_m(t)$ . In addition, the present formulation explicitly shows that the behavior of every mode is affected by all the other modes, through  $z_s$ .

Since  $\nabla \cdot \mathbf{u} = 0$  by assumption, the fluctuating velocity along the surface,  $\mathbf{u}_\perp$  is given by

$$\mathbf{u}_\perp = \sum_m \mathbf{u}_{m\perp} = \sum_m w_m(t) e^{-sk_m(z-z_s)} (-is) \sinh(i\mathbf{k}_m \cdot \mathbf{x} - i\phi_m) \hat{\mathbf{k}}_m, \quad (5.72)$$

in which  $\hat{\mathbf{k}}_m$  is a unit vector in the direction of  $\mathbf{k}_m$ . Note that  $\mathbf{u}_{m\perp}$  has this definite direction, but that the mode having a wave vector of  $-\mathbf{k}_m$  is redundant, as the hyperbolic sine changes sign, compensating for the change in unit-vector direction. One could attempt to sum over only a half space but the bookkeeping would become messy. Instead, we will sum over all directions and realize that the amplitude a measurement would detect is twice that corresponding to any one term in the sum.

We can express the position of the surface as a sum over the same modes

$$z_s = \sum_m z_m = \sum_m z_m(t) \cosh(i\mathbf{k}_m \cdot \mathbf{x} - i\phi_m), \quad (5.73)$$

in which only some modes have finite initial amplitude at  $t = 0$ . If we take the average initial position of the interface to be at  $z = 0$ , then modes with an initial amplitude small enough to be in the linear regime evolve with  $z_m(t) \propto \cosh(n_m t)$ , where  $n_m$  is the linear growth rate, and have  $w_m(t) = n_m z_m(0) \sinh(n_m t)$ .

Some further discussion of these initial values is worthwhile. Note that they involve functions of  $\pm n_m t$ . Although in Sect. 5.2.1 we took  $n > 0$  to find growing modes, we can observe that the differential equations found there for an interface separating two uniform fluids are unchanged for  $n < 0$ . Modes with  $n < 0$  decay with time and so are not relevant to the behavior after a few growth times. However, they may be important to the initial condition. Examination of our derivation in Sect. 5.2.1 shows that the implicit initial condition in that section is that of a flat interface on which a velocity perturbation has been imposed. Such an initial condition is physically sensible and might be achieved in practice, but is certainly not typical. Much more



typical is the case of (5.73), in which the interface is initially structured and the velocity is initially zero.

With these definitions, we can evaluate the  $z$  component of (5.70), using an overdot for the partial derivative in time. This gives

$$\begin{aligned} \sum_m [\dot{z}_m(t) - w_m(t)] \cosh(\mathbf{i}\mathbf{k}_m \cdot \mathbf{x} - \mathbf{i}\phi_m) &= \sum_\ell \mathbf{u}_{\ell\perp}(t, z_s) \cdot \nabla \sum_j z_j \\ &= \sum_\ell \sum_j w_\ell(t) z_j(t) (\hat{\mathbf{k}}_\ell \cdot \mathbf{k}_j) s \sinh(\mathbf{i}\mathbf{k}_\ell \cdot \mathbf{x} - \mathbf{i}\phi_\ell) \sinh(\mathbf{i}\mathbf{k}_j \cdot \mathbf{x} - \mathbf{i}\phi_j), \end{aligned} \quad (5.74)$$

which becomes, upon expanding the hyperbolic sines and cosines,

$$\begin{aligned} \sum_m [\dot{z}_m(t) - w_m(t)] \cos(\mathbf{k}_m \cdot \mathbf{x} - \phi_m) \\ &= \frac{s}{2} \sum_\ell \sum_j k_j w_\ell(t) z_j(t) (\hat{\mathbf{k}}_\ell \cdot \hat{\mathbf{k}}_j) \\ &\quad \times (\cos[(\mathbf{k}_\ell + \mathbf{k}_j) \cdot \mathbf{x} - (\phi_\ell + \phi_j)] \\ &\quad - \cos[(\mathbf{k}_\ell - \mathbf{k}_j) \cdot \mathbf{x} - (\phi_\ell - \phi_j)]). \end{aligned} \quad (5.75)$$

We want to identify the term in the sum corresponding to any specific mode  $m$ . Each possible combination of two modes  $\ell$  and  $j$  shows up four times in the sum, in consequence of summing over all directions. To be specific, a sum mode with  $\mathbf{k} = \mathbf{k}_1 + \mathbf{k}_2$  shows up twice in each term through various combinations of terms involving  $\pm$  each wave vector. The redundant mode with  $\mathbf{k} = -(\mathbf{k}_1 + \mathbf{k}_2)$  also appears four times. The result for any one of the redundant modes, summing over only one of the two wave vectors, is to introduce a factor of 2. We get

$$\begin{aligned} \dot{z}_m(t) - w_m(t) &= s \sum_j k_j w_\ell(t) z_j(t) (\hat{\mathbf{k}}_\ell \cdot \hat{\mathbf{k}}_j) \Big|_{\mathbf{k}_m = \mathbf{k}_\ell + \mathbf{k}_j; \phi_m = \phi_\ell + \phi_j} \\ &\quad - s \sum_j k_j w_\ell(t) z_j(t) (\hat{\mathbf{k}}_\ell \cdot \hat{\mathbf{k}}_j) \Big|_{\mathbf{k}_m = \mathbf{k}_\ell - \mathbf{k}_j; \phi_m = \phi_\ell - \phi_j}. \end{aligned} \quad (5.76)$$

Here the matching condition in wave vector and phase is indicated by the vertical line following each sum. This designates which terms in the sum are selected; these are the terms that contribute to mode  $m$ . The other terms in the sum are ignored. (Alternatively, one could devise some more-complicated notation related to a Kronecker delta function.)

When existing modes are creating new modes by beating together, we call them the driving modes and call the beat modes the driven modes. A consequence of the two terms in this equation is that any two driving modes produce driven modes with wave vectors that are the sum or difference of their wave vectors. Any two driving modes will produce one term driving a mode with a larger wave number, said to be upshifted and one term driving a mode with a smaller wavenumber, said to be downshifted.

In addition, the phase  $\phi_m$  of the driven mode is determined by the phases of the driving modes. A specific driven mode may already be present at some amplitude, but how this mode is affected by the driving modes will depend upon the relative phases. In experiments using initial modes to drive others, the phases are chosen. Then the phases of the driving modes determine the phase of the driven mode. In more general circumstances, such as an inertial fusion capsule, the amplitude of the driven mode might be initially increased, initially decreased, or gradually become altered in phase through the influence of the driving modes. Henceforth we will ignore any contributions from the relative phases, assuming the modes to be in phase. This allows us to rewrite (5.76), explicitly specifying  $\mathbf{k}_\ell$  in the argument of  $z_\ell$ , as

$$\begin{aligned} \dot{z}_m(t) - w_m(t) &= s \sum_j z_j(t) \\ &\times \left[ (\hat{k}_\ell \cdot \hat{k}_j) w_\ell(t, \mathbf{k}_\ell = \mathbf{k}_m - \mathbf{k}_j) - (\hat{k}_\ell \cdot \hat{k}_j) w_\ell(t, \mathbf{k}_\ell = \mathbf{k}_j - \mathbf{k}_m) \right]. \end{aligned} \quad (5.77)$$

For our purposes, it will suffice to have a second-order expression for  $w_m$ . We can obtain one by realizing that  $w_m = \dot{z}_m$  to first order. To second order in the mode amplitudes, this gives us

$$w_m(t) = \dot{z}_m(t) + s \sum_j k_j z_j(t) \left( \hat{k}_\ell \cdot \hat{k}_j \right) \left[ \dot{z}_\ell(t, \mathbf{k}_\ell) + \dot{z}_\ell(t, -\mathbf{k}_\ell) \right] \Big|_{\mathbf{k}_\ell = \mathbf{k}_m - \mathbf{k}_j}. \quad (5.78)$$

For reasons discussed above  $z_\ell(\mathbf{k}_\ell) = z_\ell(-\mathbf{k}_\ell)$ , but we leave them separate to clarify some of the steps below. We will use this relation in another boundary condition to find an equation for the overall behavior of the modes.

To make further progress, we now must return to the fundamental differential equations. For our special case of constant density, the continuity equation does not produce any contributions to mode coupling. (This is not true if there is a density gradient.) The momentum equation under these assumptions is

$$\rho \frac{\partial}{\partial t} \mathbf{u} + \rho \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p - \nabla \Psi, \quad (5.79)$$

in which  $\Psi$  is the gravitational potential, given by  $\Psi = \int \rho g dz$ . Note that  $\Psi$  has a discontinuous derivative at the interface, for our assumptions. The initial profile of pressure is determined by the initial gravitational potential, and the gradients of these profiles cancel one another in this equation. This lets us follow only the variations,  $\delta p$  and  $\delta \Psi$ . We can also expand the convective derivative, using  $\mathbf{u} \cdot \nabla \mathbf{u} = \rho \nabla \frac{u^2}{2} - \mathbf{u} \times \nabla \times \mathbf{u}$ . This allows us to see, by taking the curl of the resulting equation, that  $\nabla \times \mathbf{u}$  must remain zero if it is initially zero. Since  $\nabla \times \mathbf{u}$  is zero for our initial conditions, (5.79) becomes

$$\rho \frac{\partial}{\partial t} \mathbf{u} + \rho \nabla \frac{u^2}{2} = -\nabla \delta p - \nabla \delta \Psi. \quad (5.80)$$

As an aside, it is worth mentioning that much of the literature takes an alternative approach to this problem of a stationary, structured interface, by exploiting the fact that  $\nabla \times \mathbf{u}$ , which is known as the *vorticity*, is zero. The vorticity corresponds qualitatively to the degree of swirling present in the motion. The vorticity plays an essential role in the development of hydrodynamic turbulence, as is discussed in Sect. 5.8. We show there that, in the absence of viscosity, vorticity is frozen into the fluid volume. Thus, in this limit the volumetric vorticity is fixed in time (the vorticity on a surface is not fixed). Therefore, if one assumes that the fluid is *inviscid* (which means that the viscosity is zero) and also is incompressible, then the velocity has both zero curl and zero divergence for all time. This in turn implies that the velocity is the gradient of a potential,  $\phi_v$  and that this potential satisfies Laplace's equation,  $\nabla^2 \phi_v = 0$ . Such models are known as *potential flow* models, since the flow is described by a potential. In a potential flow model, one can write the momentum equation (2.3.2), under the present assumptions, as the gradient of an equation involving the density and gradients of pressure, velocity potential, and gravitational potential. The resulting equation corresponds to one version of Bernoulli's equation, which can also be used as a starting point. The potential-flow approach enables some simpler approaches to numerical simulation. A drawback is that such a model cannot describe any system containing actual vorticity and thus cannot follow the onset of turbulence.

Returning our attention to (5.80), our assumption of uniform density implies that the first term introduces no mode coupling, so that all the mode coupling enters here through the term involving  $u^2$ . In addition, we can identify  $\delta\Psi$  as

$$\delta\Psi = -(\rho_2 - \rho_1)g[z[H(z) - H(z - z_s)] + z_s H(z - z_s)], \quad (5.81)$$

where again  $H$  is the Heavyside step function. As in Sect. 5.1.2, (5.80) has components in the  $z$  direction and in the  $x$ - $y$  plane, written for example as

$$\rho \frac{\partial}{\partial t} w + \rho \frac{\partial}{\partial z} \frac{u^2}{2} = -\frac{\partial}{\partial z} \delta p - \frac{\partial}{\partial z} \delta\Psi, \quad \text{and} \quad (5.82)$$

$$\rho \frac{\partial}{\partial t} \mathbf{u}_\perp + \rho \nabla_\perp \frac{u^2}{2} = -\nabla_\perp \delta p, \quad (5.83)$$

in which  $\nabla_\perp$  is the gradient in the  $x$ - $y$  plane. Our first step in Sect. 5.1.2 was to take the versions of these equations corresponding to specific assumptions and to find a single differential equation. The general version of this is left for homework, but the outcome is that no mode coupling remains in the differential equation for  $w$ . This is the origin of the statement above that there is no mode coupling in the absence of a density gradient.

**Homework 5.10**

By operating on (5.82) and (5.84), create two scalar differential equations that can be subtracted to eliminate terms involving  $p$ . Compare the resulting differential equation to (5.21) and discuss.

To proceed toward a solution for the behavior with mode coupling, we proceed as we did above in finding a boundary condition across the interface. Integrating (5.82) across the interface gives

$$(\delta p_2 - \delta p_1)_{z_s} = (\rho_2 - \rho_1) g z_s, \quad (5.84)$$

while operating on (5.83) with  $\nabla_{\perp}$ , using the incompressibility condition, and subtracting across the interface gives

$$-\nabla_{\perp}^2 (\delta p_2 - \delta p_1)_{z_s} = -\frac{\partial}{\partial t} \left( \rho_2 \frac{\partial w_2}{\partial z} - \rho_1 \frac{\partial w_1}{\partial z} \right) + \rho_2 \nabla_{\perp}^2 \frac{u_2^2}{2} - \rho_1 \nabla_{\perp}^2 \frac{u_1^2}{2}. \quad (5.85)$$

Here as before the quantities are evaluated as one approaches the interface from the side designated by the subscript. Together these give

$$-(\rho_2 - \rho_1) g \nabla_{\perp}^2 z_s = -\frac{\partial}{\partial t} \left( \rho_2 \frac{\partial w_2}{\partial z} - \rho_1 \frac{\partial w_1}{\partial z} \right) + \rho_2 \nabla_{\perp}^2 \frac{u_2^2}{2} - \rho_1 \nabla_{\perp}^2 \frac{u_1^2}{2}. \quad (5.86)$$

In the absence of mode coupling, this gives the standard RT growth rate as found in Sect. 5.2.1. Mode coupling appears to second order through (5.78) for  $w_m$  and through the final two terms (from the convective derivative). One finds

$$\begin{aligned} \frac{(\rho_2 - \rho_1)}{2} \nabla_{\perp}^2 \frac{u^2}{2} \Big|_m &= \frac{(\rho_2 - \rho_1)}{2} k_m^2 \sum_j \left( 1 - \hat{k}_{\ell} \cdot \hat{k}_j \right) \dot{z}_j \\ &\times [\dot{z}_{\ell}(t, \mathbf{k}_{\ell}) + \dot{z}_{\ell}(t, -\mathbf{k}_{\ell})] \Big|_{\mathbf{k}_{\ell} = \mathbf{k}_m - \mathbf{k}_j}. \end{aligned} \quad (5.87)$$

Now we can use this equation, (5.78), and (5.86) to obtain an equation for the evolution of a mode on the interface having wave vector  $\mathbf{k}$ , as

$$\begin{aligned} \ddot{z}_k - A_n g k_m z_k &= -A_n \sum_j k_j \left[ \left( \hat{k}_{\ell} \cdot \hat{k}_j \right) z_j(t) [\ddot{z}_{\ell}(t, \mathbf{k}_{\ell}) \right. \\ &\left. + \ddot{z}_{\ell}(t, -\mathbf{k}_{\ell})] + \left( 1 + \hat{k}_{\ell} \cdot \hat{k}_j \right) \frac{\dot{z}_j(t)}{2} [\dot{z}_{\ell}(t, \mathbf{k}_{\ell}) + \dot{z}_{\ell}(t, -\mathbf{k}_{\ell})] \right] \Big|_{\mathbf{k}_{\ell} = \mathbf{k} - \mathbf{k}_j}. \end{aligned} \quad (5.88)$$

Here again one sees that in the absence of mode coupling, one recovers the usual RT growth rate. The presence of mode coupling can increase or decrease the growth of the mode relative to this, depending on the sign of the right-hand side (and thus on the phases of the modes). The right-hand side must

be positive to add to the growth of the mode (with the assumed phase). Note that the sum is over all directions, so that if the term  $(1 + \hat{k}_\ell \cdot \hat{k}_j)$  is 0 in one case, it will equal 2 for the opposing mode. The mode-coupling terms will tend to dominate if  $k_j z_j z_\ell / z_k \gg 1$ . If the driving modes are growing exponentially, the right-hand side will have terms proportional to  $n_\ell^2 e^{(n_\ell + n_j)t}$  and  $n_\ell n_j e^{(n_\ell + n_j)t}$ . In this limit, the driven mode will grow in approximately an exponential way and will have a growth rate larger than that of the driving modes by approximately  $k_j z_j z_\ell / z_k$ . Thus, mode coupling can rapidly bring the coupled modes up to an amplitude of order  $kz$  times that of the driving modes.

The modes driven as described above then act as driving modes in turn. In this way, two initial modes can overtime produce a broad spectrum of modes. These modes will have a sparse spectrum that can be constructed by taking sums and differences of multiples of the initial wave vectors. In applications, mode coupling can play a substantial role in creating more complicated structures at an RT-unstable interface.

## 5.6 The Kelvin–Helmholtz Instability

The Kelvin–Helmholtz (KH) instability, like Rayleigh–Taylor, is seen frequently in many disparate physical systems. Whenever two fluid regions flow past one another, with a sufficiently narrow transition region at their mutual boundary, fluctuations at the boundary are unstable and will grow. The transition region where the velocity changes quickly in magnitude but remains along the same axis is known as a *shear layer*. Figure 5.7 shows an example of modulations caused by a KH instability. Modulations driven by KH can routinely be seen in clouds, in flowing water, and in the ripples in the sand at the beach. They are also observed at shear layers in the magnetosphere.



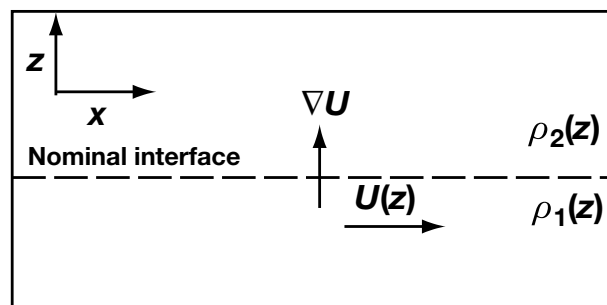
**Fig. 5.7.** The structures seen along the upper edges of these clouds were produced by the Kelvin–Helmholtz instability

Throughout astrophysics, there are many systems that produce shear layers, anytime a flow of material from one object or region passes through or around another object or region. In addition, the characteristic mushroom shape that develops at the spike tips in the RT instability is produced by KH driven by the shear between the spike material and the bubble material. (See, for example, the simulation results shown in Fig. 4.24) One can see that this process is so prevalent that it is worthwhile to understand.

### 5.6.1 Fundamental Equations for Kelvin–Helmholtz Instabilities

The fundamental equations for KH instabilities are similar to those for Rayleigh–Taylor instabilities, but have differences reflecting the presence of a nonzero velocity and velocity gradient in the initial, unperturbed state. As in the case of the Rayleigh–Taylor instability, and for the same reasons, we will develop the theory for an incompressible fluid. This instability does not require compressibility to exist. Also, the unstable behavior is not strongly modified by compressibility, although we will not prove this assertion here. We consider the system sketched in Fig. 5.8. We assume the shear layer to be planar and to lie in the  $x$ – $y$ -plane, so that the  $z$ -direction is perpendicular to it. We further assume the initial flow, designated by  $\mathbf{U}$ , to be parallel to the  $x$ -axis, and that the zeroth-order gradients of  $U$  and of  $\rho$  are parallel to the  $z$ -axis. As in Sect. 5.1.2, we designate the first-order density and pressure perturbations by  $\delta\rho$  and  $\delta p$ , respectively, and the  $x$ ,  $y$ , and  $z$  components of the first-order velocity perturbation,  $\mathbf{u}$ , by  $u$ ,  $v$ , and  $w$ . With these assumptions the continuity and momentum equations become

$$\frac{\partial\delta\rho}{\partial t} + \mathbf{U} \cdot \nabla\delta\rho + \mathbf{u} \cdot \nabla\rho = 0, \text{ and} \quad (5.89)$$



**Fig. 5.8.** Geometry for Kelvin–Helmholtz instability calculations. The densities and velocities may vary with  $z$ . The shear layer and velocity gradient may or may not be localized at the nominal interface

$$\rho \frac{\partial \mathbf{u}}{\partial t} + \rho \mathbf{U} \cdot \nabla \mathbf{u} + \rho \mathbf{u} \cdot \nabla \mathbf{U} = -\nabla \delta p - g \delta \rho \hat{z}, \quad (5.90)$$

where once again  $\hat{z}$  is a unit vector in the  $z$  direction. In addition, we have the important additional condition expressed in (5.70) above, which is to first order

$$\frac{\partial \delta \mathbf{x}_s}{\partial t} + \mathbf{U} \cdot \nabla \delta \mathbf{x}_s = \mathbf{u}_s, \quad (5.91)$$

in which  $\delta \mathbf{x}_s$  is the location of a point on the interface relative to its initial position and  $\mathbf{u}_s$  is the velocity of that point. Here  $x_s$  and  $u_s$  are both first-order quantities. This equation specifies that the interface must move with the fluid self-consistently. We will not consider mode coupling for KH instabilities, but it exists for the same reasons that produce it in RT instabilities. One specific source is the requirement that the interface remain continuous, as represented in its full nonlinear form by (5.70).

In writing (5.89) to (5.91), we have ignored surface tension for the reasons discussed in Sect. 5.1.2. We also have ignored viscosity, for which we have much less excuse. Viscosity can play a role in KH instabilities at short wavelength. However, the mathematics turns out to be particularly intractable. Nonetheless, one aspect of the influence of viscosity can be accounted for using the above equations. Viscous diffusion of momentum causes the transition region in any initially abrupt shear layer to develop a scale length of  $\sqrt{\nu t}$ . This stabilizes the KH instability for the shortest wavelengths, and the maximum wavelength that is stabilized increases with time. We consider the effect of an extended shear layer below in Sect. 5.6.3.

It is helpful to express (5.89) through (5.91) as equations for the components, and to write out the incompressibility condition, in order to obtain a set of equations we can solve to see the unstable behavior. These are as follows:

$$\rho \frac{\partial u}{\partial t} + \rho U \frac{\partial}{\partial x} u + \rho w \frac{\partial}{\partial z} U = -\frac{\partial}{\partial x} \delta p, \quad (5.92)$$

$$\rho \frac{\partial v}{\partial t} + \rho U \frac{\partial}{\partial x} v = -\frac{\partial}{\partial y} \delta p, \quad (5.93)$$

$$\rho \frac{\partial w}{\partial t} + \rho U \frac{\partial}{\partial x} w = -\frac{\partial}{\partial z} \delta p - g \delta \rho, \quad (5.94)$$

$$\frac{\partial}{\partial t} \delta \rho + U \frac{\partial}{\partial x} \delta \rho = -w \frac{\partial \rho}{\partial z}, \quad (5.95)$$

$$\frac{\partial \delta z_s}{\partial t} + U \frac{\partial}{\partial x} \delta z_s = w_s, \text{ and} \quad (5.96)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0. \quad (5.97)$$

Here only the  $z$  component of (5.91) is important in a linearized analysis. Note that all three dimensions matter for KH, unlike simple RT. This is because three directions – that of the gradients, that of  $\mathbf{U}$ , and that of  $\mathbf{k}$  – all matter

independently. Also note that the sign of gravity is such as to produce a downward acceleration. We seek surface waves growing exponentially in time, but possibly also having an oscillatory component, we assume all linearized amplitudes to be proportional to  $\exp i(k_x x + k_y y + nt)$ . This differs from our assumption in the Rayleigh–Taylor problem. Now a growing instability will be one with negative imaginary  $n$ . Our set of equations then becomes

$$i\rho(n + k_x U)u + \rho w \frac{\partial U}{\partial z} = -ik_x \delta p, \quad (5.98)$$

$$i\rho(n + k_x U)v = -ik_y \delta p, \quad (5.99)$$

$$i\rho(n + k_x U)w = -\frac{\partial}{\partial z} \delta p - g\delta\rho, \quad (5.100)$$

$$i(n + k_x U)\delta\rho = -w \frac{\partial \rho}{\partial z}, \quad (5.101)$$

$$i(n + k_x U)\delta z_s = w_s, \text{ and} \quad (5.102)$$

$$ik_x u + ik_y v = -\frac{\partial w}{\partial z}. \quad (5.103)$$

One sees that five of these six equations involve the term  $(n + k_x U)$ . In a system with uniform flow, this type of term introduces a Doppler shift into wave frequencies. Here we have the added complication that  $U$  varies with  $z$ . We simplify these equations first by obtaining from (5.98), (5.99), and (5.103)

$$-\rho(n + k_x U) \frac{\partial w}{\partial z} + \rho k_x w \frac{\partial U}{\partial z} = -ik^2 \delta p, \quad (5.104)$$

while from (5.100) and (5.101) we find

$$i\rho(n + k_x U)w = -\frac{\partial}{\partial z} \delta p - ig \frac{w}{(n + k_x U)} \frac{\partial \rho}{\partial z}. \quad (5.105)$$

Eliminating  $\delta p$  from these equations gives a single differential equation for  $w$  in terms of known parameters,

$$-k^2 \rho(n + k_x U)w + \frac{\partial}{\partial z} \left[ \rho(n + k_x U) \frac{\partial w}{\partial z} \right] - \frac{\partial}{\partial z} \left( \rho k_x w \frac{\partial U}{\partial z} \right) = g \frac{wk^2}{(n + k_x U)} \frac{\partial \rho}{\partial z}. \quad (5.106)$$

One can see that this is a second-order equation for  $w$  and thus likely to allow solutions that decay away from the interface, or that combine to satisfy specific geometric constraints. Chandrasekhar points out that it is worthwhile to separate out the role of the density in this equation, obtaining

$$\begin{aligned} & -k^2(n + k_x U)w + \frac{\partial}{\partial z} \left( (n + k_x U) \frac{\partial w}{\partial z} - k_x w \frac{\partial U}{\partial z} \right) \\ & = \frac{1}{\rho} \frac{\partial \rho}{\partial z} \left[ g \frac{wk^2}{(n + k_x U)} - \left( (n + k_x U) \frac{\partial w}{\partial z} - k_x w \frac{\partial U}{\partial z} \right) \right]. \end{aligned} \quad (5.107)$$



The right-hand side of this equation can be ignored so long as the scale length of the density profile,  $L$ , is large compared to the perturbation wavelength of interest and unless the gravitational acceleration is very large ( $> k_x^2 U^2 L$ ).

To develop solutions that involve an interface, we need boundary conditions at the interface. At this point we have incorporated (5.98) through (5.103) except for (5.102), which gives us one boundary condition as the interface position must be the same when approached from either side. This implies that  $w/(n + k_x U)$  is continuous at the interface, so

$$\frac{w_2}{(n + k_x U_2)} = \frac{w_1}{(n + k_x U_1)}. \quad (5.108)$$

Here and in the following the subscript 1 or 2 indicates the value found by approaching the interface from the side designated by the subscript and the subscript  $s$  designates the value of a continuous quantity at the interface. To find another boundary condition, we can proceed as we did in Sect. 5.1.2. We integrate (5.105) across the interface, then subtract (5.104) from itself across the interface so we can eliminate  $\delta p$ . The resulting boundary condition is

$$gk^2 \left( \frac{w}{n + k_x U} \right)_s (\rho_2 - \rho_1) = \rho_2 (n + k_x U_2) \frac{\partial w_2}{\partial z} - \rho_1 (n + k_x U_1) \frac{\partial w_1}{\partial z} + k_x \left( -\rho_2 w_2 \frac{\partial U_2}{\partial z} + \rho_1 w_1 \frac{\partial U_1}{\partial z} \right). \quad (5.109)$$

We are now prepared to consider specific cases of interest.

### 5.6.2 Uniform Fluids with a Sharp Boundary

We consider the simplest case first, to determine the most general features of this process. Consider the two fluids to have uniform densities and uniform initial flow velocity,  $U$ , supposing that the value of these parameters may change only at an interface. Further assume the gravitational acceleration to be negligible for now. We then find from (5.107) that

$$-k^2 w + \frac{\partial^2 w}{\partial z^2} = 0, \quad (5.110)$$

so that we have solutions that are a linear combination of terms proportional to  $e^{kz}$  and  $e^{-kz}$ , with coefficients that must be set to match the geometric boundary conditions. We will consider the case with the simplest algebra, in which the boundary condition is that the disturbance become negligible at large distances, so that

$$w = A_2 e^{-kz} \quad \text{for } z > 0 \quad \text{and} \quad w = A_1 e^{kz} \quad \text{for } z < 0, \quad (5.111)$$

where (5.108) implies

$$A_2 = A_1 \frac{n + k_x U_2}{n + k_x U_1}. \quad (5.112)$$

It is convenient to work in a frame of reference corresponding to the average velocity of the two regions, because the velocity difference is what drives the instability and because one often knows the velocity difference in real applications. In this case  $U_2 = \Delta U/2$  and  $U_1 = -\Delta U/2$ . With these results, (5.109) becomes

$$0 = \rho_2 (n + k_x U_2)^2 + \rho_1 (n + k_x U_1)^2, \quad (5.113)$$

which has the solution for  $n$

$$n = -k_x \frac{A_n}{2} \Delta U \pm i k_x \Delta U \frac{\sqrt{\rho_1 \rho_2}}{(\rho_1 + \rho_2)}. \quad (5.114)$$

The real part of  $n$  is finite if  $A_n \neq 0$ , so that in such cases the wave propagates along the surface in this frame of reference. The negative imaginary part of  $n$  describes the exponential growth, given our specification of the modulations. For equal densities one finds the standard and very simple result that the exponential growth rate is  $k_x \Delta U/2$ . (The factor of 2 depends on the definition of  $\Delta U$ , which varies among references.)

There are some things worth noticing about the result of (5.114). First, this process has no minimum wavenumber. Perturbations at all wavelengths are unstable (until the wavelength approaches the scale of the system, in which case this calculation becomes invalid). Shorter-wavelength perturbations have more-rapid growth rates. On the one hand, if the initial fluctuations present at a sharp interface corresponded to broadband noise, one would expect to see small-scale hair grow first, followed by the evolution of larger scales. On the other hand, one does not typically see this, which probably reflects the fact either that the initial fluctuations are larger at some specific wavelengths or that the shear layer is not indefinitely sharp. Finally, while the component of  $\mathbf{k}$  along  $\mathbf{U}$  determines the growth rate, there is no limitation on the  $y$  component of  $\mathbf{k}$ . Fluctuations whose wave vector makes some angle with  $\mathbf{U}$  grow freely, though more slowly than do fluctuations of the same wavelength for which  $\mathbf{k}$  is parallel to  $\mathbf{U}$ .

### Homework 5.11

If we take the point of view that the modulations of interest are proportional to  $e^{int}$ , then we would insist on finding negative imaginary  $n$  in order to have growth of the modulations, as opposed to damping, in time. However, this should give us pause because the complex representation is only a mathematical convenience while the physical quantities are real. Considering the real, physical quantities, what is the significance of finding positive or negative imaginary  $n$ . (The chapter in Jackson that introduces waves may be of

some help regarding the connection of real physical quantities and a complex representation.)

If we now allow for gravity but change no other assumptions, then (5.110) through (5.112) remain correct, but now (5.109) gives

$$-gk(\rho_2 - \rho_1) = \rho_2 (n + k_x U_2)^2 + \rho_1 (n + k_x U_1)^2, \quad (5.115)$$

in which we have divided out a factor of  $[kw/(n + k_x U)]_s$ . The solution for  $n$  now becomes

$$n = -k_x \frac{A_n}{2} \Delta U \pm i \frac{\sqrt{k_x^2 \Delta U^2 \rho_1 \rho_2 + gk(\rho_2^2 - \rho_1^2)}}{(\rho_1 + \rho_2)}, \quad (5.116)$$

in which if the argument of the square root is positive then there is an unstable root. Our conventions imply that  $\rho_2$  is from the “upper” region as defined relative to the gravitational acceleration. One sees that instability is always present if the upper density ( $\rho_2$ ) is higher than the lower density ( $\rho_1$ ). In this case, the KH and RT instabilities work together to produce larger growth. In contrast, when the lower density exceeds the upper density, this places a condition on the wavenumber for instability,

$$k > \frac{g(\rho_1^2 - \rho_2^2)}{\Delta U^2 \rho_1 \rho_2 \cos^2 \theta}. \quad (5.117)$$

Here  $\theta$  is defined by  $\cos \theta = k_x/k$ . Thus, the RT dynamics at a given  $k$  opposes the instability growth due to KH, but at large enough  $k$  the KH instability dominates and one will see a positive growth rate. When gravity becomes large enough (and  $\rho_1 > \rho_2$ ) the argument of the square root in (5.116) becomes negative. Then any modulations of the interface oscillate but do not grow.

### 5.6.3 Otherwise Uniform Fluids with a Distributed Shear Layer

The next level of complexity is to assume that the shear layer is not an instantaneous change in velocity, which at the microscopic level is unphysical in any case. Realistic problems with shear can become quite complex. As a first simple problem, we will suppose that the velocity shears but that the density changes abruptly at an interface. This may be relevant, for example, to the KH instability at boundaries between two fluids that are incompressible or that have only slow variations in density. The boundaries created by the RT instability or at bow shocks may be of this type. We mentioned above that the minimum width of the shear layer, in a system that has kinematic viscosity  $\nu$  and has evolved for time  $t$ , is  $\sqrt{\nu t}$ .

We assume that the right-hand side of (5.107) is small, because the density is constant or slowly varying in the sense required. Further assuming the velocity profile to be given by  $U_s(1 + z/L)$ , we can observe that the terms

involving  $\partial U/\partial z$  in this equation cancel out, and that we are left once again with (5.110), solution 5.2.23, and condition 5.2.24 on the amplitudes. Here, because there are no boundaries on the flow,  $L$  is the distance over which  $U$  changes by  $U_s$ . Note that this assumption implies working in an inertial frame for which  $U = 0$  at  $z = -L$  and that  $U_1 = U_2 = U_s$  so  $w_1 = w_2 = w_s$ . The boundary condition of (5.109) then becomes

$$A_n g k^2 + \frac{A_n k_x U_s}{L} (n + k_x U_s) + (n + k_x U_s)^2 k = 0, \quad (5.118)$$

which can be solved for  $n$  to give

$$n = -k_x U_s \left( 1 + \frac{A_n}{2kL} \right) \pm i \sqrt{g A_n k - \frac{A_n^2 k_x^2 U_s^2}{4k^2 L^2}}. \quad (5.119)$$

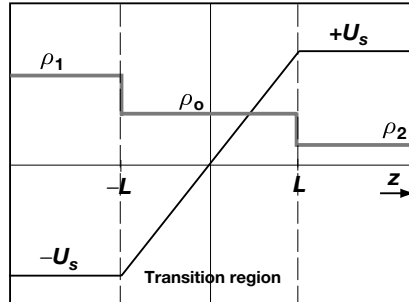
We see that  $n$  has a real part, so these modes oscillate and propagate. For instability, the argument of the square root must be positive. In particular, an interface of this type (continuous linear  $U$ , discontinuous  $\rho$ ) is always stable if the product  $g A_n$  is zero or negative. Another way to put this is that modes that perceive the region of velocity shear to be large are stabilized, and here the shear region is indefinitely large. In the next section, we will see that any boundaries, no matter how distant, will have the effect of destabilizing modes whose wavelength is of order the distance between boundaries. In contrast, when  $g A_n > 0$  the interface is unstable so long as

$$kL > \frac{A_n \cos^2 \theta U_s^2}{4Lg}. \quad (5.120)$$

This is a curious result, as the shear in this context acts to stabilize long-wavelength modes but not short-wavelength ones. Note that the shear acts to stabilize modes with  $\mathbf{k}$  along  $\mathbf{U}$ , but not modes with  $\mathbf{k}$  perpendicular to  $\mathbf{U}$ . If  $\mathbf{k}$  is aligned with  $\mathbf{U}$  and the interface decelerates over some distance  $h$  so  $g \sim U_s^2/h$ , then modes with wavelength  $\lambda > 8\pi L^2/(A_n h)$  are stable.

#### 5.6.4 Uniform Fluids with a Transition Region

The notion of a sharp interface, however, is an approximation, as molecular diffusion always will mix the materials from the two sides of the interface to some extent. This is particularly true in high-energy-density physics, where surface tension does not exist. Unfortunately, when  $U$  and  $\rho$  both vary, the solutions become much more complex. We will work out one standard case here, following Chandrasekhar, and will leave other and more realistic cases to the specialized literature and to simulations. The geometry of this case is illustrated in Fig. 5.9. One has two layers of fluid, of density  $\rho_1$  and  $\rho_2$ , separated by a transition region of width  $2L$ . The velocity of the layers  $\pm U_s$ , and with a linear velocity profile  $U(z) = U_s z/L$  connecting them through the



**Fig. 5.9.** Geometry for Kelvin-Helmholtz instability with a transition region having uniform density

transition layer. The density of the transition layer is assumed to be constant and equal to  $\rho_o = (\rho_1 + \rho_2)/2$ . This corresponds to the approximation that the transition layer is fully mixed, presumably through the action of instabilities and turbulence. We will designate the lower and upper regions, using the subscripts 1 and 2, respectively, and the transition region, using the subscript  $o$ .

We now can apply our fundamental analysis to this problem. We have three distinct regions with two boundaries, and we will once again assume that the perturbations must vanish at large  $|z|$ . Under the assumptions stated, (5.107) once again reduces to (5.110),  $\partial^2 w / \partial z^2 = k^2 w$ , in all three regions. [The right-hand side of (5.110) is zero and the other derivatives on the left-hand side cancel one another]. The solutions are  $w(z) = Ae^{\pm kz}$ , with coefficients chosen so to make  $w$  vanish appropriately in the outer regions. This gives

$$\begin{aligned} w &= A_2 e^{-kz} \text{ for } z > L, \\ w &= A_o e^{-kz} + B_o e^{kz} \text{ for } -L < z < L, \\ \text{and } w &= B_1 e^{kz} \text{ for } z < -L. \end{aligned} \quad (5.121)$$

Thus, we have four unknown amplitudes. Our first boundary condition (5.108) tells us that  $w(z)$  must be continuous at both boundaries, because  $U(z)$  is continuous by assumption. This gives us two equations,

$$A_o e^2 + B_o - B_1 = 0, \text{ and} \quad (5.122)$$

$$A_o + B_o e^2 - A_2 = 0. \quad (5.123)$$

Our second boundary condition (5.109) applies at each interface, giving two more equations,

$$0 = B_1 e^{-kL} \left[ \frac{gk^2(\rho_1 - \rho_o)}{(n - k_x U_s)} - k\rho_1(n - k_x U_s) \right] \\ + \rho_o B_o e^{-kL} \left[ nk - \left(k + \frac{1}{L}\right) k_x U_s \right] - \rho_o A_o e^{kL} \left[ nk - \left(k - \frac{1}{L}\right) k_x U_s \right], \text{ and} \quad (5.124)$$

$$0 = A_2 e^{-kL} \left[ \frac{gk^2(\rho_o - \rho_2)}{(n + k_x U_s)} - k\rho_2(n + k_x U_s) \right] \\ - \rho_o B_o e^{kL} \left[ nk + \left(k - \frac{1}{L}\right) k_x U_s \right] + \rho_o A_o e^{-kL} \left[ nk + \left(k + \frac{1}{L}\right) k_x U_s \right]. \quad (5.125)$$

As before, we can express (5.122) through (5.125) as the product of a matrix and the vector of coefficients  $(B_1, A_o, B_o, A_2)$ . The determinant of this matrix then gives the dispersion relation, which is fourth order in  $n$ .

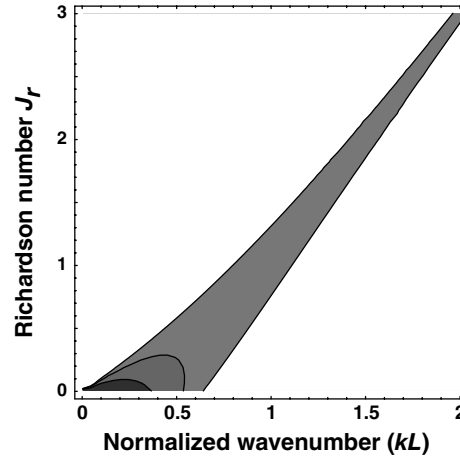
Certain quantities appear in natural combinations in these equations. It simplifies the resulting expressions to define  $\eta = kL$ ,  $\nu_g = n/(k_x U_s)$ , and  $\beta = (\rho_1 - \rho_2)/\rho_o$ . In addition, one can define a Richardson number,  $J_r$ , which measures the ratio of buoyancy to inertia, as

$$J_r = \frac{gk\eta\beta}{2k_x^2 U_s^2}. \quad (5.126)$$

For further discussion of the Richardson number, see Chandrasekhar. With these substitutions, and looking only at the modes with  $k = k_x$ , the dispersion relation becomes a fourth-order equation for the normalized growth rate  $\nu$ :

$$0 = \nu_g^4 \left[ 4\eta^2 e^{4\eta} + \frac{\beta^2 \eta^2}{4} (1 - e^{4\eta}) \right] + \nu_g^3 \beta \eta (1 - e^{4\eta}) \\ + \nu_g^2 \left[ (1 - e^{4\eta}) \left( 1 - \frac{\beta^2 \eta^2}{2} \right) + 4\eta e^{4\eta} (1 - J_r - 2\eta) \right] \\ - \nu_g \left[ \beta \eta (1 - e^{4\eta}) (1 + 2J_r) \right] - 2J_r (1 - e^{4\eta} + 2\eta e^{4\eta}) \\ + (1 - e^{4\eta}) \left[ \frac{\beta^2 \eta^2}{4} - 1 - J_r^2 \right] + 4e^{4\eta} (\eta^2 - \eta). \quad (5.127)$$

The solution of this equation, readily obtained from a computational mathematics program, has only one root with an imaginary part that is at times negative, corresponding to instability by our definition of the modes. It turns out, quite fortuitously, that the growth rate is nearly independent of  $\beta$ . As  $|\beta|$  increases, the root develops a finite real part, implying that the growing solution would oscillate once the density difference becomes larger. The growth rates can be accurately obtained from (5.127) assuming  $\beta$  to be small. This equation then becomes quadratic in  $\nu_g^2$ , and the growth rate can be displayed as contours on a plot with  $\eta$  ( $\sim kL$ ) and  $J_r$  (buoyancy) as axes. Figure 5.10 shows this.



**Fig. 5.10.** Kelvin–Helmholtz at an interface with linear velocity shear and a density transition. Working from the interior outward, the contours show a growth rate, in units of  $k_x U_s$ , of  $-0.3$ ,  $-0.1$ , and  $0$ . Surface modulations outside of the  $\nu_g = 0$  contour are damped

This plot supports a number of observations. First, if there is no gravity and thus no buoyancy, the KH instability is only unstable up to a maximum  $kL$  of about 0.65. This short-wavelength cutoff is the effect of the gradient in velocity. It says that wavelengths shorter than about  $10L$  are stabilized. As any given system evolves in time,  $L$ , being approximately  $\sqrt{\nu t}$ , increases, so that growth of the KH instability will be stopped at progressively longer wavelengths.

Second, for finite gravity, our new assumptions have introduced an additional feature that was not present in the absence of velocity shear. In the presence of a sharp interface and gravity, we found (5.117), which says approximately that  $kL > J_r$  for instability. This corresponds to the left boundary in Fig. 5.10, and determines the longest wavelength that is unstable. The right boundary is the new feature, introduced by the presence of a velocity gradient that stabilizes the shortest wavelength modes. This is the impact of a limited region of velocity shear, allowing instability for waves to which the change in velocity seems abrupt. One sees that the combination of buoyancy and shear can produce a very narrow range of unstable wavelengths.

### Homework 5.12

Suppose  $\beta$  is small enough that terms involving  $\beta$  in (5.127) can be dropped. Determine whether the two boundaries seen in Fig. 5.10 ever cross, completely eliminating the instability.

Finally, it is of interest to compare the results of these last two calculations. On the one hand, with a velocity gradient that extended over all space and a sharp density change at the interface, we found instability only in the presence of gravity and only when the interface is RT unstable ( $A_n > 0$  with our definitions). In effect, the velocity gradient acted to stabilize all the modes we would normally describe as Kelvin–Helmholtz modes. In the presence of shear but only across a transition layer, we find instability whether or not there is gravity and for either direction of the density gradient, but only over a range of wavelengths longer than some multiple of the velocity scale length. In addition, the shear acts to stabilize the RT modes, producing a long-wavelength limit like that we saw in Sect. 5.6.2.

## 5.7 Shock Stability and Richtmyer–Meskov Instability

The two instabilities we have now considered develop within some enduring state of a fluid system. Rayleigh–Taylor requires sustained acceleration, while Kelvin–Helmholtz depends on sustained shear. In both cases, an incompressible model produces an excellent conceptual model for observed phenomena, even when the actual fluids are compressible. Even so, it was clear in Chap. 4 that real high-energy-density systems nearly always involve some combination of shock waves, rarefactions, and interfaces. As a result, the structure of such a system is sensitive to the stability properties of these phenomena. This motivates the present section. The introduction of compressibility does greatly complicate the mathematics. In addition, at this writing the understanding of these phenomena is still evolving in the literature. As a result, the discussion here will be more qualitative than in some other sections of the book.

### 5.7.1 Shock Stability

We consider first what may happen to a shock wave that has lateral structure. We will describe this as a rippled shock and assume the rippling (in the  $z$  direction) to be proportional to  $\cos(kx)$ , corresponding to a two-dimensional ripple with no dependence on  $y$ . As usual, more-complex structures can be treated as a sum over such plane waves. We will also suppose that the rippling is of a long-enough wavelength that we can think about its effects using our analysis of oblique shocks in Sect. 4.1.5. A rippled shock can be produced by pushing on a fluid with a rippled piston, or by allowing a planar shock to interact with a rippled interface. Here we focus only on the shock; later we will consider the interface. We will assume the ripple to be of initial amplitude  $a_o$ , and to be small (so that  $a_o k \ll 1$ ). We will also assume the shock wave to be strong, in order to simplify the mathematics.

Figure 5.11 illustrates the deflection of the flow that occurs at a rippled shock. Here a fluid moving in the  $-z$  direction approaches a shock whose



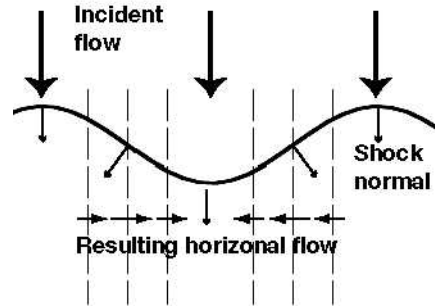


Fig. 5.11. Sketch showing the horizontal flow produced by a rippled shock

$z$  location is given (in the shock frame) by  $a = a_o \cos(kx)$ . This results in a deflection of the flow away from the shock normal. The shock normal is indicated in the figure by arrows attached to the shock. The deflection has three consequences. First, material flows toward the lagging section of the shock; the horizontal arrows in the figure indicate this component of the flow. Second, sound waves are driven in the shocked material. Third, the shock transition is affected by the change in deflection as a function of position. We consider these in turn.

The shock normal vector is shown in the figure and given to first order in  $a_o k$  by

$$\mathbf{n} = -\hat{x}(a_o k)\sin(kx) - \hat{z}. \quad (5.128)$$

$$\tan\phi_1 \approx \phi_1 = (a_o k)\sin(kx), \quad (5.129)$$

from which the small-angle formula for the deflection  $\psi$  of the flow away from the  $-z$  direction,  $\psi = \phi_1(\rho_2/\rho_1 - 1)$  gives  $\psi = 2\phi_1/(\gamma - 1)$ . As is discussed in Sect. 4.1.5, the local transverse component of the flow is unchanged by the shock, while the local normal component of the flow is reduced.

### Homework 5.13

Analyze the shock conditions for a small-amplitude ripple and show that the change due to the ripple in the  $\hat{z}$  component, relative to that from a planar shock, is second order in the ripple amplitude [i.e., generalize (5.130)].

To first order in  $a_o k$ , the immediate postshock fluid velocity is given by

$$\mathbf{u}_2 = \hat{x}u_s(a_o k)\sin(kx) - \hat{z}u_s \left( \frac{\gamma - 1}{\gamma + 1} \right), \quad (5.130)$$

in which the shock speed is  $u_s$  and the postshock velocity, in the shock frame, is  $\mathbf{u}_2$ . The lateral ( $x$ ) component of this,  $u_{2x}$ , is proportional to  $a_o k \sin(kx)$ ,

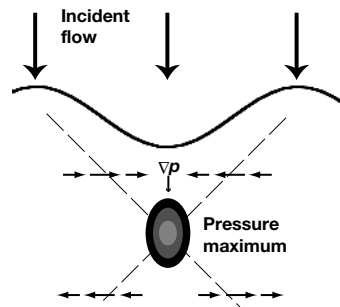
illustrated by the horizontal arrows in Fig. 5.11. We can estimate the consequences of the lateral flow by the following simple estimate. We make the oversimplified assumption that the shape of the shock remains sinusoidal and that the amplitude is reduced as material flows laterally. The rate of mass flow per unit length per unit density from the leading to the lagging sections, on each side of the minimum of  $a(x)$ , designated as  $z_{\min}$ , is approximately the wave amplitude times the transverse velocity at the mean interface position, or  $a_o u_s (a_o k)$ . This has units of area per unit time. The mass per unit length per unit density between  $z_{\min}$  and  $z = 0$ , which has units of area, within half of the lagging region of the shock, is  $a_o/k$ . If this flow of material were the only factor, one would have

$$\frac{d}{dt} a_o = -(a_o k)^2 u_s, \tag{5.131}$$

from which, with the initial amplitude given by  $a_{oo}$ , one finds

$$\frac{a_o}{a_{oo}} = \frac{1}{1 + (a_{oo} k)(k u_s t)}. \tag{5.132}$$

This corresponds to a steady decay of the ripple in the shock on a timescale given by the number of ripple wavelengths the shock has propagated. The decay is quite rapid, so that by the time the shock has propagated a few ripple wavelengths the amplitude will be greatly reduced. This result has the correct trend (rapid damping) but is wrong in two respects – the shock wave damps as  $1/t^{3/2}$  rather than  $1/t$  and it oscillates as it damps. Both effects are a result of the second factor we mentioned above, discussed next.

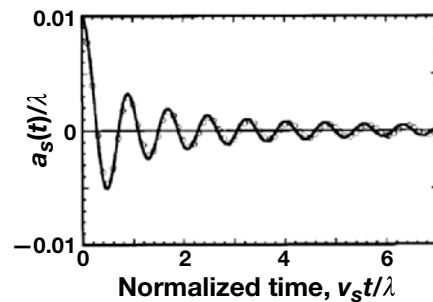


**Fig. 5.12.** The lateral flow produced by a rippled shock causes a pressure maximum to form downstream of the shock

If one considers the horizontal arrows in Figs. 5.11 and 5.12, one may note that they also correspond to a standing sound wave. As the downstream material flows away from the shock, the transverse flow of fluid will compress the material until the lateral flow stagnates, after which the fluid will recoil.

The fluid will proceed to slosh back and forth laterally as it flows away from the shock, producing a sequence of stagnations. Thus, while any shock wave is a source of sound waves, a rippled shock wave radiates an organized pattern of sound waves. Note that the first pressure maximum produced by the flow will lie underneath the lagging portion of the shock wave in Fig. 5.12. As the shock wave flattens, this will result in an upward pressure gradient. This increases the rate at which the shock flattens, and also causes the shock wave to overshoot and to oscillate.

If the formation of this pressure maximum were strictly out of phase with the flattening of the shock, then one might find that the shock wave would oscillate indefinitely. However, the pressure maximum will form in one-quarter cycle for the sound wave, which is  $\lambda/(4c_s)$ , where the wavelength of the ripple is  $\lambda$ . The shock wave does not flatten this quickly from lateral flow alone, so the pressure gradient from the sound wave acts to increase the rate of flattening but also weakens as the shock becomes flatter. As a result, the pressure gradient causes an overshoot but not to the full initial amplitude. This, however, is not simple to model mathematically. We show instead, in Fig. 5.13, numerical results showing the decay of a shock wave of amplitude  $a_o = 0.01\lambda$ .



**Fig. 5.13.** The decay of a rippled shock, adapted from Ishizaki *et al.*, Phys. Rev. E 53, R5592 (1996)

The third effect mentioned above is the change in the shock properties due to the ripple. At any given location, the shocked material is diverted toward the unshocked material, altering to some degree the conditions of the adjacent element of material as it is shocked. For ordinary materials and polytropic gases this is a small effect, but for a sufficiently pathological material it can lead the shock to become unstable. Landau and Lifshitz discuss the necessary conditions.

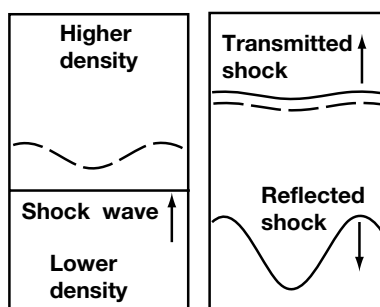
A qualitative summary is then as follows. Shocks are ordinarily stable. If they become rippled, the ripple damps out as the shock propagates. The shock oscillates as it damps (sometimes this is described as *superstable* behavior). The ripple becomes negligible as the shock propagates.

### 5.7.2 Interaction of Shocks with Rippled Interfaces

In high-energy-density experiments, one is often concerned with the interaction of a shock with an interface and with the structure that may be introduced by that interaction. Here we consider first what structures result from the interaction of a shock with a rippled interface. For modeling, we will assume that the analysis of Sect. 4.4 can be applied point by point along the ripple. However, our conclusions will be more general, as these depend mainly on the relative speed of the various waves in the problem. At the interface of interest, the density may increase or the density may decrease.

Figure 5.14 shows what usually happens when the density increases at a rippled interface. We anticipate that there will be a transmitted shock and a reflected shock, as discussed in Chap. 4. As the shock crosses the interface, the reflected shock moves backward with a faster velocity than the incoming shock. As a result, the phase of the ripple of this wave is the same as that of the interface ripple while the initial ripple amplitude on this shock is larger than the ripple amplitude on the interface; the ratio equals the ratio of reflected-shock velocity to interface velocity. For any ordinary equation of state, the postshock velocity of the interface and the transmitted shock velocity are each smaller than that of the initial shock wave. A first consequence is that the ripple on each of them remains in phase with the ripple on the interface. A second consequence is that the amplitude of the modulation decreases, in this case in proportion to the ratio of postshock interface velocity to incoming shock velocity. The velocity of the transmitted shock will typically be somewhat larger than that of the interface. As a result, the initial modulations of the transmitted shock will be somewhat larger than that of the interface, although these will damp rapidly.

There are some differences in the response when a shock reaches an interface where the density decreases. Figure 5.15 illustrates this case. We will assume that the conditions are such that there will be a transmitted shock and a reflected rarefaction wave; the specialized exceptions of Sect. 4.4.1 are

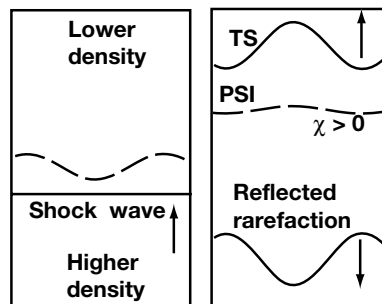


**Fig. 5.14.** Behavior when a shock reaches a rippled interface where the density increases

straightforward and we ignore those others that might correspond to a very unusual equation of state. The reflected rarefaction wave again moves faster than the shock wave, though not by much; it moves at the sound speed of the initially shocked matter. As a result, the ripple on the reflected rarefaction wave remains in phase with the ripple on the interface. Velikovich and Phillips show that such reflected rarefaction waves are weakly unstable. The perturbation amplitude of the “trailing edge” (the one near the interface) grows linearly with time. There are standing but damped sound waves, emitted downstream and propagating in the rarefaction fan toward the interface.

In contrast, the transmitted shock nearly always moves faster than the incident shock. As a result, the ripple of this shock is typically inverted in phase relative to the ripple of the interface. This is equivalent to saying that  $\alpha > 0$  in Sect. 4.4.6. However, the postshock behavior of the interface depends on whether its postshock velocity is larger or smaller than the incident shock velocity. The case  $\chi > 0$  corresponds to a larger postshock velocity, which will occur for strong shocks if the density ratio is large enough. As the density ratio becomes smaller, and depending on the EOS, eventually one will have  $\chi < 0$ , and the interface will not be inverted as the shock passes.

These dynamics of the rarefaction wave have interesting consequences for experiments that view such behavior from the side (from above the page here). It is common in such experiments to use a radiographic diagnostic whose x-rays are preferentially absorbed by the material on one side of the interface. If the downstream side is diagnosed in this way, then the signal decreases exponentially as the areal density (mass/area) increases along any given line of sight. In consequence, the diagnostic may be more sensitive to the higher-density material near the head of the rarefaction wave (if the transmission is relatively high) or to the lower-density material near the interface (if the diagnostic x-rays are strongly absorbed). Thus, for large density ratios (so  $\chi > 0$ ) the structure that the diagnostic detects may be in phase or out

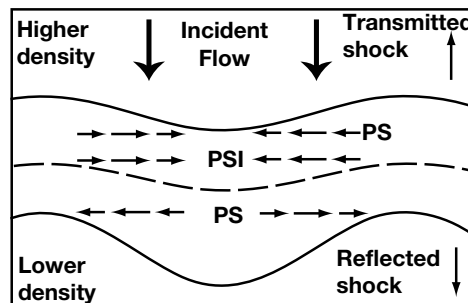


**Fig. 5.15.** Behavior when a shock reaches a rippled interface where the density decreases. The transmitted shock is labeled TS and the postshock interface is labeled PSI

of phase with the modulations at the interface, or may change phase within the image.

### 5.7.3 Postshock Evolution of the Interface; Richtmyer Meshkov Instability

At this point we have seen that modulations in shock waves are typically damped, so modulations introduced by an interface will die away in time. We have also seen that the heads of rarefaction waves are stable, so modulations introduced by an interface will not grow further. The remaining issue involved in understanding such systems is the postshock evolution of the interface. It turns out that structure on the interface grows in time after the shock passes. The origin of this is easily seen in Fig. 5.16. This figure is centered on a trough in the interface modulation. The behavior of the shock waves is clear from our discussion above: the reflected shock wave deflects the flow away from the trough, while the transmitted shock wave deflects material toward the trough. As the shock crosses the interface, the postshock velocity slows. This causes a deflection of the flow away from the normal to the interface or toward the trough. This corresponds to  $\alpha < 0$  in the analysis of Sect. 4.4.6. Thus, one can see that this postshock flow of material will deepen the valleys and raise the peaks of the initial modulations.



**Fig. 5.16.** Lateral velocities after a shock interacts with an interface where the density increases

This process is known as the Richtmyer–Meshkov instability. We will designate it by the initials RM. This label is firmly entrenched in the literature, but strikes the author of this book as problematic. One definition of an instability is “a process through which the rate of increase of the magnitude of some physical quantity increases in time“. This is typified by the equation  $df/dt = \gamma f$ , in which  $\gamma$  is the (exponential) growth rate of the quantity  $f$ . Instabilities by such a definition inherently involve some feedback mechanism. In the case of Rayleigh–Taylor, for example, this mechanism is the increase

in the net buoyant force resulting from increased interpenetration of the two materials. By this definition the RM process is not an instability. A weaker definition of an instability, which RM would satisfy, is “a process through which the magnitude of some physical quantity increases in time.” However, many other phenomena, not described as instabilities, would satisfy this definition. The main point of this discussion is that, unlike every other process described in this book as an instability, one will search in vain for a feedback mechanism that increases the rate of RM growth.

There is also a further conceptual difficulty associated with the RM process. The RM instability is often described as the impulsive limit of the RT instability, from the point of view that it corresponds to the limit of RT as the variation of the acceleration in time approaches the delta function corresponding to the shock. This description originates with the original paper of Richtmyer in 1960, but he and others have recognized problems with it. The evolution of the structure occurs after the shock passes, and thus is not the limiting case of growth that occurs during acceleration. Correspondingly, one does not do theory of RM by taking a limit of RT theory. In addition, as Velikovich described in 1996, one can produce RM, at least in principle, in a system with two rarefaction waves and no initial acceleration of the interface.

Independent of this history, the best intuitive sense of the RM process can be found by thinking of the flow that will develop following the initial conditions created by the shock. For the case of Fig. 5.16, we can use this intuition and the analysis of Sect. 4.4.6 to develop a semiquantitative description of the growth of modulations, as follows.

The four regions seen in Fig. 5.16 correspond from bottom to top to regions *a*, *R*, *d*, and *c* of Fig. 4.30 and Sect. 4.4.6. The small-angle limits of (4.127) and (4.130) and give

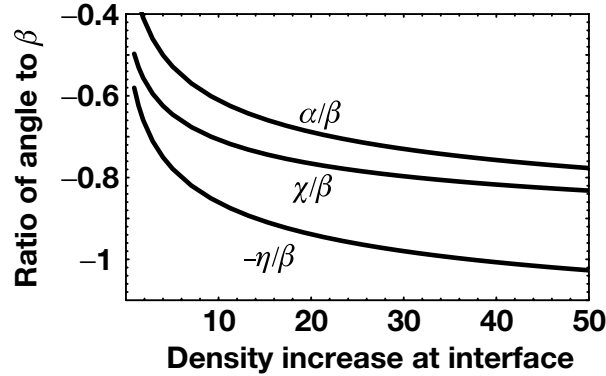
$$\chi = \frac{2\alpha - (\gamma - 1)\beta}{\gamma + 1}, \text{ and} \quad (5.133)$$

$$\chi = \frac{2\eta(\gamma + 1) - \beta(\gamma - 1)^2}{(\gamma + 1)^2}, \quad (5.134)$$

respectively. Recall that  $\beta$  is the angle between the initial shock normal and the local interface normal. Setting the pressures in the two postshock regions equal, in the small-angle limit, gives

$$\frac{\rho_c}{\rho_b} \left(1 + \frac{\alpha}{\beta}\right)^2 = \frac{\gamma + 1}{\gamma - 1} \left(\frac{\gamma - 1}{\gamma + 1} + \frac{\eta}{\beta}\right)^2, \quad (5.135)$$

These three equations can be solved to find the ratio of  $\alpha$ ,  $\eta$ , and  $\chi$  to  $\beta$  as a function of  $\gamma$  and the density ratio at the interface,  $\rho_c/\rho_b$ . Also note that in the geometry of (5.128) and for small  $\beta$  one has  $\beta = (a_0 k) \sin(kx)$ . One ends up, for any given value of  $\gamma$ , with a figure like Fig. 5.17.



**Fig. 5.17.** The dependence of the postshock angles of deflection on the density ratio at an oblique interface where the density increases, for strong shocks. The angles of the transmitted shock, the interface, and the reflected shock are  $\alpha$ ,  $\chi$ , and  $\eta$ , respectively, with  $\eta$  defined in the opposite direction in Sect. 4.4.6

#### Homework 5.14

Solve (5.133) through (5.136) to find the ratio of  $\alpha$ ,  $\eta$ , and  $\chi$  to  $\beta$ . Plot the results for various values of  $\gamma$  and comment on what you observe.

Equations (4.125) and (4.128), after transformation back into the lab frame, give the lateral deviation of the flow as the x-component of the velocity vectors  $\mathbf{u}_{d\text{lab}}$  and  $\mathbf{u}_{R\text{lab}}$ . In the small-angle limit, these are

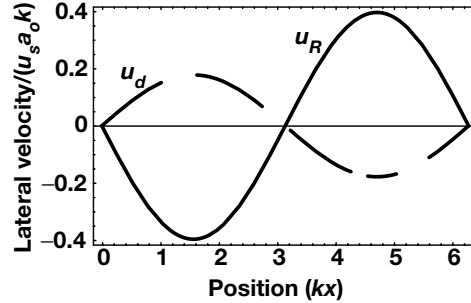
$$\mathbf{u}_{d\text{lab}} = \frac{-2\beta}{(\gamma+1)}u_s \left[ \frac{\alpha}{\beta} \left( 1 + \frac{\alpha}{\beta} \right) \right] \hat{x} + \frac{2}{(\gamma+1)}u_s \left[ 1 + \frac{\alpha}{\beta} \right] \hat{y} \quad \text{and} \quad (5.136)$$

$$\begin{aligned} \mathbf{u}_{R\text{lab}} = & \frac{2\beta}{(\gamma+1)^2}u_s \left[ \frac{\eta}{\beta} \left( (\gamma-1) + \frac{\eta}{\beta}(\gamma+1) \right) \right] \hat{x} \\ & + \frac{u_s}{(\gamma+1)^2} \left[ 4\gamma - 2\frac{\eta}{\beta}(\gamma+1) \right] \hat{y}, \end{aligned} \quad (5.137)$$

respectively. By substituting the solution to (5.133) to (5.135) into these two equations, one can find the deviations in velocity introduced by the shocks. For any given value of the density ratio  $\rho_c/\rho_b$ , one can plot the lateral deviation as a function of distance as Fig. 5.18 shows.

Referring again to Fig. 5.16, one can see that there is a significant difference between the behavior at the interface and the behavior at the shock. The flow at the interface acts to increase (rather than to decrease) the size of the perturbation. We can make an approximate calculation, similar to the one we did for the rippled shock, to estimate how rapidly the ripple amplitude will increase. An important difference is that the lateral velocity is set by the initial amplitude and does not evolve further as the ripple changes,





**Fig. 5.18.** The lateral flow velocities produced by a shock at an interface where the density increases are shown. This circumstance produces shear flow at the interface

except perhaps due to the effects of the sound waves emanating from the shock waves, which are not accounted for in the present estimate. The lateral velocity produced by the reflected shock tends to be the larger of the two, so we will assume that it is responsible for the flow. Using this velocity to be  $u_{\text{perp}}$ , taking  $\eta/\beta \sim 0.9$  as a typical value, and recalling that  $\beta = a_{oo}k$  for small angles, one can show

$$u_{\perp} \approx \frac{3.4\gamma}{(\gamma+1)^2} (a_{oo}k) u_s. \quad (5.138)$$

From the point of view that this inward flow of material from each side produces a corresponding increase in the full amplitude ( $2a_o$ ) of the ripple, we then estimate

$$\frac{d}{dt} a_o = \frac{3.4\gamma}{(\gamma+1)^2} k u_s a_{oo}. \quad (5.139)$$

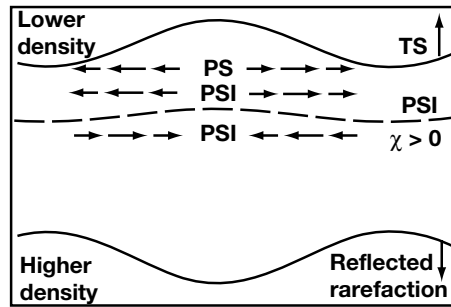
Key qualitative features of this estimate are that the interface ripple grows linearly in time, and that the rate of growth is proportional to the initial normalized amplitude,  $a_{oo}k$ . These features are present in more sophisticated estimates in the literature, such as the widely used formula due to Richtmyer,

$$\frac{d}{dt} a_o = k A^* u_{\text{ps}} a_{\text{ps}}, \quad (5.140)$$

in which the postshock velocity of the interface in the lab frame is  $u_{\text{ps}}$ , the postshock amplitude of the ripple is  $a_{\text{ps}}$ , and the postshock Atwood number at the interface is  $A^*$ . However, no such formula is completely general and cases where this one does not work have been identified.

We have just discussed Richtmyer–Meshkov growth as a consequence of the lateral velocities induced by the shock. An alternative way to view such growth is in terms of the *vorticity* that the shock generates. In fluid dynamics the vorticity is  $\nabla \times \mathbf{u}$ . An initial, uniform postshock flow has no vorticity, but vorticity is generated as the shock passes a rippled interface. The shock

produces tubes of vorticity with alternating direction when it passes an interface with the two-dimensional structure we have been discussing. Vorticity corresponds to rotational flow; it is conserved in the absence of dissipation. Thus, another way to view the growth of the ripple amplitude is that the rotation in opposing vortex tubes funnels material into the maxima of the ripple.



**Fig. 5.19.** Lateral velocities after a shock interacts with an interface where the density decreases. Here PSI and PS stand for postshock interface and postshock, respectively

Now we turn to the behavior when a shock reaches a rippled interface where the density decreases. This can be analyzed in a similar way, using the equations for a rarefaction from Sect. 4.4.6. Figure 5.19 illustrates the qualitative behavior, for the case of a large enough density decrease that  $\chi > 0$ . One can see that here again the amplitude of the ripple at the interface will grow with time. The qualitative behavior is the same in this case as in the previous one. A small ripple grows linearly with time, with the rate of increase of the amplitude being proportional to the initial normalized amplitude and the shock velocity. In the language of vorticity, the shock wave deposits vorticity on both sides of the interface. After this, the transmitted shock continues to deposit vorticity in the newly shocked fluid. In contrast, the rarefaction wave does not deposit additional vorticity in the fluid it affects.

If the density decrease is smaller, so that  $\chi < 0$ , then the interface at first retains the phase of the initial ripple. The flows then cause the interface to invert before the ripples grow larger. For this case, there is a standard theoretical estimate due to Meyer and Blewett, which involves the average of  $a_{oo}$  and  $a_{ps}$ . It is

$$\frac{d}{dt}a_o = kA^*u_{ps} \frac{1}{2}(a_{ps} + a_{oo}), \tag{5.141}$$

in which one should note that  $A^* < 0$ .

**Homework 5.15**

Evaluate the small-angle limit of the equations for a shock at an oblique interface with a density decrease, and produce a plot similar to Fig. 5.19 for this case.

**Homework 5.16**

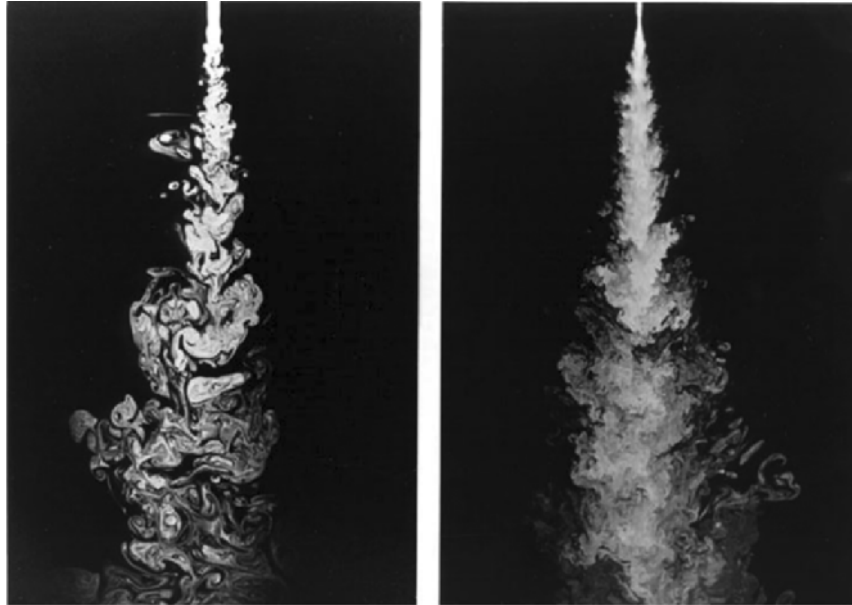
Consider the qualitative behavior of the postshock interface when there is a rarefaction but  $\chi < 0$ . Redraw Fig. 5.19 for this case. Discuss the evolution of the interface.

In more complete theoretical treatments of the RM process, one finds that the growth at the interface can be expressed as a consequence of the sound waves generated by the shocks near it. Both transmitted and reflected shock waves contribute in this way. A reflected rarefaction, however, does not produce sound waves that affect the interface growth.

**5.8 Hydrodynamic Turbulence**

We often see phenomena that one might describe as turbulent. This is particularly true when two distinctly observable fluids, such as clouds and air or cream and coffee, mix. But flow in a single fluid also can become in some sense turbulent, as does the airflow behind an airplane wing or a racecar. There are a number of possible definitions of *turbulence*, and one finds that the word has distinctly different meaning in different areas of plasma physics and hydrodynamics. As a result, when reading a wide range of literature one should be somewhat wary of this term. A fairly general definition of turbulence is “the presence of structures having a range of spatial scales that are smaller than the spatial scales of the motions that provide the energy source producing the structures.” This may not satisfy the extreme hydrodynamicist, who may insist that a system to be turbulent must have evolved to a state that is independent of its initial conditions. Whether such a state is practically realizable is not so clear. Here we will not trouble ourselves further with definitions. Rather, we will examine the properties of hydrodynamic systems that lead to the presence of structure over a range of small spatial scales. The book by Tennekes and Lumley provides an excellent introduction to hydrodynamic turbulence. They emphasize that such turbulence is a property of fluid flows and not of the underlying fluid itself.

The basic notion behind descriptions of turbulence is that energy is introduced to a system by some process, such as the RT or KH instability, that this enables processes to occur which produce much smaller-scale fluctuations, and that these fluctuations eventually lose their energy by viscous dissipation. Because vorticity once generated spreads by viscous diffusion and is removed only through viscous heating, swirling patterns of motion generally



**Fig. 5.20.** Images of a slice through round turbulent jets in liquids, illuminated by a laser (Dimotakis, 2005). (a)  $Re \approx 2,500$ . (b)  $Re \approx 10,000$

characterize turbulent systems. Indeed, the presence of varying patterns of vorticity on a range of spatial scales is considered to be an essential property of turbulent hydrodynamic flows.

A simple example of turbulent flows are the jets shown in Fig. 5.20. The Reynolds number  $Re$  increases from left to right in the figure. In all similar cases, the jet first produces the KH instability. This sets the stage for further instabilities and for the development of smaller scale structures. At low  $Re$ , the flow remains dominated by large-scale structures. As  $Re$  increases, the flow develops finer-scale structures and the distribution of these structures becomes more uniform.

As we shall see, turbulence is typically a property of flows with large  $Re$ . This may seem paradoxical at first. In Chap. 2 we found that  $Re$  is the ratio of the convective momentum transport to viscous momentum transport. We argued that  $Re$  is nearly always large in high-energy-density flows, and correspondingly that the Euler equations are typically a good basis for analysis. In detail, however, this argument works only for phenomena whose spatial scale is not too small. One can construct a Reynolds number from any sensible length and velocity scales so that different aspects of a system can have different Reynolds numbers. If we focus on small-enough spatial scales, the

corresponding phenomena do experience strong dissipation and cannot be described by the Euler equations. This is what makes it possible for a turbulent flow to dissipate energy.

Before we proceed with some discussion of incompressible hydrodynamic turbulence, it is worthwhile to consider how this might apply to plasma systems. In a sufficiently collisional plasma, the fluid motion is hydrodynamic and collisional damping dominates the dissipation of energy. However, as the plasma becomes less collisional, collective effects begin to occur and the compressive fluctuations in the plasma begin to produce significant electric fields. These electric fields accelerate particles, providing a source of energy dissipation that is distinct from viscous effects. At this writing, the competition between these sources of dissipation is not well understood. This competition could alter the structure of turbulence in plasmas as compared to that in purely hydrodynamic fluids.

Returning to the point of view that turbulent flows are dominated by rotating motions that we can call *vortices*, we can idealize these motions, using the (oversimplified) model of rotating toroids. These donut-shaped structures rotate about the axis of the donut. They can have any aspect ratio, being thin rings, fat rings, or elongated and nearly cylindrical structures. The rotation is essential, though, as this is what makes  $\nabla \times \mathbf{u}$  nonzero so that there is vorticity. Such vortices can have a range of sizes but the smallest possible vortex is one that is damped by viscosity in of order one rotation. If we use  $w$  to represent the rotational velocity and  $\lambda$  to represent the diameter of the vortex, then the rotational timescale is  $\lambda/w$  while the timescale for viscous damping is  $\lambda^2/\nu$ , where  $\nu$  is again the kinematic viscosity. Setting these timescales equal gives  $w\lambda/\nu \sim 1$  for the smallest vortex. Thus, the Reynolds number constructed from the characteristic scales of the smallest vortex is of order unity. The reader may recall from Chap. 2 that the typical Reynolds number describing high-energy-density flows is at least several orders of magnitude larger than 1. The consequence is that the smallest vortices are some orders of magnitude smaller than the characteristic scale of the entire system.

A next step in this discussion is to consider the overall rate at which damping must dissipate energy. The largest vortices produced in the system are known as the *eddies*. The eddies typically span the turbulent zone. In simple cases such as KH or RT they are created through the evolution of the large structures generated by these instabilities. We will designate the characteristic speed of the material in the eddy as  $w_e$  and the characteristic diameter of the eddy as  $\ell$ . An observed property of turbulence, and an assumption in traditional turbulence theory, is that these large eddies dissipate their energy on a timescale of order 1 circulation time. Their specific energy is of order  $w_e^2$ , so the specific energy dissipation rate (a power per unit mass) is of order  $w_e^3/\ell$ . We will designate this turbulent dissipated power by  $P_t$ . However, the inherent viscous damping of these structures is small. To see this, one can note that their viscous timescale is  $\ell^2/\nu$ , so their viscous

damping rate is  $w_e^2\nu/\ell^2$ . Thus, the ratio of the viscous damping rate to  $P_t$  is  $\nu/(w_e\ell) = 1/Re$ . Taken together with the previous paragraph, the implication is that dynamical processes must create small-scale structures to dissipate the energy deposited in the eddies by the global processes in the system (such as KH or RT).

The order-of-magnitude size of these smallest structures is one of the *Kolmogorov scales*. These are the length scale,  $\eta_k$ , the time scale,  $\tau_k$ , and the velocity scale,  $u_k$ , that can be constructed from the specific energy dissipation rate and the viscosity. One has

$$\eta_k = (\nu^3/P_t)^{1/4}, \quad \tau_k = (\nu/P_t)^{1/2}, \quad \text{and} \quad u_k = (\nu P_t)^{1/4}. \quad (5.142)$$

To see how small these scales are physically, one can substitute for  $P_t$  and obtain results in terms of the Reynolds number corresponding to the eddies,  $w_e\ell/\nu$ . This gives

$$\eta_k = \ell/Re^{3/4}, \quad \tau_k = (\ell/w_e)/Re^{1/2}, \quad \text{and} \quad u_k = w_e/Re^{1/4}. \quad (5.143)$$

Thus, for a Reynolds number of  $10^5$ , the size of the smallest vortex will be of order 6,000 times smaller than the size of the largest eddy. Note that the Reynolds number corresponding to the Kolmogorov scales satisfies the condition we developed above, having  $\eta_k u_k/\nu = 1$ .

We will explore how structures can form on such scales below. To prepare for this, we first will compare them with some other characteristic dimensions and then discuss the dynamics of the fluid in more detail. An eddy may initially form with sharp edges, as during the roll-up produced by Kelvin–Helmholtz, as at the spike tips in Fig. 5.1 and at the shear layer in Fig. 5.7. As the eddy evolves, viscous diffusion smoothes the edge of the shear layer (see Sect. 5.6.3). This diffusion produces a laminar boundary layer within which there is a finite transverse gradient of the velocity. The scale length of this boundary layer is  $\sqrt{\nu t}$ . On the timescale of the large eddy,  $\ell/w_e$ , this boundary layer scale length is thus  $\ell/\sqrt{Re}$ . Comparing this with (5.143), one sees that the Kolmogorov scale length where the dissipation occurs is smaller than the boundary-layer scale length, and that the difference between them increases as  $Re$  increases. Externally driven instabilities do not readily occur on the small scales that exist within such a boundary layer, where there is a continuous gradient in flow velocity. KH, for example, is stabilized by this gradient. Thus, the fluctuations within the boundary layer should evolve through local fluid dynamics until they dissipate. The full thickness of the boundary layer, throughout which the velocity gradients may limit the instabilities, is a few times  $\ell/\sqrt{Re}$ .

To be able to go further in our description, we need to work with the fluid equations. The relevant equations are (2.27) for momentum and the mechanical energy equation that can be constructed from it by taking the dot product with  $\mathbf{u}$ , keeping the terms involving viscosity but dropping all

the terms involving radiation or other forces. One also assumes incompressibility and for simplicity assumes constant  $\rho$ , constant  $\nu$ , and that the second coefficient of viscosity is 0. Then one has

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = \frac{-1}{\rho} \nabla p + 2\nu \nabla \cdot \underline{\mathbf{s}}, \text{ and} \quad (5.144)$$

$$\frac{1}{2} \frac{\partial u^2}{\partial t} + \mathbf{u} \cdot \nabla \left( \frac{u^2}{2} \right) = \frac{-1}{\rho} \mathbf{u} \cdot \nabla p + 2\nu \nabla \cdot (\mathbf{u} \cdot \underline{\mathbf{s}}) - 2\nu (\underline{\mathbf{s}} \cdot \cdot \underline{\mathbf{s}}). \quad (5.145)$$

in which one has used the incompressibility condition to simplify  $\underline{\boldsymbol{\sigma}}_\nu$  from (2.36), and defined the *strain rate tensor* as  $\underline{\mathbf{s}}$ , given by

$$\underline{\mathbf{s}} = \frac{1}{2} (\nabla \mathbf{u} + (\nabla \mathbf{u})^T). \quad (5.146)$$

With the elements of  $\underline{\mathbf{s}}$  given as  $s_{ij}$ , the expression  $\underline{\mathbf{s}} \cdot \cdot \underline{\mathbf{s}}$  is the sum over both indices of  $s_{ij}s_{ij}$ .

### Homework 5.17

Develop (5.144) and (5.145) from the equations in Chap. 2.

Since fluid turbulence develops within a fluid flow, it is useful to analyze these equations as the sum of terms describing the mean flow and terms describing the (turbulent) fluctuations. We take  $\mathbf{u} = \mathbf{U} + \mathbf{w}$ ,  $p = P + \delta p$ , and  $\underline{\mathbf{s}} = \underline{\mathbf{S}} + \delta \underline{\mathbf{s}}$ , in which the first, uppercase quantity is the mean value and the second term is the fluctuating term. We substitute these definitions into (5.144) and (5.145) and average over a time long compared with the dissipation time for any specific eddy. The fluctuating terms average to zero individually. However, products of the fluctuating quantities do not, in general, average to 0, but instead have values depending on the degree of correlation of these quantities. Indicating such an average by an overbar, and assuming the overall system to be in steady state for simplicity, we obtain equations for the mean flow and for the turbulent fluctuations. The equation for the momentum and energy of the mean flow are

$$\rho(\mathbf{U} \cdot \nabla) \mathbf{U} = \frac{-1}{\rho} \nabla P + 2\nu \nabla \cdot \underline{\mathbf{S}} - \nabla \cdot (\rho \overline{\mathbf{w}\mathbf{w}}) \quad \text{and} \quad (5.147)$$

$$\mathbf{U} \cdot \nabla \left( \frac{U^2}{2} \right) = \nabla \cdot \left( \frac{-P}{\rho} \mathbf{U} + 2\nu \mathbf{U} \cdot \underline{\mathbf{S}} - \overline{\mathbf{w}\mathbf{w}} \cdot \mathbf{U} \right), \quad (5.148)$$

$$-2\nu (\underline{\mathbf{S}} \cdot \cdot \underline{\mathbf{S}}) + \overline{\mathbf{w}\mathbf{w}} \cdot \cdot \mathbf{S}$$

respectively, while the equation for the turbulent energy is

$$\mathbf{U} \cdot \nabla \left( \frac{w^2}{2} \right) = -\nabla \cdot \left( \frac{1}{\rho} \overline{\mathbf{w}\delta p} - 2\nu \overline{\mathbf{w} \cdot \delta \underline{\mathbf{s}}} + \frac{1}{2} \overline{w^2 \mathbf{w}} \right). \quad (5.149)$$

$$-\overline{\mathbf{w}\mathbf{w}} \cdot \cdot \underline{\mathbf{S}} - 2\nu \overline{\delta \underline{\mathbf{s}} \cdot \cdot \delta \underline{\mathbf{s}}}$$

If one pursues the literature of fluid dynamics, one finds various intermediate quantities that are given names and contribute to the jargon. These include, for example, the general *stress tensor*, which includes both pressure and viscous stress terms. We are now prepared to discuss the dynamics of the turbulent flow in more detail.

### Homework 5.18

Derive (5.147) through (5.149).

First consider the mean momentum. Applying the scaling analysis discussed in Sect. 2.3 shows that the viscous term in (5.147) is of order  $1/Re$  relative to the convective term and so typically is negligible. The rightmost term in this equation is the divergence of a tensor. This term describes the forcing of the mean flow by the fluctuations. This tensor, having first been developed by Reynolds in 1895, is known as the *Reynolds stress tensor*. It quantifies the effects of the turbulent fluctuations on the mean flow. Unfortunately, the magnitude of this term is not obvious. The turbulent velocity  $w$  should be smaller than  $U$  and the eddy diameter  $\ell$  smaller than the global scale length of the flow  $L$ . In actual turbulence it is common to see  $w/U \sim \ell/L \sim$  a few percent. This has the implication that the final term in (5.147) is a few percent of the term on the left-hand side. This is often much larger than the viscous loss term but remains small enough that the flow changes gradually on the scale of  $L$ .

The story is similar with regard to the mean energy, as expressed by (5.148). The viscous terms are of order  $1/Re$  relative to the energy. The terms involving the correlations of the turbulent velocities are typically much larger, but remain small enough that the energy of the mean flow is only gradually reduced.

The equation for the turbulent energy (5.149) has more to tell us. The first line can be rearranged as the divergence of a vector that contains energy fluxes,  $p dV$  work, and energy transport by viscous stresses. These sum to zero over any volume within which the fluctuating mechanical energy is unchanging, as will be the case in steady turbulence. The second line of this equation includes the terms identified with the production of turbulent energy (the first term) and with the dissipation of turbulent energy (the second term). In a crude scaling sense  $S \sim U/L \sim w_e/\ell$ , so the magnitude of the first term in the second line is  $w_e^3/\ell$ . This equals the specific energy dissipated by turbulence that we obtained above from general arguments.

The final term in (5.149) represents the dissipation by the turbulence, and is the significant new result we obtain from this analysis. Assuming the turbulence to be isotropic, one can show with some algebra that

$$2\nu\overline{\delta\mathbf{s} \cdot \delta\mathbf{s}} = 15\nu\overline{(\partial w_1/\partial x_1)^2}, \quad (5.150)$$

in which the subscript 1 designates the first vector component. (For a detailed development of the equations related to turbulence, see Hinze.) Fluctuations



at all spatial scales contribute to  $\delta s$ , so that  $\delta s$  is not given by  $w/\ell$ . Instead, the length-scale associated with  $\delta s$  must be smaller than  $\ell$ . Defining this length scale as  $\lambda_T$ , and balancing the production of turbulence with its dissipation, we have

$$w^3/\ell = 15\nu w^2/\lambda_T^2, \text{ so that} \quad (5.151)$$

$$\lambda_T/\ell = \sqrt{15}/Re^{1/2}. \quad (5.152)$$

The length scale  $\lambda_T$  is known as the Taylor microscale, named after G.I. Taylor, who first defined it. However, the curious aspect of the Taylor microscale is that it is not a physical distance that characterizes the turbulence. Rather, it is the maximum size at which the energy from the large eddies can be dissipated by viscosity, if the turbulent fluctuation velocity does not change as the scale decreases. Tennekes and Lumley would prefer to see  $\lambda_T$  used only in combination with  $w$  to give a dissipation rate. Vortices at this scale are able to dissipate all the energy from the eddies. Thus, this is a reasonable estimate of the scale below which the behavior is not influenced by the large-scale dynamics that drives the eddies. However, this dissipation cannot occur within one vortex circulation timescale for eddies with rotation speed  $w$  and size scale  $\lambda_T$ . Thus, such vortices fail to satisfy the assumption that any turbulent vortex dissipates its energy in one turnover time. If this presumed property of turbulence, which is experimentally supported, is valid, smaller vortices will continue to form until the Kolmogorov scale is reached.

A notable feature of the Taylor microscale is that it is of the same order as the laminar boundary layer thickness developed during one eddy timescale, discussed above. Any structures that endure for an eddy timescale will develop such boundary layers. This adds a physical basis to the notion that the global unstable dynamics might have limited effects below this scale. To emphasize this correspondence, Dimotakis calls this the Liepmann–Taylor microscale.

Thus, we have a picture in which the fluid dynamics causes the transfer of energy to smaller-scale vortices, although we have yet to discuss the dynamics of this. The result is that the global unstable dynamics produces eddies and also produces vorticity, at least on surfaces. The vorticity spreads by viscous diffusion. The energy created by the global dynamics is transferred to smaller and smaller vortices, which become independent of the global processes once the vortex size drops below the Taylor microscale. The energy is then eventually dissipated when it reaches the Kolmogorov scale. As  $Re$  increases, the Taylor microscale and Kolmogorov scale become increasingly separated.

In between the Taylor microscale and the Kolmogorov scale, only the dynamics of the vortices governs the flow of energy to smaller scales. Since the equations do not depend fundamentally on the scale, one might hope to find a fairly simple scaling for the changes in the eddy properties as their diameter decreases. Working from a fairly general set of limiting assumptions, reviewed, for example, in Chap. 8 of Tennekes and Lumley, Kolmogorov showed that the

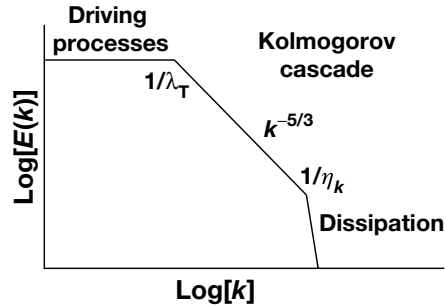


Fig. 5.21. Structure of the turbulent spectrum of kinetic energy

wavenumber spectrum of the kinetic energy,  $E(k)$ , is proportional to  $k^{-5/3}$ . The meaning of this statement is that  $E(k)dk$  is the kinetic energy of the vortices whose characteristic wavenumber is within  $dk/2$  of  $k \sim 2\pi/\lambda$ , and that this is proportional to  $k^{-5/3}$ . The corresponding scaling of the fluctuating strain rate is  $\delta s \propto k^{2/3}$ , so the fluctuating strain rate is largest at the smallest scales. The exponent of  $5/3$  is not a universal constant; one will see other values in both theory and observations for systems that satisfy assumptions different from those used by Kolmogorov.

Figure 5.21 illustrates the qualitative structure of the resulting spectrum of  $E(k)$ . The spectrum at the lowest wavenumbers is determined by the processes that create the turbulent energy, in addition to the processes that transfer it to smaller scales. Thus, the structure of this part of the spectrum may vary with conditions. It is shown as flat in the figure. As  $k$  increases so that the vortex size  $\lambda$  drops below  $\lambda_T$ , the spectrum becomes a *Kolmogorov spectrum*, with a slope of  $-5/3$ . Then as the vortex size approaches the Kolmogorov dissipation scale, the energy is dissipated and  $E(k)$  decreases more rapidly. On the basis of a review of data, *Dimotakis* concludes that dissipative effects begin to alter the spectrum for vortex scales  $\lambda \leq 50\eta_k$ . The region of the spectrum that has a power-law shape is known as the *inertial range*. This reflects the point of view that the inertial dynamics of the fluid is responsible for producing this part of the spectrum. Based on this discussion, an inertial range should appear once  $Re$  becomes large enough that  $50\eta_k < \lambda_T$ . This requires a value of  $Re$  above about  $10^4$ .

An important feature of turbulent systems is the presence or absence of a *mixing transition*. This is generally observed to occur at some value of  $Re$  which depends on details. Once the mixing transition has occurred, the turbulence causes rapid mixing of the two interacting fluids and rapid diffusion of each into the other. This transition is of significant practical importance for systems such as chemical processors, intended to generate copious interactions between the molecules in two fluids. This may also be important for high-energy-density systems, as one may in various contexts desire to encourage or to discourage such mixing. A conjecture due to *Dimotakis* is that the

mixing transition corresponds to the development of an inertial range, and that achieving  $Re > 10^4$  is a necessary condition for this development.

We close this section with a brief discussion of the dynamics that produces the flow of energy from larger to smaller vortices. To see these dynamics, it is useful to recast the momentum equation, (5.144) in two ways. We keep all the same assumptions including incompressibility. First, one can manipulate the convective derivative and the viscous term to highlight the effect of the vorticity  $\boldsymbol{\omega} = \nabla \times \mathbf{u}$  in this equation. We obtain

$$\frac{\partial \mathbf{u}}{\partial t} = -\nabla \left( \frac{p}{\rho} + \frac{u^2}{2} \right) + \mathbf{u} \times \boldsymbol{\omega} - \nu \nabla \times \boldsymbol{\omega}. \quad (5.153)$$

This equation illustrates one important effect of vorticity. The fluid velocity is redirected in the direction of  $\mathbf{u} \times \boldsymbol{\omega}$ . This is easy to understand if one returns to our analogy that vortices are spinning donuts. Recall the behavior of topspin shots in tennis, curve balls in baseball, or slice shots in golf. In all these cases, a spinning object creates lift by increasing the flow velocity on one side and decreasing it on the other, which creates a pressure difference through the Bernoulli effect. Thus, vortices redirect the flow in a direction perpendicular to the flow and to the vorticity vector.

Second, we can take the curl of (5.144) to develop a dynamic equation for the vorticity itself. This gives

$$\frac{\partial \boldsymbol{\omega}}{\partial t} = \nabla \times (\mathbf{u} \times \boldsymbol{\omega}) + \nu \nabla^2 \boldsymbol{\omega}. \quad (5.154)$$

This equation is identical to (2.48), describing the behavior of the magnetic field. The analogy between vorticity and magnetic field is often exploited for both physical explanations and mathematical analysis. Here we note that the vorticity will diffuse if there is no net flow or if its spatial scales are small enough. Otherwise, the first term on the right-hand side causes the vortices to move with the flow. Thus, in the same sense that magnetic field in a plasma is *frozen in*, the vorticity in a fluid is frozen in. In addition, this equation makes it clear that the role of viscosity with regard to momentum is to create diffusion. This transfers momentum (and vorticity) from the structure being damped into the surrounding fluid. The energy involved in vortex motion can be dissipated by its conversion to thermal energy, described by the viscous heating term in the energy equation.

### Homework 5.19

To be more precise about this point, one should recognize that what moves with the fluid is the vorticity passing through a surface  $S$ . Prove this by taking the time derivative of the integral of  $\boldsymbol{\omega} \cdot d\mathbf{S}$  over a surface  $S$  that moves with the fluid and may change its shape in time. Relate the result

to (5.154). Hint: The key here is the evaluation of the partial derivative in time of the surface as a contour integral involving the edge of the surface.

The consequence of (5.153) and (5.154) together is that vortices do not allow a fluid to flow through them undisturbed. They deflect the fluid, and stay with it to deflect it further. However, it remains the case that changes in the flow can affect the local value of  $\boldsymbol{\omega}$ . This is due to the effects of the gradients in velocity. To see these effects more clearly, it is helpful to recast (5.154) as follows:

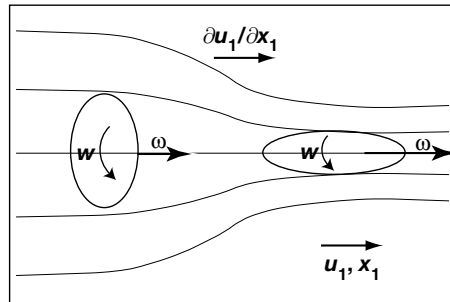
$$\frac{\partial \boldsymbol{\omega}}{\partial t} + \mathbf{u} \cdot \nabla \boldsymbol{\omega} = \boldsymbol{\omega} \cdot \underline{\mathbf{s}} + \nu \nabla^2 \boldsymbol{\omega} . \quad (5.155)$$

Here we see that only if the strain rate is zero (and viscous damping remains negligible) does the local vorticity move with the fluid without changing. To see what kinds of changes may occur, we consider the effects of finite strain rate on the vortices.

First, the off-diagonal elements of  $\underline{\mathbf{s}}$  act to rotate the vortices. This is simple to understand. We discussed the lift generated by the interaction of  $\boldsymbol{\omega}$  and  $\mathbf{u}$ . If  $\mathbf{u}$  changes along the vortex, then the lift varies along the vortex, which will produce a torque and cause a rotation of it. Since vortices generally involve derivatives of  $\mathbf{u}$  in all three directions, the distribution of vorticity tends to become isotropic as vortices come to dominate the dynamics. As a result, small-scale turbulence is typically isotropic even when the driving instability at the global scale may not be.

Second, the diagonal elements of  $\underline{\mathbf{s}}$  produce changes along the direction of the vortices. These act either to stretch or to compress the vortices. These effects are illustrated in Fig. 5.22. As is illustrated, stretching or compressing changes the size of a vortex. This is a simple consequence of the fluid flow. When, for example, fluid in a pipe speeds up to pass through a narrower section, a cylindrical element of fluid is stretched in length but shrinks in diameter. When this happens to a vortex, the rotation speed also must change to conserve angular momentum. As a result, when the fluid dynamics stretches a vortex, the vorticity increases. Furthermore, note that the increase in vorticity is very rapid. For example, in the case of Fig. 5.22, the vorticity is in the  $x_1$  direction, and the nonzero element of the strain rate tensor that produces the vortex stretching is  $s_{11} = \partial u_1 / \partial x_1$ . Through (5.155), this produces exponential growth of the vorticity (for constant strain rate). This might potentially produce a turbulent state, as exponential growth often leads to large amplitudes. However, any fluctuations in the strain rate as the flow developed would tend to prevent this outcome.

Even so, vorticity in turbulent systems often increases explosively, through the combination of two effects. The first effect is the one we just discussed – the amplification of vorticity through its interaction with the strain rate. The second effect is the unstable growth of the strain rate through a “secondary instability.” The simple instabilities, such as RT or KH, tend to produce very ordered two-dimensional or three-dimensional flows. On the jets



**Fig. 5.22.** Vortex stretching. Changes in fluid velocity cause vortices to stretch or shrink

of Fig. 5.22, for example, the KH instability produces curled structures that wrap around the column of the jet to form loops. These have vorticity on their surfaces, which soon diffuses into the fluid near these surfaces. The direction of this vorticity is azimuthal (it wraps around the jet). Initially, the azimuthal strain rate is zero – the system is cylindrically symmetric. However, many symmetric, two-dimensional systems are unstable to fluctuations in the third dimension. In the case of the jets, these fluctuations modulate the surfaces in the azimuthal direction, and these modulations grow exponentially. The key consequence is that these growing modulations create a finite azimuthal strain rate that also grows exponentially. At this point the vorticity amplifies exponentially from the exponentially growing strain. A very turbulent flow develops almost immediately.

### Homework 5.20

Obtain (5.153) through (5.155) from the momentum equation.

The above discussion enables us to understand how the dynamics of the fluid creates a distribution of vortices on all scales down to the Kolmogorov length. The vortices larger than any given scale produce the strain rate that is experienced by the smaller vortices. Thus, vortices at any given scale are both stretched and rotated through the influence of the larger scales. Tennekes and Lumley show that any given scale is most strongly affected by slightly larger scales. As a result, the flow of energy and vorticity to smaller scales can accurately be described as a *cascade*. Komogorov first described this dynamics. For this reason, the spectrum of  $E(k)$  observed in turbulent systems is often described as a *Kolmogorov cascade*.

We need to address a few more details in order to conclude this discussion. First, note that vortices are inherently three-dimensional objects. They have structure in all three directions. As a result, vorticity and its effects cannot be captured by one-dimensional or two-dimensional calculations or simulations. This makes the accurate modeling of turbulence a very challenging problem.

Second, in the above we have used the notion of fluctuating vorticity. This can be misleading, as we also saw that the vorticity through a given surface is conserved unless viscous diffusion matters. The vorticity in turbulence fluctuates because vortices move and change their shape, not because there is a vorticity oscillation. Another way to put this is that vortex motion is not wave motion and does not involve the oscillation of physical quantities.

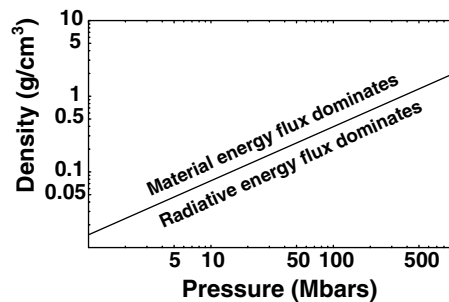
Finally, there is the question of how the vortices at small scales begin. We have seen that vortices on a given scale can affect those on a smaller scale, but this assumes that the smaller scale vortices are already present. There is certainly some thermal level of vorticity, but growing this to large amplitude would be a very long process. The global instabilities such as KH do deposit vorticity on surfaces, and perhaps these or other more-complex processes are responsible for the initial production of vorticity at small scales. One may hope and expect that further research will clarify the details of the transition to turbulence.

This takes us to the end of our discussion of unstable hydrodynamic behavior. Given that acceleration, deceleration, shear layers, and shock waves can each produce such behavior, it is no surprise that hydrodynamic instabilities are common in high-energy-density experiments. We proceed now to turn our attention to another aspect of high-energy-density systems. They are often quite hot, and being hot they tend to radiate profusely. We begin to cope with this in the next chapter.

## 6 Radiative Transfer

Thus far we have examined primarily systems that are purely hydrodynamic. In so doing, we have ignored a major aspect of many high-energy-density systems: radiation. It is easy to see why radiation often matters. At any given pressure, the temperature increases as density decreases, and there is some density below which radiative fluxes will exceed material fluxes. If we suppose that  $Z + 1 = A$ , for simplicity, then the temperature is given by  $T = m_p p / (\rho k_B)$ . The characteristic radiative flux is  $\sigma T^4$ , which can be compared to a characteristic material energy flux  $\rho \epsilon c_s$ . The actual material energy flux may differ from this by some factor, but the threshold density below which radiative fluxes exceed thermal fluxes depends only on the one-fourth power of this factor. Figure 6.1 shows the density at which  $\sigma T^4$  equals  $\rho \epsilon c_s$ , for pressures from 1 Mbar to 1000 Mbars. Radiative effects matter in gases and foams toward the low end of this regime, and in solid-density materials toward the high end. We consider some other comparisons based on temperature, and when radiation pressures matter, at the beginning of Chap. 7.

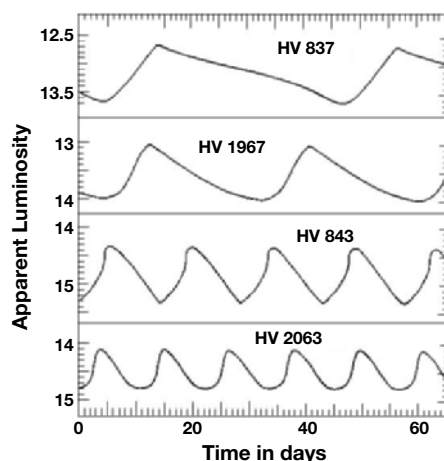
The first problem to address in understanding radiation is radiative transfer, which is the transport of energy and momentum through a physical system by radiation, including the interactions with matter. Radiative transfer is familiar to us in everyday life. For example, suppose radiation from the sun, with its spectral peak at frequencies we perceive as green, is transmitted



**Fig. 6.1.** Energy flux regimes. For any given density, the radiative energy flux exceeds the material energy flux at a high enough material pressure

through the atmosphere, heating a black asphalt driveway. On a hot, bright day we can feel the radiation emitted from this black surface, if we are smart enough to put our hand or foot near it before we step on it. We experience another aspect of radiative transfer, namely scattering, when looking through a fog bank at a bright light.

The emission, absorption, and transfer of radiation are central to much of astrophysics. Shu, for example, devoted one entire volume of his two-volume set, *The Physics of Astrophysics*, to radiation. For example, fusion generates heat in the core of a star that in the end must be radiated from its surface. The stellar structure depends strongly on the absorption and emission of radiative energy. In some variable stars, structure in the absorption causes the luminosity of the star to oscillate. This occurs for example in Cepheid variable stars, which are often used as distance indicators because in order to oscillate their luminosity must remain in a narrow and well-known range. Figure 6.2 shows an example of luminosity oscillations from four Cepheid variable stars. The period of oscillation varies, and can be used to establish their absolute luminosity based on the empirically established relation between period and luminosity. The luminosity oscillations are a consequence of oscillations in the temperature dependence of the absorption length for radiation within the star, which leads the surface of the star to regularly become hotter and cooler. The inverse of the absorption length is known as the *opacity*, which is discussed at length below. The period–luminosity relation for Cepheid variable stars is now understood, thanks to more-sophisticated versions of the calculations described in this chapter, verified by laboratory measurements of the opacity, some of which are shown in Sect. 6.2.3.



**Fig. 6.2.** Luminosity variations in several Cepheid variable stars. Adapted from the Michigan Math and Science Scholars Summer Program



## 6.1 Basic Concepts

In order to work with radiation, its transport, and its effects on matter, one must first face the task of describing the radiation itself. This is a task at least as complicated as the description of the particles. Like particles, the radiation can fill space, vary in time, and propagate in any direction. The radiation does have a unique velocity, which is simpler than the situation with particles, but this is more than offset by the fact that the radiation can have any frequency and can interact through several mechanisms, some of which would be described in mechanics as “inelastic.” In addition, since radiation does move at the speed of light, it is a bit easier for relativistic effects to matter. Nonetheless, by proceeding step by step we can develop useful descriptions. As we do so, we will use primarily a subset of the notation in the book by Mihalas and Mihalas, where one can find a much longer and much more complete discussion, especially in the area of relativistic effects.

### 6.1.1 Properties and Description of Radiation

Here we will build up our description from fundamentals. This corresponds to the development of fluid theories beginning with the Boltzmann equation. The analog of the distribution function, for radiation, is the *spectral radiation intensity*,  $I_\nu$ , which has units of energy per unit area per unit time per unit solid angle per unit frequency, or  $\text{ergs cm}^{-2} \text{ s}^{-1} \text{ sr}^{-1} \text{ Hz}^{-1}$  in cgs units. Thus, within differential elements of area (perpendicular to the direction of propagation), time, solid angle, and frequency given by  $dA$ ,  $dt$ ,  $d\Omega$ , and  $d\nu$ , respectively, the increment of energy delivered is

$$\text{energy} = I_\nu dA dt d\Omega d\nu. \quad (6.1)$$

It should be evident that the spectral intensity is fundamentally related to the Poynting vector. Working out the relationship is an interesting problem, but not one that we need to take up here. A fundamental and complete description of radiation would have to describe the variations in  $I_\nu$  as a function of all these variables. Fortunately, much can be understood by working with less-complete models. We will identify the *radiation intensity*,  $I_R$ , as the energy per unit area per unit time per unit solid angle in the radiation. That is,

$$I_R(\mathbf{x}, t, \boldsymbol{\Omega}) = \int_0^\infty I_\nu(\mathbf{x}, t, \boldsymbol{\Omega}, \nu) d\nu, \quad (6.2)$$

with cgs units of  $\text{ergs cm}^{-2} \text{ s}^{-1} \text{ sr}^{-1}$ . The total intensity emitted by a black body is the *thermal intensity*,  $B(T)$ , given by

$$B(T) = \sigma T^4 / \pi \text{ ergs cm}^{-2} \text{ s}^{-1} \text{ sr}^{-1}. \quad (6.3)$$

Useful values of  $\sigma$  are  $1.03 \times 10^{12} \text{ ergs cm}^{-2} \text{ s}^{-1} \text{ eV}^{-4}$  or  $1.03 \times 10^5 \text{ J cm}^{-2} \text{ s}^{-1} \text{ eV}^{-4}$ .

**Homework 6.1**

Integrate the thermal intensity over  $2\pi$  steradians to find the total radiation power per unit area from a surface at temperature  $T$ .

The radiation intensity is not necessarily uniform in direction, as hotter regions generally emit more thermal radiation (though not necessarily more radiation in atomic line emission, as we discuss below). It turns out that quantities known as the *mean spectral intensity*,  $J_\nu$ , and *mean intensity*,  $J_R$ , are quite useful. These are defined as

$$J_\nu(\mathbf{x}, t, \nu) = \frac{1}{4\pi} \int_{4\pi} I_\nu(\mathbf{x}, t, \boldsymbol{\Omega}, \nu) d\Omega \quad (6.4)$$

and

$$J_R(\mathbf{x}, t) = \frac{1}{4\pi} \int_{4\pi} I_R(\mathbf{x}, t, \boldsymbol{\Omega}) d\Omega. \quad (6.5)$$

We will see how these quantities are important later in this chapter when we discuss radiative energy transport.

No matter what the distribution of the radiation in angle, its energy density is an important property that appears in some of the dynamic equations. In general the density of something is a ratio of flux to velocity, but in particular the mathematics depends upon the details. When material particles move in many directions, as in a gas, their total energy density is much larger than the directed energy of motion of the gas viewed as a fluid. Similarly, the energy density of the radiation is not the net radiation flux divided by the propagation speed. Instead, the *radiation energy density*,  $E_R$ , is the integral over solid angle of the radiation intensity divided by the group velocity. Thus,

$$E_R(\mathbf{x}, t) = \frac{1}{c} \int_{4\pi} I_R(\mathbf{x}, t, \boldsymbol{\Omega}) d\Omega = \frac{4\pi}{c} J_R, \quad (6.6)$$

in which we have made the nearly-always-valid assumption that the group velocity is isotropic and is equal to  $c$ . There is of course a corresponding *spectral radiation energy density*, given by

$$E_\nu(\mathbf{x}, t, \nu) = \frac{1}{c} \int_{4\pi} I_\nu(\mathbf{x}, t, \boldsymbol{\Omega}, \nu) d\Omega = \frac{4\pi}{c} J_\nu. \quad (6.7)$$

The above quantities are similar to integrals of a particle distribution function to find total density, distributions of speeds, and so on. There is a choice in both cases regarding what variables to use to define the distribution. Particle distributions could be defined by their energy density in a space of position, direction, and energy, but it is generally more intuitive to use the number density in a space of position and velocity. In contrast, one can treat photons in terms of their number density, by dividing  $I_\nu$  by  $ch\nu$ , and this is at times useful. But it is generally more intuitive to work with their energy density in a space of position, direction, and energy.

Continuing the analogy with distributions of particles, we next discuss the moments of the distribution of photons. By direct analogy, one would say that the photon flux is

$$\int_{4\pi} \frac{I_\nu(\mathbf{x}, t, \boldsymbol{\Omega}, \nu)}{ch\nu} \mathbf{v} d\Omega, \quad (6.8)$$

in which  $\mathbf{v}$  is the photon velocity vector. But the *radiation energy flux*,  $\mathbf{F}_R$ , is generally more useful, and we know that the speed is  $c$ , so we write

$$\mathbf{F}_R(\mathbf{x}, t) = \int_{4\pi} I_R(\mathbf{x}, t, \boldsymbol{\Omega}) \mathbf{n} d\Omega, \quad (6.9)$$

in which  $\mathbf{n}$  is a unit vector in the direction of propagation for any value of  $\Omega$ . Thus it varies as one integrates. The  $z$ -component of  $\mathbf{n}$ , for example, is  $\cos \theta$  in a standard spherical coordinate system. The cgs units of radiation energy flux are  $\text{ergs s}^{-1} \text{cm}^{-2}$ . It will come as no surprise that the *spectral radiation energy flux* is

$$\mathbf{F}_\nu(\mathbf{x}, t, \nu) = \int_{4\pi} I_\nu(\mathbf{x}, t, \boldsymbol{\Omega}, \nu) \mathbf{n} d\Omega, \quad (6.10)$$

with units  $\text{ergs s}^{-1} \text{cm}^{-2} \text{Hz}^{-1}$ . The radiation flux is a particularly important quantity, because there is a large and important regime in which transport of energy by radiation is crucial even though the energy density and pressure of the radiation are negligible. The radiation flux is related to the radiation momentum density. The total radiation momentum density is  $\mathbf{F}_R/c^2$ , and the spectral radiation momentum density is  $\mathbf{F}_\nu/c^2$ . Thus, the total radiation momentum transport across an element of area,  $d\mathbf{A}$ , is  $\mathbf{F}_R \cdot d\mathbf{A}/c$ .

## Homework 6.2

Using the particle treatment of the radiation, derive an expression for the total radiation momentum density, and show that it equals  $\mathbf{F}_R/c^2$ .

As is the case with particle distributions, one can define further moments of the radiation distribution function indefinitely. In practice, the second moment is as far as one typically needs to go. The *spectral radiation pressure tensor*,  $\underline{\mathbf{P}}_\nu$ , is defined in dyadic notation by

$$\underline{\mathbf{P}}_\nu(\mathbf{x}, t, \nu) = \frac{1}{c} \int_{4\pi} I_\nu(\mathbf{x}, t, \boldsymbol{\Omega}, \nu) \mathbf{n} \mathbf{n} d\Omega. \quad (6.11)$$

This is clearly a symmetric tensor since reversing the order of the components of  $\mathbf{n}$  does not change the integral. The integral of (6.11) over frequency is the *total radiation pressure tensor*,  $\underline{\mathbf{P}}_R$ . The transport of momentum by radiation, in the absence of matter, is fundamentally described by

$$\frac{1}{c^2} \frac{\partial \mathbf{F}_\nu}{\partial t} = -\nabla \cdot \underline{\mathbf{P}}_\nu. \quad (6.12)$$

This is again perfectly natural, as one can see by integrating over a finite volume and using Gauss' theorem. It is worth noting that the left-hand side of this equation is nearly always negligible in systems involving both radiation and matter. The radiative contribution to the energy flux in a system of radiation and matter is often large, but the material momentum nearly always dominates over the radiation momentum. Even so, the radiative momentum source (the right-hand side of this equation) can be the dominant momentum source for the matter.

The radiation field often is symmetric in one of three ways that produce simpler results for the radiation pressure. In general, the *scalar spectral radiation pressure*  $p_\nu$  is defined by

$$\begin{aligned} p_\nu(\mathbf{x}, t, \nu) &= \frac{2\pi}{c} \int_0^\pi I_\nu(\mathbf{x}, t, \theta, \nu) \cos^2(\theta) \sin(\theta) d\theta \\ &= \frac{2\pi}{c} \int_{-1}^1 I_\nu(\mathbf{x}, t, \mu, \nu) \mu^2 d\mu. \end{aligned} \quad (6.13)$$

Here we have evaluated the  $zz$  component of  $\underline{\mathbf{P}}_\nu$ , which equals the  $xx$  and  $yy$  components. The integral of (6.13) over frequency gives the (total) *scalar radiation pressure*,  $p_R$ . If the radiation field is isotropic, then  $\underline{\mathbf{P}}_\nu$  is evidently diagonal with three equal, nonzero elements. One then has

$$p_\nu(\mathbf{x}, t, \nu) = \frac{2\pi I_\nu(\mathbf{x}, t, \nu)}{c} \int_{-1}^1 \mu^2 d\mu = \frac{1}{3} E_\nu, \quad (6.14)$$

which is the simplest example in which one can see this relation between pressure and energy density. In the isotropic case the divergence of the pressure tensor in the momentum equation becomes the gradient of the scalar pressure,  $\nabla \cdot \underline{\mathbf{P}}_\nu = \nabla p_\nu$ , just as occurs with material pressures. (One might protest that a truly isotropic radiation field cannot have pressure gradients, because local isotropy cannot be maintained without having a spatially uniform radiation intensity. This is mathematically true. However, in practice, significant pressure gradients can correspond to negligible anisotropy.) Treating the radiation field as isotropic is for example justified in the diffusion regime, which is of great importance and which we will discuss at length below.

### Homework 6.3

Derive (6.14).

A second useful, symmetric case is the planar case, in which  $I_\nu$  varies only in angle relative to one direction, and is isotropic in the two orthogonal directions. In this case we choose the direction of variation as the  $z$  axis and

write  $I_\nu = I_\nu(z, t, \mu, \nu)$  where again  $\mu = \cos \theta$ . In this case the  $zz$  element of  $\underline{P}_\nu$  is again  $p_\nu$ , and the  $xx$  element,  $P_{xx}$ , is

$$\begin{aligned} P_{xx} &= \frac{1}{c} \int_0^{2\pi} \int_{-1}^1 I_\nu(z, t, \mu, \nu) (1 - \mu^2) d\mu \cos^2 \phi d\phi \\ &= \frac{1}{2} E_\nu - \frac{1}{2} p_\nu = p_\nu - \frac{1}{2} (3p_\nu - E_\nu), \end{aligned} \quad (6.15)$$

which is also equal to  $P_{yy}$ . Note that for an isotropic intensity or for any angular distribution that yields  $E_\nu = 3p_\nu$ , the pressure again reduces to a scalar. In the planar case, the only nonzero derivatives are in the  $z$  direction, so

$$\nabla \cdot \underline{P}_\nu = (\partial p_\nu / \partial z) \hat{z}, \quad (6.16)$$

where  $\hat{z}$  is a unit vector in the  $z$  direction. In the third useful case, of spherical symmetry, the diagonal components corresponding to the polar and azimuthal angles equal  $P_{xx}$ , and the radial component is again  $p_\nu$ . In this case the only nonzero derivatives are in the radial direction, so

$$\nabla \cdot \underline{P}_\nu = [\partial p_\nu / \partial r + (3p_\nu - E_\nu) / r] \hat{r}, \quad (6.17)$$

in which  $\hat{r}$  is a unit vector in the radial direction.

Now consider in general terms the ratio of  $p_\nu$  to  $E_\nu$ . This ratio is known as the *Eddington factor*,  $f_\nu = p_\nu / E_\nu$ . The Eddington factor depends on the angular variation of  $I_\nu$ , as is clear in (6.13). In the limit of a plane wave at frequency  $\nu$ , with irradiance (power per unit area)  $I$ , one would have  $E_\nu = I/c$  and  $p_\nu = (I/c^2)c = I/c$  so in this case  $p_\nu = E_\nu$  and  $f_\nu = 1$ . A sufficiently beam-like intensity distribution can have  $f_\nu \sim 1$ . The limit where the radiation propagates freely with little interaction is known as the *free-streaming limit*. In the free-streaming limit,  $f_\nu$  will approach 1 as the distance from the source increases. It is also clear from (6.13) that  $f_\nu$  decreases as the distribution spreads in angle, reaching 1/3 when the distribution becomes isotropic. In natural systems,  $f_\nu$  typically varies between 1/3 and 1. Mathematically, a ‘‘pancake-like’’ distribution, in which most of the energy was transverse to the symmetry axis, would produce an Eddington factor below 1/3. This could be achieved in an experiment designed for that purpose and might occur in nature if an extended, hot, source region were sandwiched between two strongly absorbing regions.

It should be evident that one can define a total radiation pressure tensor,  $\underline{P}_R$ , and a scalar radiation pressure,  $p_R$ , by integrating the relations above over frequency. This permits one to define an overall Eddington factor,  $p_R / E_R$ . Some computational approaches to radiation transport are formulated in terms of an Eddington factor, which can be an effective way to improve the accuracy of a calculation without always dealing explicitly with all the possible directions of radiation propagation.

### 6.1.2 Thermal Radiation

Thermal radiation is very important even in systems with very nonequilibrium radiation. The reason is that the electrons are responsible for the emission of radiation, and the electrons very often develop a Maxwellian or piecewise-Maxwellian distribution. In such cases, the spectral intensity of the emitted radiation in some frequency range is proportional to the equilibrium spectral radiation intensity at the temperature of the electrons that are responsible for the emission. Nearly all texts on modern physics or statistical mechanics derive the properties of thermal equilibrium (or *blackbody*) radiation, so we need not repeat this here. By considering the relative probability that a state will be occupied, the density of states in phase space, and the two possible polarizations, one can show that the *spectral thermal radiation intensity*,  $B_\nu(T)$ , is,

$$B_\nu(T) = \frac{2h\nu^3}{c^2} \frac{1}{e^{h\nu/(k_B T)} - 1}, \quad (6.18)$$

in which  $h$  is Planck's constant. Here the subscript indicates that  $B_\nu(T)$  is frequency dependent. Integrating over  $\nu$ , one finds that the *total thermal radiation intensity*,  $B(T)$ , is

$$B(T) = \sigma T^4 / \pi, \quad (6.19)$$

in which  $\sigma$  is the Stefan–Boltzmann constant. The energy density of the radiation is

$$E_R(T) = \frac{4\pi}{c} B(T) = \frac{4}{c} \sigma T^4. \quad (6.20)$$

Thermal radiation must be isotropic, so the pressure is a scalar and  $p(T) = E_R(T)/3$ .

### 6.1.3 Types of Interaction Between Radiation and Matter

Having defined the variables necessary to describe the radiation in isolation, we now are ready to ask how radiation and matter interact. This will prepare us to consider the combined problem, known as the *radiative transfer problem*. One can identify three fundamental types of interaction between radiation and matter. The first of these involves *bound–bound transitions*, which one encounters in elementary physics upon being introduced to the Bohr atom. The electron in a Bohr atom is bound to the atom, but can transition among the energy levels of the atom, known as “states.” The lowest energy state is the *ground state* and all bound, higher-energy states are *excited states*. Radiation can be emitted when the electron “decays” from a higher-energy state to a lower-energy one. This is *bound–bound emission*, which is responsible for the familiar Lyman and Balmer spectral series. The inverse process, compared to decay, is excitation, in which an electron is given energy and moves

from a lower-energy state to a higher-energy one. In general, both excitation and decay require the involvement of at least one additional particle, other than the electron and the atom or ion, to conserve energy and momentum. This additional particle can be any type of particle including a photon. In practice, certain particles tend to dominate the rate at which a specific decay or excitation occurs. In low-density plasmas, a decay is nearly always radiative and an excitation nearly always is produced by an electron collision. In high-energy-density plasmas, both collisional and radiative decay often matter, collisions may move the electron from one excited state to another, and radiative excitation will also be important if the radiation is near equilibrium with the matter.

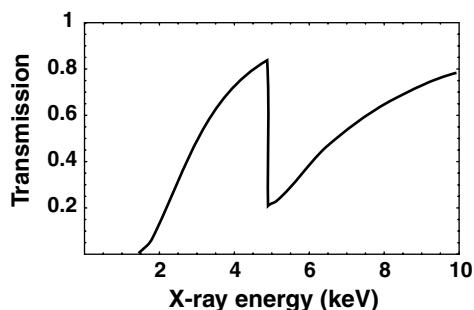
Bound-bound emission produces line radiation, whose frequency  $\nu$  is given by  $E_{\Delta} = h\nu$ , where  $E_{\Delta}$  is the energy difference between the levels and  $h$  is Planck's constant. The spectral width is quite narrow. Energy emitted from one particle can be absorbed by a second particle, but only if the frequency of the radiation as seen by the second particle overlaps a bound-bound transition in the second particle. Since any Doppler shift changes the frequency seen by the second particle, the transfer of energy by repeated absorption and emission involving bound-bound transitions can become complex.

#### Homework 6.4

From the uncertainty principle, the spectral width in frequency,  $\Delta\nu$ , of an emission line is roughly the inverse of the decay time. For a typical decay time of 1 ns, find the normalized spectral width  $\Delta\nu/\nu$ , for emission lines in the visible and in the soft x-ray with a photon energy of 100 eV.

The second fundamental type of radiation-matter interaction involves the *bound-free transition*. The limiting energy of the bound states is the ionization energy, given in the Bohr model in the limit that the principal quantum number goes to infinity. In plasmas, continuum lowering may reduce this, as was discussed in Chap. 3. When an electron is given energy that moves it above the ionization energy, it becomes a *free electron*. It is then said to be in *the continuum*, so called because the energy of the allowed states can vary continuously. Just as in the case of bound-bound transitions, an electron can decay from or be excited to a continuum state as a result of an interaction with any particle including photons.

Thus, a photon can be absorbed by an atom or ion, releasing one of the electrons through a transition from a bound state to a free state. This is a major contribution to the absorption of x-ray photons by materials. Figure 6.3 shows the transmission of a thin layer of titanium. There is an abrupt decrease in transmission at 4.7 keV, which is the lowest energy at which the x-ray photon can pull an electron from the bound state whose principal quantum number is  $n = 1$  (the *K-shell*) into the continuum. This absorption feature is known as the *K-edge*. As the photon energy increases above the K-edge, any absorbed photon will place the electron into a higher-energy



**Fig. 6.3.** The transmission of a 5  $\mu\text{m}$  thick titanium slab to x-rays

continuum state. The cross section for this process decreases as the energy of the continuum state increases, which leads to the increase in transmission as energy increases above the K-edge. The energy of the K-edge increases with increasing atomic number. The next edge, with lower ionization energy, is the *L-edge*. It corresponds to the extraction of an electron from the  $n = 2$  shell. Higher- $Z$  materials such as tin or rhodium have an L-edge in the 3 to 4 keV energy range. All solid materials become strongly absorbing at x-ray energies below 1 keV, where bremsstrahlung (free-free) absorption becomes large.

Alternatively, a photon can be emitted when an electron in the continuum recombines with an ion. This produces an x-ray line near the K-edge, as it is a much stronger process for electrons in continuum states with near zero energy. On the low-energy side of this line, one may observe structure resulting from electrons that recombine into an excited state and then decay into the ground state. On the high-energy side of this line, one may observe a continuous feature from electrons that make free-bound transitions from higher-energy states in the continuum. The spectrum of free-bound radiation, at energies just above the x-ray line, can in some cases be used as a temperature diagnostic.

The third category of radiation-matter interaction involves *free-free transitions*. These transitions move an electron from one continuum state to another. The interaction of a free electron with any other particle (including a photon) produces a free-free transition. Such transitions often produce photon emission or absorption. Two of the most common and important free-free interaction processes are *bremsstrahlung* emission and *inverse bremsstrahlung* absorption. In bremsstrahlung, a particle (in practice, an electron) is accelerated by interaction with another charged particle (in practice, a nucleus), and this results in the emission of photons. This is the primary source of continuum emission from hot dense matter. In inverse bremsstrahlung, a photon (or light wave) moves an electron past a nucleus. The interaction with the nucleus randomizes the motion of the electron, which has the effect of extracting energy from the light. The absorption coefficient for inverse



bremsstrahlung is discussed in Sect. 9.2. The high-energy limit of inverse bremsstrahlung is Compton scattering, in which the photon–particle energy exchange is quantized. Another free–free emission mechanism, important in magnetized plasmas, is *synchrotron emission*.

#### 6.1.4 Description of the Net Interaction of Radiation and Matter

Fortunately, one often need not explicitly account for every distinct interaction of radiation and matter. Instead, one can obtain an adequate description of many systems by considering the net total emission, absorption, and scattering. We develop such a description here. Plasmas emit radiation, both directly through the interactions of the particles, such as bremsstrahlung, and indirectly by scattering radiation in angle and/or energy. We will write the *spectral emissivity*,  $\eta_\nu$ , as

$$\eta_\nu(\mathbf{x}, t, \mathbf{n}, \nu) = \eta_{\nu\text{th}}(\mathbf{x}, t, \mathbf{n}, \nu) + \eta_{\nu\text{sc}}(\mathbf{x}, t, \mathbf{n}, \nu), \quad (6.21)$$

which has (cgs) units  $\text{ergs cm}^{-3} \text{ s}^{-1} \text{ sr}^{-1} \text{ Hz}^{-1}$ . In some writings, the term *spectral emission coefficient* is used rather than spectral emissivity. Here the *spectral thermal emissivity* is  $\eta_{\nu\text{th}}$ , which is an approximation assuming that the particles have a single, Maxwellian energy distribution. A more general and complete expression would explicitly include all the processes by which all the particles in the system can emit radiation, including for example line emission following collisional excitation and bremsstrahlung emission by high-energy tails on the electron distribution. Note that the integral of this term over frequency and solid angle gives the power loss rate of the matter in the plasma due to radiation. The *spectral scattering emissivity* is  $\eta_{\nu\text{sc}}$ , which includes all processes that scatter radiation in angle or energy. We will not pursue this in any depth here, but in general this emissivity at a given angle or energy depends on an integral over the radiation intensity present at other angles or energies. Unlike the quantities discussed previously in this section, this implies that the integral of (6.21) over frequency or angle is not straightforward, unless one can simply approximate the scattering term or ignore it for some reason.

The rate of absorption of energy from radiation depends inherently on the radiation intensity, so the energy absorbed must be an expression involving the radiation intensity. We express the energy absorbed per unit volume per unit time per unit solid angle per unit frequency as  $\chi_\nu I_\nu$ , in which  $\chi_\nu$  is the *spectral total opacity* in units of  $\text{cm}^{-1}$ , also at times known as the *spectral extinction coefficient*. In analogy with the case of emission, there is a *spectral absorption opacity*,  $\kappa_\nu$ , for absorption by the particles, which contributes to heating of the matter, and there is a *spectral scattering opacity*,  $\sigma_\nu$ , for scattering by the particles, which changes the direction (and possibly the energy) of the radiation. Thus

$$\chi_\nu(\mathbf{x}, t, \mathbf{n}, \nu) = \kappa_\nu(\mathbf{x}, t, \mathbf{n}, \nu) + \sigma_\nu(\mathbf{x}, t, \mathbf{n}, \nu). \quad (6.22)$$

Here again, the general interaction of radiation and matter may be much more complex. In principle it may involve dielectric tensors and powers of the electric field of the light waves. But for nearly all problems in high-energy-density physics, it is sufficient to take the energy absorption rate to be linearly proportional to the radiation intensity, as we do here.

In the following we will not specifically treat the emission, absorption, and scattering of radiation by spectral lines. In some contexts, these can be approximated as an overall effective emissivity and opacity. In other contexts, they must be treated explicitly. The methods will be similar to those discussed here, but it will be necessary to treat each line discretely and to associate Doppler shifts with all relative material motions.

Kirchoff's law states that emission and absorption of radiation must be equal in equilibrium, which today we would view as an application of the principle of detailed balance to radiation emission and absorption. Mathematically, we would write this as

$$\eta_{\nu\text{th}}(\mathbf{x}, t) = \kappa_{\nu}(\mathbf{x}, t)B_{\nu}(T). \quad (6.23)$$

*Local thermodynamic equilibrium* (LTE) is a state in which each species in the plasma, including the radiation, has an equilibrium distribution of energies, and in which the temperatures of these distributions are all equal. Plasmas are in LTE if the photon mean free path for absorption (and the collisional mean free paths) are very small compared to the gradient scale length of the temperature and if any variations in time are slow compared to the time required for an equilibrium distribution of ionization and excitation to be established. Thus, a system must be quite dense or very large to establish LTE. Some high-energy-density systems can be accurately described as LTE. In some experiments, for example, a low- $Z$  envelope is used to confine a higher- $Z$  material of interest, and this entire sample is maintained within an equilibrium radiation environment long enough to be uniformly heated. This places the material of interest in LTE.

Relation (6.23) is useful under other circumstances. Collisions are usually rapid enough that the distribution of electrons in a high-energy-density system is Maxwellian. The thermal radiation emitted by these electrons is accurately described by (6.23). The situation with regard to atomic line emission is more complicated, however. The intensity produced by line emission cannot rise above  $B_{\nu}(T)$ , assuming the electron distribution is Maxwellian. Also the emission from any excited states whose populations are maintained at their equilibrium (Saha) values by electron collisions will be described by (6.23). However, the excited states that interact with the tail of the electron distribution, and with photons having long mean free paths, typically are not in their equilibrium distributions. These states are not in LTE, (6.23) does not accurately describe their emission, and an explicit non-LTE (NLTE) calculation of their populations is necessary to accurately describe them. Unfortunately, NLTE calculations are costly computations, so it has often proven

necessary to use (6.23) even when it is not strictly applicable. Finally, note that in many astrophysical systems photons are not confined; such systems will nearly always be far from LTE.

## 6.2 Radiation Transfer

### 6.2.1 The Radiation Transfer Equation

The radiation transfer equation is no more than an accounting of the change in radiation intensity due to sources and losses of radiation in a specific element of phase space. In this case the phase space includes an element of solid angle,  $d\Omega$ , about a direction,  $\mathbf{n}$ , an element of frequency,  $d\nu$ , about a frequency,  $\nu$ , and an element of volume of length  $ds$  and cross-sectional area  $dA$  beginning at position  $\mathbf{x}$ . The net rate that energy from this element of phase space adds to the radiation intensity in some direction is the difference between the rate that energy enters the element at  $\mathbf{x}$  and  $t$  and the rate that energy leaves it at  $\mathbf{x} + \mathbf{n}ds, t + \Delta t$ , where  $\Delta t = ds/c$ . Mathematically, we can write this as

$$\begin{aligned} & \left[ I_\nu \left( \mathbf{x} + \mathbf{n}ds, t + \frac{ds}{c}, \mathbf{n}, \nu \right) - I_\nu(\mathbf{x}, t, \mathbf{n}, \nu) \right] dAd\Omega d\nu \\ &= \left[ \frac{1}{c} \frac{\partial I_\nu}{\partial t} + \frac{\partial I_\nu}{\partial s} \right] dsdAd\Omega d\nu, \end{aligned} \quad (6.24)$$

which has units of energy per unit time. What causes this increase in intensity is the difference between the total rate of energy emission from the medium into the element of phase space, which includes scattering from other elements of phase space, and the total rate of energy removal from the radiation by absorption or by scattering into other elements of phase space. In terms of the total spectral emissivity,  $\eta_\nu$ , and the total opacity,  $\chi_\nu$ , we can write

$$\left[ \frac{1}{c} \frac{\partial}{\partial t} + \frac{\partial}{\partial s} \right] I_\nu(\mathbf{x}, t, \mathbf{n}, \nu) = \eta_\nu(\mathbf{x}, t, \mathbf{n}, \nu) - \chi_\nu(\mathbf{x}, t, \mathbf{n}, \nu) I_\nu(\mathbf{x}, t, \mathbf{n}, \nu). \quad (6.25)$$

This is the “classical” equation of radiation transfer. It is perhaps most accurately thought of as an equation for photons treated as particles, which is an excellent approximation for the x-ray photons that carry the thermal energy in high-temperature plasmas. Wave effects, including diffraction, refraction, and polarization are ignored. Note that, unlike most of the equations above, one cannot obtain an equation for the total intensity by integrating this equation without making a severe simplifying assumption about the frequency dependence of  $\chi_\nu$ . Also note that this form of the equation will be most useful when scattering, including absorption and reemission, is not a central feature of the radiation dynamics. If, for example, the emissivity includes significant scattering from other angles (or other frequencies), then the

emissivity involves an integral over  $I_\nu$  times a scattering coefficient and (6.25) becomes an integro-differential equation. For example, the problem of radiation transport through an expanding envelope, as in a supernova, introduces just this sort of complexity. This issue is the origin of various standard averaged opacities, such as the Rosseland mean and the Planck mean, discussed below.

It is worthwhile for ease of applications to elaborate on  $\partial/\partial s$ . From the chain rule we can write

$$\frac{\partial I_\nu}{\partial s} = \frac{\partial I_\nu}{\partial \mathbf{x}} \cdot \frac{\partial \mathbf{x}}{\partial s} = \mathbf{n} \cdot \nabla I_\nu + \frac{\partial \mathbf{n}}{\partial s} \cdot \frac{\partial I_\nu}{\partial \mathbf{n}}, \quad (6.26)$$

in which  $\partial/\partial \mathbf{x}$  is equivalent to the gradient operator. The second equality follows because the  $j$ th component of the position vector,  $x_j$ , along  $\mathbf{n}$  is  $x_j = s \cos \alpha_j$ , where the direction cosine for the  $j$ th component of  $\mathbf{x}$  is  $\alpha_j$ . Of course,  $\mathbf{n}$  is the unit vector composed of these direction cosines. In a Cartesian coordinate system,  $\alpha_j$  is fixed and only the first term on the right is nonzero. In curvilinear coordinate systems,  $\alpha_j$  varies along  $s$ . This can lead to very complicated expressions in general cases. In a standard spherical coordinate system, for example, one needs three variables ( $r, \theta, \phi$ ) to specify the location of a point on the ray and in addition two variables (a polar angle  $\Theta$  and an azimuthal angle  $\Phi$ ) to specify the direction of the ray with respect to the local radial direction. In a spherically symmetric system, such as a star treated as a symmetric object, the location is fully specified by  $r$ . At any specific point, the radiation intensity varies with direction, but it is symmetric about the local radius vector. As a result, one needs a single angle,  $\Theta$ , to specify the local direction of the ray. Defining  $\mu = \cos \Theta$ , one can show that

$$\left[ \frac{1}{c} \frac{\partial}{\partial t} + \frac{\partial}{\partial s} \right] I_\nu(\mathbf{x}, t, \mathbf{n}, \nu) = \left[ \frac{1}{c} \frac{\partial}{\partial t} + \mu \frac{\partial}{\partial r} + \frac{(1 - \mu^2)}{r} \frac{\partial}{\partial \mu} \right] I_\nu(r, t, \mu, \nu). \quad (6.27)$$

### Homework 6.5

Derive (6.27)

### 6.2.2 Radiative Transfer Calculations

We have at last come to the end of our first task relating to radiative transfer. We have defined the properties of radiation and of its interactions with matter. We now face the problem of actually describing this interaction, and of developing applied equations that will prove useful in various limits. In many circumstances, the time derivative in (6.25) can be neglected, as the motion of the radiation is effectively instantaneous in comparison with that of the matter. Especially in these cases, but even in general, it can be useful to normalize (6.25) by the opacity. To do so, we introduce a new variable

known as the *optical depth*,  $\tau_\nu$ , which is inherently a function of frequency. We define an infinitesimal increment of optical depth as

$$d\tau_\nu = \chi_\nu ds. \quad (6.28)$$

Thus, the optical depth, at frequency  $\nu$ , between point  $s$  and point  $s_o$ , is

$$\tau_\nu = \int_s^{s_o} \chi_\nu(s') ds'. \quad (6.29)$$

In applications, sign conventions are usually chosen so that the optical depth is a positive quantity. This often takes care of itself. For example, if radiation is propagating in the  $-z$  direction then  $ds = -dz$  and  $\tau_\nu$  becomes the integral of the opacity from smaller to larger  $z$ . A layer of material is said to be *optically thick* (at some frequency) if  $\tau_\nu \gg 1$ , and *optically thin* if  $\tau_\nu \ll 1$ . All materials are optically thick at long enough wavelengths and optically thin at short enough wavelengths. (We should also mention that the use of optical depth and related terms is not always consistent in the literature and can be misleading. In some experimental work, the transmission of a layer may be given as its “optical depth.” In contrast, the optical depth of (6.29) is the natural logarithm of the transmission. In optics, one encounters filters designated by “O.D.” However, these initials stand for “optical density”, not optical depth. Optical density is consistently defined as the logarithm to the base 10 of the transmission.)

To complete the normalization of (6.25), we define the source function,  $S_\nu$ , as

$$S_\nu = \eta_\nu / \chi_\nu. \quad (6.30)$$

As an application of Kirchoff’s law, discussed above, the source function due to a Maxwellian distribution of electrons is

$$S_\nu = B_\nu(T). \quad (6.31)$$

When this is the case, one can see by normalizing (6.25) that the spectral intensity will be driven toward this equilibrium value as optical depth becomes large. We will frequently constrain the electron distribution to be Maxwellian, as opposed to specifying that the system be in equilibrium. For (6.31) to describe the emission, it is a necessary and sufficient condition that the electron distribution be Maxwellian. This often occurs, and (6.31) often describes the source function, even in systems that are far from complete equilibrium.

When we proceed to calculate the angular dependence of the spectral intensity, the definition of  $\tau_\nu$  in (6.29) becomes impractical, because it gives a different value of  $\tau_\nu$  for each direction of propagation through a layer of some thickness. For planar systems in which quantities vary along the  $z$  axis, it is practical to define  $d\tau_\nu = -\chi_\nu dz$ , so that  $d\tau_\nu = -\mu\chi_\nu ds$ , where  $\mu = \cos\theta$ , with  $\theta$  being the polar angle relative to the  $z$  axis. Then (6.25) becomes

$$\mu \frac{\partial I_\nu}{\partial \tau_\nu} = I_\nu - S_\nu. \quad (6.32)$$

Next we suppose that the thermal emission is given by (6.23), and that the scattering is isotropic and elastic (or “coherent”), so that it does not change the photon energies. Then we have

$$\eta_\nu = \kappa_\nu B_\nu + \sigma_\nu J_\nu \quad (6.33)$$

so

$$S_\nu = (\kappa_\nu B_\nu + \sigma_\nu J_\nu) / (\kappa_\nu + \sigma_\nu). \quad (6.34)$$

Using this definition of  $\eta_\nu$ , one can find from (6.25),

$$\frac{1}{c} \frac{\partial I_\nu}{\partial t} + \frac{\partial I_\nu}{\partial s} = \kappa_\nu (B_\nu - I_\nu) + \sigma_\nu (J_\nu - I_\nu). \quad (6.35)$$

Integrating this equation in frequency one obtains

$$\frac{1}{c} \frac{\partial I_R}{\partial t} + \frac{\partial I_R}{\partial s} = \int \kappa_\nu (B_\nu - I_\nu) d\nu + \int \sigma_\nu (J_\nu - I_\nu) d\nu. \quad (6.36)$$

It is often sensible to approximate the second term on the right-hand side in this equation as negligible, either because the intensity distribution is nearly isotropic, so  $I_\nu = J_\nu$ , or because the system is optically thin so that it changes  $I_R$  primarily by emission, or because the scattering is small relative to the absorption. (However, in some astrophysical systems scattering is large relative to absorption, especially when the radiation is dominantly line radiation.) In the static limit, and with this approximation, we have

$$\frac{\partial I_R}{\partial s} = \kappa (B(T) - I_R), \quad (6.37)$$

in which  $\kappa$  is a nonlinearly averaged absorption coefficient, approximately equal to the Planck mean opacity, defined and discussed just below. Equation (6.37) may be needed to determine the angular variation in (and integral of) the radiation intensity reaching some surface of interest in an application. One important example is the calculation of the radiation intensity emerging from an optically thin layer of material.

If instead one integrates (6.36) over all solid angle, then the second term on the right vanishes identically. Note that this is equivalent to taking the zeroth moment over angle of (6.25). In Cartesian coordinates, we find

$$\frac{\partial E_R}{\partial t} + \nabla \cdot \mathbf{F}_R = 4\pi \int_0^\infty d\nu \kappa_\nu (B_\nu - J_\nu) \equiv 4\pi \kappa (B - J_R), \quad (6.38)$$

which relates the overall absorption of radiation to changes in the radiation flux and energy density. Note that  $\nabla \cdot \mathbf{F}_R = \int (\partial I_R / \partial s) d\Omega$ . Equation (6.38) is an equation for the radiation energy, although conceptually it is the analogue

of the continuity equation for mass, in which the rate of change of a density is related to the divergence of a flux and to sources. It is convenient, but also intuitively sensible, that the scattering terms cancel out of this result. The energy density is very often negligible, in which case this becomes a fairly simple equation for  $\nabla \cdot \mathbf{F}_R$ . It is also important to note that (6.38) is only valid for a static medium. In a moving medium, there are both a convective energy flux and  $pdV$  work associated with the radiation, as discussed in Chap. 7.

Equation (6.38) also defines an *absorption opacity*,  $\kappa$ . One sees that  $\kappa$  is again a nonlinear average over frequency of  $\kappa_\nu$ . This is one of several similar averages that one encounters in simple calculations. The average in (6.37) is also labeled  $\kappa$ , even though it represents a nonlinear average over  $B_\nu - I_\nu$  rather than  $B_\nu - J_\nu$ . The reason we make no distinction here is that in practice one seldom calculates either average. Unfortunately, one cannot evaluate  $\kappa$  as defined unless one has already solved the problem of the radiation transport and knows  $I_\nu$  and  $J_\nu$ . So to obtain practical solutions one must somehow approximate  $\kappa$ . It turns out that, for systems to which it makes sense to apply (6.37) or (6.38),

$$\kappa \approx \kappa_P \equiv \frac{1}{B(T)} \int_0^\infty d\nu \kappa_\nu B_\nu. \quad (6.39)$$

Here  $\kappa_P$  is the *Planck mean opacity*, which depends only on the equilibrium properties of a material, is often tabulated, and may be available in an approximate functional form. Of course, one could develop iterative solutions to either (6.37) or (6.38), in which one determined  $\kappa$  from an initial solution assuming  $\kappa = \kappa_P$ . But if one actually needed to do this, one might be better advised to employ a more-sophisticated radiation transport calculation from the start.

One can proceed to take higher moments of the radiation transfer equation, to determine additional features of the radiation transport. We will not make use of any of these in what follows, but for reference will include and discuss here the radiation momentum transport equation, which is the first moment in angle of (6.25), again integrated also in frequency. This gives

$$\frac{1}{c^2} \frac{\partial \mathbf{F}_R}{\partial t} + \nabla \cdot \mathbf{P}_R = \frac{1}{c} \int_0^\infty d\nu \int_{4\pi} d\Omega (\eta_\nu - \chi_\nu I_\nu) \mathbf{n}. \quad (6.40)$$

This equation greatly simplifies in most situations, as the first term on the left is only significant for relativistic motions and as the emission nearly always is isotropic and averages to zero. One then has on the right-hand side with the radiation force density,  $\mathbf{f}_R$ , expressing the momentum imparted to the matter by the radiation, as

$$\mathbf{f}_R = \frac{1}{c} \int_0^\infty d\nu \int_{4\pi} d\Omega (\chi_\nu I_\nu) \mathbf{n}. \quad (6.41)$$

For isotropic scattering, and in the Eddington approximation, the spectral force density is related to the spectral radiation pressure and spectral radiation energy density by

$$\nabla p_\nu = \nabla (f_\nu E_\nu) = -\frac{\chi_\nu}{c} \mathbf{F}_\nu. \quad (6.42)$$

This is the intuitively obvious result. Note also that this set of approximations corresponds to *diffusive* radiation transport, in which the flux of radiation energy is proportional to the gradient of the density of radiation energy.

### Homework 6.6

Take moments of the radiation transfer equation to derive (6.38) and (6.40).

Note at this point that the situation with regard to closure of the radiation equations is identical to that for the particle equations. Each moment equation introduces the next higher moment, so one will need some additional assumption, such as an “equation of state” for the radiation in order to close the equations. The use of an Eddington factor, relating radiation pressure and energy density, is one way to achieve this closure.

It is both straightforward and worthwhile to obtain solutions to the time-independent radiation transfer equation. First, note that (6.25) has an integrating factor, which is just  $\exp(-\tau_\nu)$ . With this realization, integrating and simplifying this equation gives

$$I_\nu(\mathbf{x}_o + \mathbf{n}s) = e^{-\tau_\nu(0,s)} I_\nu(\mathbf{x}_o) + \int_0^s S_\nu(\mathbf{x}_o + \mathbf{n}s') e^{-\tau_\nu(s',s)} ds', \quad (6.43)$$

where the optical depth  $\tau_\nu(a,b)$  is the integral of  $\chi_\nu$  along  $\mathbf{n}$  from  $s = a$  to  $s = b$ . This shows how the intensity at some propagation distance  $s$  from an initial point  $\mathbf{x}_o$  (where  $s = 0$ ) is the transmitted intensity from  $\mathbf{x}_o$  plus the attenuated contribution from the emission at each intervening point.

For planar systems, with  $ds = dz$  and  $d\tau_\nu = \chi_\nu dz$ , and  $\tau_\nu = 0$  at a specified value of  $z$ , one obtains

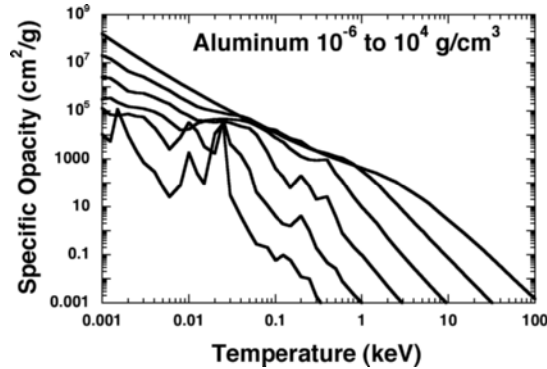
$$I_\nu(\tau_\nu) = e^{-|\tau_\nu - \tau_o|/\mu} I_\nu(\tau_o) + \int_{\tau_\nu}^{\tau_o} S_\nu(\tau') e^{-|\tau' - \tau_\nu|/\mu} \frac{d\tau'}{\mu}. \quad (6.44)$$

This equation is essentially identical to the previous one, except that it accounts for the variation in optical depth with angle through the introduction of  $\mu = \cos \theta$ . For applications in which an approximate treatment of the total intensity is relevant, one can define  $d\tau = \kappa dz$  and integrate (6.37) to find

$$I_R(\tau) = e^{-|\tau - \tau_o|/\mu} I_R(\tau_o) + \int_{\tau_o}^{\tau} B(\tau') e^{-|\tau - \tau'|/\mu} \frac{d\tau'}{\mu}. \quad (6.45)$$

We will use this in the next chapter.





**Fig. 6.4.** The specific Planck mean opacity of Al versus electron temperature, with the curves from bottom to top showing densities in  $\text{g}/\text{cm}^3$  of  $10^{-6}$ ,  $10^{-4}$ ,  $10^{-2}$ , 1,  $10^2$ , and  $10^4$  (based on LANL SESAME table 13710)

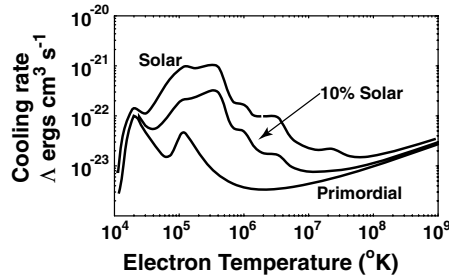
### 6.2.3 Opacities in Astrophysics and the Laboratory

Opacities are often tabulated as specific opacities, which can be approximated as power laws in density and temperature. The density dependence of the specific opacity is typically weak at densities of interest for high-energy-density physics. Figure 6.4 shows the Planck mean opacity of Aluminum from one standard set of tables (this is LANL SESAME table 13710). There are a number of interesting features in this table. First, note that in the range of density ( $1\text{--}100 \text{ g}/\text{cm}^3$ ) and temperature ( $10 \text{ eV--}1 \text{ keV}$ ) typically of interest in high-energy-density experiments, the density dependence of the opacity is very weak. This is the origin of the standard formula given below. Second, note that the absorption peak near  $30 \text{ eV}$  in temperature associated with the second ionization (producing neon-like aluminum) is so strong that in this temperature range the opacity depends very weakly on density even at lower densities. Third, at lower densities and at higher temperatures the curves separate, and the opacity becomes proportional to density squared. This is the regime in which two-body interactions dominate the absorption (and emission).

Here are some approximations to the specific Planck mean opacity,  $\kappa_m$ , based on a standard set of tables (the SESAME tables):

$$\begin{aligned} \kappa_m &\approx 2 \times 10^5 T_{eV}^{-1} \text{ cm}^2/\text{g} && \text{for CH} \\ \kappa_m &\approx 3 \times 10^6 T_{eV}^{-1} \text{ cm}^2/\text{g} && \text{for Al} \\ \kappa_m &\approx 3 \times 10^9 T_{eV}^{-2} \text{ cm}^2/\text{g} && \text{for Xe.} \end{aligned} \quad (6.46)$$

The scaling of the opacity is different in low-density systems, where the absorption and emission may be dominated by bound-bound transitions and line radiation. The net emission from low-density astrophysical plasmas is often described using a *cooling function*. The cooling function  $\Lambda$  is the power loss per unit volume per unit electron density per unit ion density. Thus, the



**Fig. 6.5.** Typical astrophysical cooling functions. The overall shape does not vary, but the location of the minimum depends upon the density of elements above He. The primordial case has only H and He while the other two cases have solar or 10% solar concentrations of such heavier elements, as indicated. Adapted from *Sutherland and Dopita*

power loss per unit volume is  $n_e n_i \Lambda$ , and one can see that  $\Lambda$  has units of  $\text{ergs cm}^3/\text{s}$  or equivalent. The discussion in Sect. 7.2.2 shows that the relation of the absorption opacity to the cooling function is  $\kappa = n_e n_i \Lambda / (2\sigma T^4)$ . Using typical numbers of  $n_e \sim n_i \sim 10 \text{ cm}^{-3}$ ,  $\Lambda \sim 10^{-22}$ ,  $\text{ergs cm}^3/\text{s}$ , and  $T = 10 \text{ eV}$ , one finds  $\kappa \sim 5 \times 10^{-37} \text{ cm}^{-1}$ , or about  $10^{-19} \text{ parsec}^{-1}$ . Figure 6.5 shows typical astrophysical cooling functions, based on results in *Sutherland and Dopita*. These particular results correspond to a model assuming that the distribution of ionization states is in an equilibrium determined by collisions.

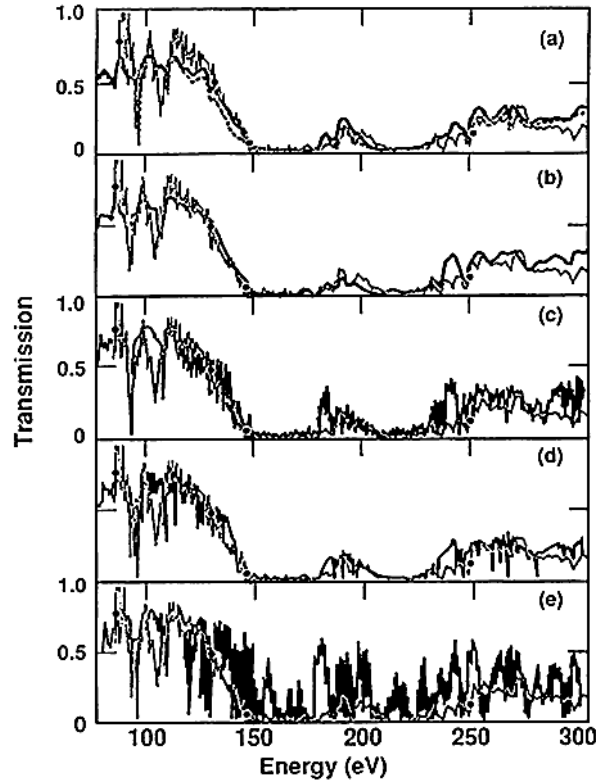
Laboratory measurements can determine opacities under material conditions that are the same as those present in some astrophysical systems. Indeed, they are essential for this. We saw in the case of EOS that such measurements under identical conditions was all one might hope for. The situation with regard to opacity is somewhat better, because opacities depend fundamentally upon quantum-mechanical processes within atoms. (In contrast, the EOS depends in part upon chemical interactions among groups of atoms.) The quantum-mechanical processes can be reliably scaled from one atom to another in computations. This is especially true along *isoelectronic sequences*, for which the number of electrons attached to the nucleus is the same. Such scalings may break down when the difference in nuclear charge becomes too large, introducing new issues such as relativistic effects into the calculation. The net effect is that laboratory measurements are essential in determining those opacities that can be measured. In addition, these results can validate computational approaches to calculating opacity of other elements, when such calculations can be scaled to the experiment.

The radiative transfer of energy through a star or a supernova is an example of a process that is complicated and three-dimensional, that is difficult to model, and that cannot be evaluated in a static experiment. Exploding stars create a homologous expansion, with velocity,  $v$ , radial distance,  $r$ , and time,  $t$ , related by  $v = r/t$ . As a result, each radiating region resides in a

velocity gradient and sees plasma receding from it in all directions. In other words, the absorbing regions are always red shifted relative to the emitting regions. The relative motion of any two locations creates Doppler shifts that move any specific emission line out of resonance with itself and (perhaps) into resonance with other lines. For photons emitted in one region to escape the star, they have to pass through “windows” in opacity, where the absorption probability is low. An adequate radiative transfer calculation must include the effects of the Doppler shifts in the opacity line and edge locations, due to the expansion. In due course, laboratory observations may prove to be of great value because of the near-impossibility of incorporating a fully correct treatment of radiative transfer into a computer simulation of an entire system. Experimental examples will be needed to validate (or invalidate) various possible approaches.

At this writing a number of experiments have been conducted to measure the LTE opacities of a variety of materials (e.g., Fe, Ge, Na, Al) at temperatures in the range of 10–75 eV and densities of 10–50 mg/cm<sup>3</sup>, using either lasers or *Z*-pinches as the energy source. We discussed above the difficulty of obtaining LTE conditions and this is a key issue for all of these experiments. The most common approach uses hohlraums (see Sect. 8.2) to provide an equilibrium radiation environment free of energetic electrons or strong nonthermal emission, either of which could alter the conditions of the sample. Measurement of this temperature is an essential detail. The sample is “tamped” by surrounding it with a low-*Z* material. This prevents rarefactions from reaching the sample and constrains it so that its density remains uniform and changes slowly. At a chosen moment, one produces a source of high-energy radiation that enters the hohlraum through a small hole, irradiates the sample, and exits the hohlraum through another hole. By making spectral measurements of this radiation, both through the sample and around the hohlraum, one can determine the spectral transmission through the sample and thus the total spectral opacity.

As an example (1992), we show in Fig. 6.6 the results of this type of measurement of the opacity of Fe at  $T_e = 59$  eV and  $\rho = 11$  mg/cm<sup>3</sup>. The electron temperature was measured using the spectrum from a Na dopant. The 1D radiographic spatial imaging gave the sample density of the thin Fe foil (sandwiched between tamping layers). Hence, the opacity of Fe was measured for known conditions of  $T_e$  and  $\rho$ . The experimental results shown in Fig. 6.6 were compared with several different opacity calculations employing different approximations. The conclusion of this work was an unambiguous demonstration of the need to include quantum-mechanical term splitting in the opacity calculations. Models that neglect this, such as DCA (panel e), significantly underpredict the opacity.



**Fig. 6.6.** Iron transmission spectra. Comparison of data (*gray line*) and calculations for a plasma of 80.2% Fe plus NaF, of density  $0.0113 \text{ g cm}^{-3}$  and areal density  $339 \mu\text{g cm}^{-2}$  at a temperature of 59 eV. The various models are described in Springer et al. (1992). Reproduced with permission

#### 6.2.4 Radiation Transfer in the Equilibrium Diffusion Limit

The discussion to this point has *not* assumed that the radiation is in equilibrium with the matter, or that the radiation mean free path is short. That is to say, we have not assumed that the system is in LTE. Yet there are cases in which an LTE description is justified, such as a stellar interior or an experiment in which an equilibrium region has been created. There are also far more cases in which an LTE description may be the only way to obtain an approximate answer through a tractable calculation. To describe these systems, we will consider the radiation properties to be a function of optical depth. We will take the point of view that the temperature can vary slowly, but only so slowly that the temperature change is negligible over a distance of one radiation mean free path (i.e.,  $\chi_\nu^{-1}$ ). In LTE, as in other cases with Maxwellian electron distributions, the radiation source is the thermal

source,  $S_\nu(\tau_\nu) = B_\nu(\tau_\nu)$ . If the temperature were constant, then the solution to (6.32) would be that  $I_\nu(\tau_\nu) = B_\nu(\tau_\nu)$ . As a result  $J_\nu(\tau_\nu) = B_\nu(\tau_\nu)$  and (6.35) would imply that there is no radiation flux. But suppose instead that there is a temperature gradient, small in the sense described above so that  $S_\nu$  can be described by a Taylor expansion relative to some initial location,  $\tau_\nu$ , as

$$S_\nu(\tau'_\nu) = \sum_{n=0}^{\infty} \frac{\partial^n B_\nu(\tau_\nu)}{\partial \tau_\nu^n} \frac{(\tau'_\nu - \tau_\nu)^n}{n!}. \quad (6.47)$$

Equation (6.44) can then be integrated with  $\tau_\nu \rightarrow \infty$  to find  $I_\nu(\tau, \mu)$ , again with  $\mu = \cos \theta$ , giving

$$I_\nu(\tau_\nu, \mu) = B_\nu(\tau_\nu) - \mu \frac{\partial B_\nu(\tau_\nu)}{\partial \tau_\nu} + \mu^2 \frac{\partial^2 B_\nu(\tau_\nu)}{\partial \tau_\nu^2} + \dots \quad (6.48)$$

Note that for forward-going radiation ( $\mu > 0$ ), if  $\partial B_\nu / \partial \tau_\nu > 0$  then  $I_\nu$  is smaller than the local value of  $B_\nu$ . This is as it should be. One can then apply the definitions of  $J_\nu, P_\nu$ , and  $F_\nu$  to obtain

$$J_\nu(\tau_\nu) = B_\nu(\tau_\nu) + \frac{1}{3} \frac{\partial^2 B_\nu(\tau_\nu)}{\partial \tau_\nu^2} + \dots, \quad (6.49)$$

$$p_\nu(\tau_\nu) = \frac{4\pi}{3c} B_\nu(\tau_\nu) + \frac{4\pi}{5c} \frac{\partial^2 B_\nu(\tau_\nu)}{\partial \tau_\nu^2} + \dots, \quad (6.50)$$

and

$$F_\nu(\tau_\nu) = -\frac{4\pi}{3} \frac{\partial B_\nu(\tau_\nu)}{\partial \tau_\nu} - \frac{4\pi}{5} \frac{\partial^3 B_\nu(\tau_\nu)}{\partial \tau_\nu^3} - \dots \quad (6.51)$$

### Homework 6.7

Beginning with (6.47), derive (6.48) to (6.51).

Note that the leading term in the spectral radiation flux is the first derivative of  $B_\nu$ , which is small compared to  $B_\nu$ . However, the radiation flux is  $c$  times the pressure (or  $c/3$  times the energy density), while the material energy fluxes are a much smaller multiple of the material energy density. As a result, here again the radiation flux can become important at much lower temperatures than those at which the radiation pressure does. If we recognize that  $B_\nu$  is a function of  $T$  only and that  $T$  varies with position, and we keep only the first term in (6.51), then we are led to write the total radiation energy flux as

$$\mathbf{F}_R = -\frac{4\pi}{3} \left[ \int_0^\infty \frac{1}{\chi_\nu} \frac{\partial B_\nu}{\partial T} d\nu \right] \nabla T. \quad (6.52)$$

This involves a rather different average than the Planck mean and absorption opacities discussed above. Specifically, the Rosseland *mean opacity* is defined by

$$\frac{1}{\chi_R} \equiv \frac{\int_0^\infty \frac{1}{\chi_\nu} \frac{\partial B_\nu}{\partial T} d\nu}{\int_0^\infty \frac{\partial B_\nu}{\partial T} d\nu} = \left[ \frac{\partial B}{\partial T} \right]^{-1} \int_0^\infty \frac{1}{\chi_\nu} \frac{\partial B_\nu}{\partial T} d\nu, \quad (6.53)$$

from which

$$\mathbf{F}_R = -\frac{4\pi}{3} \frac{1}{\chi_R} \frac{\partial B}{\partial T} \nabla T = -\frac{16\sigma T^3}{3\chi_R} \nabla T \equiv -\kappa_{\text{rad}} \nabla T. \quad (6.54)$$

We will call this the *Rosseland heat flux*. This equation defines the coefficient of radiative heat transport,  $\kappa_{\text{rad}}$ , valid only in the equilibrium diffusion limit. It is worth noting that the Rosseland heat flux is the equilibrium heat flux,  $\sigma T^4$ , multiplied by a small quantity. The small quantity is the fractional change in temperature per unit optical depth, defined here as  $(3/16) \chi_R$  times distance. Some characteristic values of  $\chi_R/\rho$ , in  $\text{cm}^2/\text{g}$ , for near-solid densities, are

$$\begin{aligned} \chi_R/\rho &\approx 2 \times 10^6 \rho^{1/7} T_{eV}^{-2} && \text{for CH} \\ \chi_R/\rho &\approx 3 \times 10^6 T_{eV}^{-4/3} && \text{for Al} \\ \chi_R/\rho &\approx 2 \times 10^8 T_{eV}^{-2} && \text{for Xe} \\ \chi_R/\rho &\approx 6 \times 10^6 \rho^{0.3} T_{eV}^{-3/2} && \text{for Au.} \end{aligned} \quad (6.55)$$

Again these are from the SESAME tables (but from Lindl for Au). Note that with these values  $\kappa_{\text{rad}}$  scales as  $T^4$  to  $T^5$ . In astrophysical regimes where the cooling function has a negative slope,  $\kappa_{\text{rad}}$  scales somewhat more rapidly, as  $T^6$  to  $T^7$ .

The thermal diffusive limit is actually more restrictive than it would appear from the above derivation. If we substitute from (6.49) for  $J_\nu$  into (6.38), which assumes only that the scattering is isotropic, and if we further assume that the variation in the opacities on the scale of the temperature gradient is negligible, we find

$$\nabla \cdot \mathbf{F}_R = -\frac{4\pi}{3} \nabla \cdot \left[ \int_0^\infty d\nu \frac{\kappa_\nu}{\chi_\nu^2} \frac{\partial B_\nu}{\partial T} \right] \nabla T. \quad (6.56)$$

This result is only consistent with (6.52) if the scattering opacity is much smaller than the absorption opacity, so that  $\kappa/\chi_\nu \sim 1$ . Pause a moment here. This is truly a bizarre result, because the derivations for equilibrium diffusion seem completely general. Yet the calculation leading to (6.56) is more fundamental, and only produces the same result in the case of small scattering. The solution to this dilemma is as follows. LTE can only exist, in the presence of a temperature gradient, if scattering is much smaller than absorption. Otherwise the photons are transported down the density gradient much faster than the material is heated, which will drive the radiation “temperature” out of equilibrium with the material temperature. Absorption may be larger than scattering if free–free transitions, notably bremsstrahlung and inverse bremsstrahlung, dominate the radiation–matter interactions. This may

often be the case in high-energy-density plasmas. But when bound-bound transitions dominate the radiation-matter interactions then scattering will dominate and LTE will be much less likely to occur.

### Homework 6.8

Derive (6.56). Discuss the result.

### 6.2.5 Nonequilibrium Diffusion and Two-Temperature Models

The consequence of the discussion above is that there may be circumstances in which the radiation transport *is* diffusive but the radiation and the matter *are not* in equilibrium. Such systems are in fact quite common, both in the laboratory and in astrophysics. The simplest treatment of them treats the radiation transport as diffusive but does not assume that  $E_R = 4\pi B(T)/c$ , where  $T$  is the material temperature. Thus, from (6.42), the spectral radiation flux is

$$\mathbf{F}_\nu = -\frac{c}{3\chi_\nu} \nabla E_\nu, \quad (6.57)$$

and the total radiation flux is

$$\mathbf{F}_R = -\frac{c}{3} \int_0^\infty \frac{\nabla E_\nu}{\chi_\nu} d\nu = -\frac{c}{3\bar{\chi}} \nabla E_R, \quad (6.58)$$

which defines an averaged opacity,  $\bar{\chi}$ , that might be approximated as the Rosseland mean opacity.

In this case we will anticipate our work in Chap. 7, including fluid motion and radiation-fluid energy exchange in our result. For this purpose, we can extract the radiation terms from the equation for the internal energy of the gas and radiation (7.4), ignore heat conduction, use (6.58) for the radiation flux, take  $p_R = E_R/3$ , and include the radiation-matter energy exchange to obtain

$$\frac{DE_R}{Dt} - \frac{4E_R}{3\rho} \frac{D\rho}{Dt} = \nabla \cdot \left( \frac{c}{3\bar{\chi}} \nabla E_R \right) + 4\kappa_P \sigma T^4 - c\kappa_E E_R, \quad (6.59)$$

in which  $D/Dt$  is the convective derivative (see Chap. 7) and

$$\kappa_E = \int_0^\infty \kappa_\nu E_\nu d\nu / \int_0^\infty E_\nu d\nu \quad (6.60)$$

is yet another averaged opacity, perhaps of order  $\kappa_P$ . It is common to assign a “temperature” to the radiation, defined by  $E_R = 4\sigma T_R^4/c$ , so that (6.59) becomes an equation for the radiation temperature,

$$\frac{D}{cDt} (\sigma T_R^4) - \frac{4\sigma T_R^4}{3\rho} \frac{D\rho}{cDt} = \nabla \cdot \left( \frac{4\sigma T_R^3}{3\bar{\chi}} \nabla T_R \right) + \kappa_P \sigma T^4 - \kappa_E \sigma T_R^4. \quad (6.61)$$

There is a corresponding, simultaneous equation for the internal energy of the gas:

$$\rho \left[ \frac{D\epsilon}{Dt} - \frac{p}{\rho^2} \frac{D\rho}{Dt} \right] = 4\kappa_E \sigma T_R^4 - 4\kappa_P \sigma T^4, \quad (6.62)$$

which we can recognize as the generalization of the purely hydrodynamic equation for energy to include sources and losses of energy from exchange with radiation.

### 6.3 Relativistic Considerations for Radiative Transfer

While there are some systems in the universe, such as pulsar envelopes or experiments with lasers at  $10^{21}$  W/cm<sup>2</sup>, that are manifestly relativistic, most laboratory and astrophysical systems seem at first glance manifestly nonrelativistic. This, however, is often not true in at least two senses. First, because spectral lines are very narrow, often having a normalized line width of order  $10^{-4}$ , Doppler shifts can complicate the transport of energy by line radiation at velocities as small as  $10^{-4}c$ . Second, the Lorentz transformation between an observer and a fluid, or between different parts of a fluid, introduces terms in all orders of  $v/c$ , where  $v$  is a velocity difference between frames of reference. But the moments of the nonrelativistic transfer equations already contain terms that differ from one another by  $v/c$ , and the terms that are first order in  $v/c$  at times become the dominant ones (for example if  $F_R/E_R < v/c$ ). As a result the leading relativistic terms, which are of order  $v/c$ , may contribute as much to the radiative transfer as the nonrelativistic terms in the equations. It turns out that the terms which matter, to this order, are just those one would find from Galilean relativity.

We will not attempt a derivation of all the relations among relativistic radiative transfer equations here. This is a large project, carried out for example in *Mihalas and Mihalas* and references therein. Our goal, instead, is to introduce the relativistic effects, discuss their origin, and discuss the equations that result.

In discussing relativistic effects, we will write equations that relate quantities in two frames of reference. The first frame, designated by the subscript  $o$ , is the frame that is at rest locally within the fluid. This is the frame in which the microscopic interactions of radiation and matter are correctly described by nonrelativistic equations, sometimes known as the *proper frame*. For example, emission from random processes is isotropic only in this frame. The second frame, designated by no special subscript, is in motion with velocity  $\mathbf{v}$  relative to the first frame. Recalling that the phase space of radiation intensity involves radiation (or photons) of frequency  $\nu$  and direction  $\mathbf{n}$ , the Doppler shift and aberration are given by

$$\nu_o = \gamma_r \nu (1 - \mathbf{n} \cdot \mathbf{v}/c) \quad (6.63)$$



and

$$\mathbf{n}_o = \frac{\nu}{\nu_o} \left[ \mathbf{n} - \gamma_r \frac{\mathbf{v}}{c} \left( 1 - \frac{\gamma_r \mathbf{n} \cdot \mathbf{v}/c}{\gamma_r + 1} \right) \right] \quad (6.64)$$

in which  $\gamma_r$  is the relativistic contraction factor,  $\gamma_r = 1/\sqrt{1 - v^2/c^2}$ . The inverse transformations of these quantities are

$$\nu = \gamma_r \nu_o (1 + \mathbf{n} \cdot \mathbf{v}/c) \quad (6.65)$$

and

$$\mathbf{n} = \frac{\nu_o}{\nu} \left[ \mathbf{n}_o + \gamma_r \frac{\mathbf{v}}{c} \left( 1 + \frac{\gamma_r \mathbf{n}_o \cdot \mathbf{v}/c}{\gamma_r + 1} \right) \right]. \quad (6.66)$$

There are some aspects of these equations worth noting from the standpoint of radiation transport. First, the Doppler shift depends on direction, so that local emission in some frequency band appears distributed into a range of frequencies when viewed in a moving frame of reference. Second, the measured direction depends on the frame of reference, so that isotropic emission in the local frame of reference does not appear isotropic in another frame of reference. The well-known “beaming effect” is present in (6.66) – as  $\gamma_r$  becomes large, the radiation all is observed to lie near the direction of  $\mathbf{v}$  in the moving frame.

It is very helpful in working with relativistic systems to identify which quantities are Lorentz invariant, as this greatly facilitates the conversion between frames of reference. From the above equations, one can show that

$$\nu d\nu d\Omega = \nu_o d\nu_o d\Omega_o. \quad (6.67)$$

### Homework 6.9

Demonstrate this.

In addition, photon number in a given volume must be independent of frame of reference, from which one can obtain several relations of use in radiative transfer. Specifically,

$$I_\nu(\mu, \nu)/\nu^3 = I_{\nu_o}(\mu_o, \nu_o)/\nu_o^3, \quad (6.68)$$

$$\eta(\mu, \nu)/\nu^2 = \eta_o(\mu_o, \nu_o)/\nu_o^2, \quad (6.69)$$

and

$$\nu \chi(\mu, \nu) = \nu_o \chi_o(\mu_o, \nu_o). \quad (6.70)$$

### Homework 6.10

Given these relations, show that the radiative transfer equation is relativistically invariant.

It may also be worthwhile for reference to provide here the result of the Lorentz transformation from the frame moving with the material to the frame of an observer moving at velocity  $v$  relative to the material, in planar geometry, for the moments of  $I_R$ . These are

$$\begin{aligned} E_R &= \gamma_r^2 \left( E_{R_o} + 2\frac{v}{c} \frac{F_{R_o}}{c} + \frac{v^2}{c^2} p_{R_o} \right) \approx E_{R_o} + 2\frac{v}{c} \frac{F_{R_o}}{c} \\ F_R &= \gamma_r^2 \left[ \left( 1 + \frac{v^2}{c^2} \right) F_{R_o} + vE_{R_o} + vp_{R_o} \right] \approx F_{R_o} + vE_{R_o} + vp_{R_o} \\ p_R &= \gamma_r^2 \left( p_{R_o} + 2\frac{v}{c} \frac{F_{R_o}}{c} + \frac{v^2}{c^2} E_{R_o} \right) \approx p_{R_o} + 2\frac{v}{c} \frac{F_{R_o}}{c}, \end{aligned} \quad (6.71)$$

in which the second approximate equality gives the result to order  $v/c$ .

In the context discussed above, our interest here is in the relativistic effects that are first-order in  $v/c$ . Dealing with strong relativistic effects is beyond our scope. The fact that the radiative transfer equation is relativistically invariant has less utility than we might wish for, because the emission and opacity are very inconvenient in frames of reference in which they are not isotropic in angle and frequency. This leads us to always want to evaluate these quantities as functions in the local frame. So our first task is to obtain a relation between the spectral intensity in an arbitrary inertial frame and the plasma properties in a local frame, to first order in  $v/c$ . Accordingly, we take

$$\nu = \nu_o(1 + \mathbf{n} \cdot \mathbf{v}/c), \quad (6.72)$$

from which one can relate the emission or opacity in the moving frame to the corresponding quantity in the local frame, evaluated at the same frequency, as

$$\chi(\mathbf{n}, \nu) = \chi_o(\nu) - (\mathbf{n} \cdot \mathbf{v}/c) [\chi_o(\nu) + \nu_o(\partial\chi_o/\partial\nu)] \quad (6.73)$$

and

$$\eta(\mathbf{n}, \nu) = \eta_o(\nu) + (\mathbf{n} \cdot \mathbf{v}/c) [2\eta_o(\nu) - \nu_o(\partial\eta_o/\partial\nu)]. \quad (6.74)$$

From these, one can obtain the radiative transfer equation, in Cartesian coordinates, for an inertial frame, as

$$\begin{aligned} \frac{1}{c} \frac{\partial I_\nu(\mathbf{n}, \nu)}{\partial t} + \mathbf{n} \cdot \nabla I_\nu(\mathbf{n}, \nu) &= \eta_o(\nu) - \chi_o(\nu) I_\nu(\mathbf{n}, \nu) + \\ &\left( \frac{\mathbf{n} \cdot \mathbf{v}}{c} \right) \left[ 2\eta_o(\nu) - \nu \frac{\partial \eta_o}{\partial \nu} + \left( \chi_o(\nu) + \nu \frac{\partial \chi_o}{\partial \nu} \right) I_\nu(\mathbf{n}, \nu) \right]. \end{aligned} \quad (6.75)$$

### Homework 6.11

Derive (6.73), (6.74), and (6.75). Discuss the limits on  $v/c$  for this specific description if the emission and absorption are dominated by (a) continuum emission or (b) line emission.

In (6.75) we have achieved our goal of relating the spectral intensity in a moving, inertial frame to the plasma properties in a local frame where they are angularly symmetric. By taking moments of this equation, as above one can obtain the following equations for the radiation energy and momentum:

$$\begin{aligned} \frac{\partial E_R}{\partial t} + \nabla \cdot \mathbf{F}_R &= \int_0^\infty [4\pi\eta_o(\nu) - c\chi_o(\nu)E_\nu] d\nu + \\ &\quad \frac{\mathbf{v}}{c} \cdot \int_0^\infty \left[ \chi_o(\nu) + \nu \frac{\partial \chi_o}{\partial \nu} \right] \mathbf{F}_\nu d\nu \end{aligned} \quad (6.76)$$

and

$$\begin{aligned} \frac{1}{c^2} \frac{\partial \mathbf{F}_R}{\partial t} + \nabla \cdot \mathbf{P}_R &= \frac{-1}{c} \int_0^\infty \chi_o(\nu) \mathbf{F}_\nu d\nu + \frac{4\pi\mathbf{v}}{c^2} \int_0^\infty \eta_o(\nu) d\nu \\ &\quad + \frac{\mathbf{v}}{c} \cdot \int_0^\infty \left[ \chi_o(\nu) + \nu \frac{\partial \chi_o}{\partial \nu} \right] \mathbf{P}_\nu d\nu. \end{aligned} \quad (6.77)$$

In the first of these equations, for example, the second term on the right-hand side is new by comparison with (6.38). It can be essential. In the diffusion regime, for example the two components of the integral in the first term on the right-hand side are nearly in balance, differing only because of the relativistic shift of  $E_\nu$  between the reference frame and the local frame. The net remaining value of the first term is of the same order as the second term.

The ability to work with radiation in an inertial frame of reference moving at a fixed velocity relative to a specific volume of plasma may be of use, but is in fact insufficient for typical radiation hydrodynamic problems. What one actually needs is the ability to *always* treat the radiation in the local frame of reference, so the emission and opacity are isotropic, even as the plasma velocity changes from place to place. An analysis in which one continuously transforms the frame of reference as the radiation moves through the plasma is described as an analysis in the *comoving frame*. This name is rather misleading however, as it represents no fixed frame of reference and as any given frame of reference may be accelerating and therefore not an inertial frame. Instead, the “comoving frame” represents a continuously varying sequence of frames of reference that are always at rest in the local plasma. This has the effect of introducing terms into the radiation transfer equation that depend upon the local fluid velocity,  $\mathbf{u}$ , derivatives of the local fluid velocity, and the local acceleration,  $\mathbf{a}$ . We will not work through this derivation here (see *Mihalas and Mihalas*) but will provide the resulting first two moments of the radiative transfer equation, which are

$$\begin{aligned} &\rho \frac{D}{Dt} \left( \frac{E_{R0}}{\rho} \right) + \nabla \cdot \mathbf{F}_{R0} + \mathbf{P}_{R0} : \nabla \mathbf{u} + \frac{2}{c^2} \mathbf{a} \cdot \mathbf{F}_{R0} \\ &= \int_0^\infty [4\pi\eta_o(\nu_o) - c\chi_o(\nu_o)E_{\nu_o}] d\nu_o \end{aligned} \quad (6.78)$$

and

$$\begin{aligned} & \frac{\rho}{c^2} \frac{D}{Dt} \left( \frac{\mathbf{F}_{\text{Ro}}}{\rho} \right) + \nabla \cdot \underline{\mathbf{P}}_{\text{Ro}} + \frac{1}{c^2} \mathbf{F}_{\text{Ro}} \cdot \nabla \mathbf{u} + \frac{1}{c^2} (E_{\text{Ro}} \mathbf{a} + \mathbf{a} \cdot \underline{\mathbf{P}}_{\text{Ro}}) \\ & = -\frac{1}{c} \cdot \int_0^\infty \chi_o(\nu_o) \mathbf{F}_{\nu_o} d\nu_o. \end{aligned} \quad (6.79)$$

In these equations the velocity terms mentioned above have been compactly expressed through the division by  $\rho$  in the convective derivative and the terms involving the tensor  $\nabla \mathbf{u}$ . Also,  $\underline{\mathbf{P}}_{\text{Ro}} : \nabla \mathbf{u}$  is dyadic notation and could also be written as  $(\underline{\mathbf{P}}_{\text{Ro}} \cdot \nabla) \cdot \mathbf{u}$ . Recall that these equations are only accurate to order  $u/c$ . *Mihalas and Mihalas* give more general results in the nonaccelerating limit of special relativity.

Comparing (6.78) with (6.38) and its derivation, one can see that one would recover the latter from the former for a stationary medium with isotropic scattering. The terms involving  $\rho$  and  $\mathbf{u}$  in these equations arise from the flux of radiative energy due to fluid motion and from adiabatic work done by the radiation and the fluid upon one another. We could have obtained these terms from a nonrelativistic derivation, but would have been unsure of their correctness. The good news is that Galilean intuition suffices for systems in which  $u \ll c$ . As mentioned above, these effects may not be negligible. In particular  $uE_{\text{Ro}}$  can exceed  $F_{\text{Ro}}$  under some circumstances. To properly treat radiation in higher-velocity systems would require that one revisit the derivation of radiative transfer under the conditions of interest.

### Homework 6.12

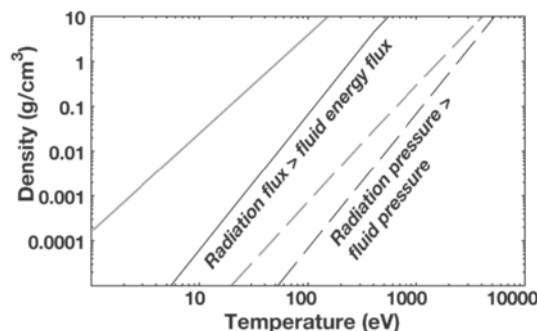
Rework (6.78) into the form of a conservation equation and discuss the meaning of the terms that result.

## 7 Radiation Hydrodynamics

It is fair to say that we never directly experience radiation hydrodynamic phenomena – that is, phenomena in which the radiation directly participates in the hydrodynamic evolution of a system. We do experience consequences of radiative heat transport, as for example when heating by solar irradiation produces wisps of fog above a wet road. And we are aware of some systems, such as solar sails, in which radiation directly causes material motion. But as we shall see, radiation hydrodynamic phenomena require temperatures of millions of degrees, more or less, so they are outside the realm of our direct experience.

One would like to know when radiation affects hydrodynamics in important ways. This requires either that the radiation flux becomes comparable with the material energy flux or that the radiation pressure becomes comparable with the material pressure. Thus our first goal is to see when radiation hydrodynamics matters.

To find the conditions under which radiation affects hydrodynamics by a simple calculation, one must make some assumptions. Assume that the electron temperature and the radiation temperature are comparable and equal to  $T$ . If the ion temperature matters, which it often does not, we also assume it to be not too far from  $T$ . Further assume that the systems of interest are optically thick, which matters in determining the radiation flux but also in keeping the temperatures comparable. Under these conditions, the radiation flux is  $\sigma T^4$ . We can evaluate the material energy flux by examining (2.4), finding it to be  $\rho u[\epsilon + (u^2/2) + (p/\rho)]$ , where  $u$  is some characteristic velocity. Here we take  $u$  to equal the sound speed  $c_s$  – any other reasonable number will be within a small enough multiple of  $c_s$  that the results will not be affected. For the specific energy of the material,  $\epsilon$ , we use the hydrogenic model discussed in Chap. 3. For specific conditions, we can then identify the boundary in temperature above which the radiation flux dominates. Similarly, taking the radiation pressure from (6.20) and the material pressure from (3.1), we can identify the boundary in temperature above which the radiation pressure dominates.



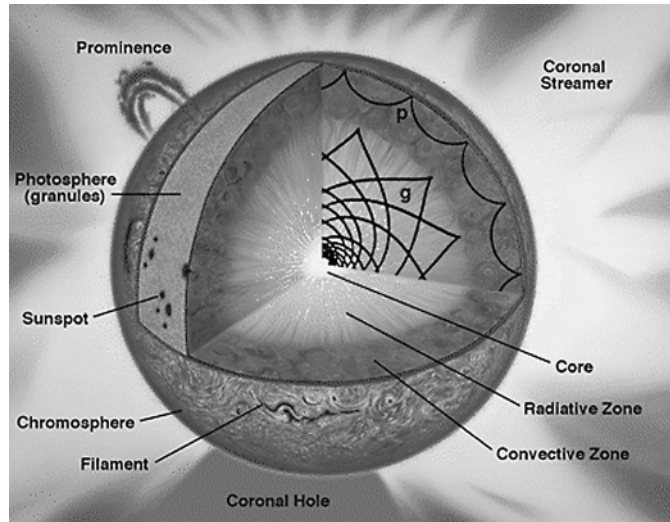
**Fig. 7.1.** Radiation hydrodynamic regimes. The gray curves are for Xe and the black curves are for CH

### Homework 7.1

Carry out the calculations just described and compare the behavior of pure hydrogen as opposed to  $C_1H_1$  (used in Fig. 7.1).

Figure 7.1 shows the results of such a calculation for  $C_1H_1$ , assumed to be fully ionized, and for Xe, assumed to be ionizing with  $Z = 0.63\sqrt{T}$ . In the region to the right of a solid line, the radiation flux exceeds the material energy flux. In the region to the right of a dashed line, the radiation pressure exceeds the material pressure. This is the radiation-dominated regime discussed briefly in Chap. 3. One sees that Xe is more radiative than  $C_1H_1$ ; the Xe curves are displaced to lower temperature. This is no surprise. Xe has  $A = 131.3$  and  $Z_{\text{nuc}} = 54$ . Over this range of temperatures it still has electrons in  $s, p, d$ , and  $f$  shells, allowing very many x-ray transitions, while the C has only six electrons. The  $C_1H_1$  is assumed to be fully ionized; accounting for the internal energy of the C properly would move the solid black curve slightly to the right. Leaving aside the fine details, one can see that radiation fluxes become important at temperatures of tens of eV, and that the exact value depends on details. The plasma becomes radiation-dominated at temperatures of hundreds of eV, again with exact values depending on details. In the important case of plasmas near  $1 \text{ g/cm}^3$  in density, it takes temperatures of about 2 keV to make the plasma radiation-dominated.

Stellar interiors include regions in which radiation affects the hydrodynamics. This is not surprising as the essential behavior of stars is to release energy in their cores and then to radiate it away from their surfaces. As a result, radiative fluxes must exceed material energy fluxes, and they do, for example, in the sun. Figure 7.2 shows results of a simulation of the sun. A convective zone exists because diffusion of the radiation is not fast enough to transport the energy generated by nuclear fusion to the solar surface. Larger, hotter, stars include regions in which the radiation pressure exceeds the material pressure. In addition, all supernovae heat the stellar interior into the radiation-dominated regime, where it stays until it cools sufficiently through



**Fig. 7.2.** Cut-away view of the Sun. Energy is released by fusion in the core, propagates by radiative transfer in the radiative zone, and drives radiation-hydrodynamic convection in the convection zone. From Sun, in the Encyclopedia of Planetary Sciences, Chapman & Hall, 1997, used with permission

volumetric expansion. Some dense astrophysical environments, such as neutron stars and black hole regions, can be strongly radiation-dominated, and even relativistic. Here we will not consider the relativistic cases, as laboratory experiments are a long way from accessing them.

Interstellar astrophysical systems, including the interstellar medium, interstellar shocks, and molecular clouds, have densities more than 15 orders of magnitude below those shown in Fig. 7.1. The curves there make it appear that such systems would be in a radiative regime. However, such interstellar astrophysical systems violate a key assumption of this figure. They always have very small optical depth. The radiative flux from an optically thin system, for thermal emission, equals  $\kappa d \sigma T^4$ , where  $\kappa$  is the absorption opacity of the system and  $d$  is its size. However, many optically thin systems, especially in astrophysics, produce primarily line emission, in which case  $\kappa d$  would be an appropriate average over the spectral variation of the optical depth and thermal spectrum. (The relation of  $\kappa$  and the cooling function,  $\Lambda$ , is discussed in Sect. 7.2.2) Moreover, at low density the opacity decreases as density squared while the material energy fluxes decrease linearly with density. In this regime, the temperature required to enter radiative regime must increase as density decreases. Curiously, this increases the shock velocity required for radiation fluxes to be significant into the range of  $>100$  km/s, which is just where it is for laboratory experiments with foams or dense gas. The small optical depth and lack of sources or boundaries also implies that the radiation “temperature” remains decoupled from and much smaller than

the electron temperature. As a result, the radiation pressure never exceeds the material pressure in interstellar astrophysical systems and the radiation-dominated regime is genuinely difficult to reach in experiments.

However, the radiation flux may at times be essential to the dynamics of such systems even at densities below  $10^{-20}$  g/cm<sup>3</sup>. This can occur when the energy input from the interactions of matter, such as  $pdV$  work, becomes smaller than the radiative cooling. An astrophysical shock wave compresses and heats the material it shocks, but after that there is often no further energy input. The material cools by radiation, however long this may take. The shocked layer of material produced by a supernova is driven for a long time, gaining energy from the pressure of the hot “bubble” created by the explosion. But eventually this ends, and the layer subsequently cools by radiation, eventually decreasing in temperature and increasing in density by orders of magnitude. We discuss the dynamics of these astrophysical phenomena later in the chapter, when we discuss optically thin radiative shocks.

## 7.1 Radiation Hydrodynamic Equations

Our first task is to develop the equations we will need to account for radiation hydrodynamic phenomena in the high-energy-density regime. The simple equations of Chap. 2 (2.27 and 2.28) are useful for considering the relative contributions of the various processes that may occur in an energetic fluid or plasma, all of which are included, for example, in many simulation codes. However, these equations are rarely practical for simple calculations. In the context of the present chapter we will ignore viscous effects and usually heat conduction – these play a very limited role in most systems hot enough for radiative effects to matter. We will also necessarily work with only the simplest models of radiation transport. However, there certainly are cases in which more-complicated models are needed to obtain an accurate description. A wider range of such models is discussed in Mihalas and Mihalas.

### 7.1.1 Fundamental Equations

We will treat radiation here primarily as a fluid, able to carry momentum and energy. When radiation becomes important, it effectively introduces an additional fluid species into the plasma. For nonrelativistic systems with isotropic radiation fields, the momentum equation remains quite simple. The momentum of the radiation is negligible compared to that of the fluid in this limit. The effect of the isotropic radiation field on the matter can be expressed as a gradient in the scalar radiation pressure. The momentum equation then becomes

$$\rho \frac{\partial}{\partial t} \mathbf{u} + \rho \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla(p + p_R), \quad (7.1)$$



in which  $p_{\text{R}}$  is the scalar radiation pressure. There are usually no other significant forces in laboratory systems, but there might be other forces such as gravitation in an astrophysical problem. In this case the right-hand side will have additional terms.

To obtain useful energy equations, we need generalized versions of (2.13) and (2.14). To obtain them, we rearrange (2.28), dropping the terms involving viscosity, to find

$$\begin{aligned} \frac{\partial}{\partial t} \left( \frac{\rho u^2}{2} + \rho\epsilon + E_{\text{R}} \right) + \nabla \cdot \left[ \rho \mathbf{u} \left( \epsilon + \frac{u^2}{2} \right) + (E_{\text{R}} + p + p_{\text{R}}) \mathbf{u} \right] \\ = -\nabla \cdot (\mathbf{F}_{\text{R}} + \mathbf{Q}). \end{aligned} \quad (7.2)$$

If there are additional forces such as gravitation, they will introduce terms relating to both potential energy and work into this equation. Note that this equation involves fundamentally a Galilean treatment of the radiation energy density. In particular, it convects with velocity  $\mathbf{u}$ . This is certainly not valid for strongly relativistic fluid velocities (or frames of reference), but is accurate to order  $u/c$ , as we found in Sect. 6.3.

From the dot product of  $\mathbf{u}$  with the momentum equation, one can find the mechanical energy equation

$$\rho \left( \frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla \right) \left( \frac{u^2}{2} \right) = \rho \frac{D}{Dt} \left( \frac{u^2}{2} \right) = -\mathbf{u} \cdot \nabla (p + p_{\text{R}}), \quad (7.3)$$

in which we introduce the common notation that  $D/Dt$  represents the total derivative. Subtracting this equation from (7.2), gathering all the derivatives of density, which turn out to cancel through the use of the continuity equation, and also using the continuity equation to replace  $\nabla \cdot \mathbf{u}$ , one is left with this equation for the internal energy of the gas and the radiation energy:

$$\rho \frac{D}{Dt} \epsilon + \frac{D}{Dt} E_{\text{R}} - \frac{(E_{\text{R}} + p + p_{\text{R}})}{\rho} \frac{D}{Dt} \rho = -\nabla \cdot (\mathbf{F}_{\text{R}} + \mathbf{Q}). \quad (7.4)$$

This equation can be rearranged into several useful forms. One that we will make use of involves the total energy density, and is

$$\frac{D}{Dt} (\rho\epsilon + E_{\text{R}}) - \frac{(\rho\epsilon + p) + (E_{\text{R}} + p_{\text{R}})}{\rho} \frac{D}{Dt} \rho = -\nabla \cdot (\mathbf{F}_{\text{R}} + \mathbf{Q}). \quad (7.5)$$

Note that up to this point, we have assumed only that the system is not relativistic. We have made no assumptions about the equation of state. In nearly all applications one could also drop  $\mathbf{Q}$  from this equation, but we keep it because it will be mathematically simple to do so. We also note that formally the radiation quantities in these equations should be evaluated in a frame of reference moving with the fluid – the *comoving frame* discussed in Sect. 6.3. For problems that can be addressed with the above equations, one

can generally ignore this distinction at least until it is necessary to consider what will be detected by an external observer.

Equation (7.5) can be expressed more simply if there is a simple relation between pressure and energy density. Let us explore this. On the one hand, if the radiation pressure is negligible and the matter is a polytropic gas, then  $\rho\epsilon = p/(\gamma - 1)$  and the left-hand side of (7.5) reduces to that of (2.14) with an appropriate value of  $\gamma$ . On the other hand, if the material pressure is negligible and the radiation pressure is dominant, then we showed in Chap. 3 that the same relation applies, so one also recovers the left-hand side of (2.14) with  $\gamma = 4/3$ . One can also see from (7.3) and (7.4) that, when radiation is dominant, the momentum and energy equations can be placed in the form of the Euler equations, with radiation pressure replacing the material pressure.

### 7.1.2 Thermodynamic Relations

For the intermediate cases we return to the discussion of Sect. 3.3 for media with variable polytropic indices. We will evaluate the behavior for thermal radiation, thus assuming optically thick, near-equilibrium plasmas. The total pressure,  $\tilde{p}$ , is the sum of the plasma pressure and the radiation pressure, so the equation of state is

$$\tilde{p} = \frac{\rho(1+Z)k_B T}{Am_p} + \frac{4\sigma}{3c}T^4. \quad (7.6)$$

The total specific energy density,  $\tilde{\epsilon}$ , is the sum of the contributions from thermal particles, radiation, and internal energy, again designated as  $R(T)$ .

$$\tilde{\epsilon} = \frac{3}{2} \frac{(1+Z)k_B T}{Am_p} + \frac{R(T)}{Am_p} + \frac{4\sigma}{\rho c}T^4. \quad (7.7)$$

We then can use a hydrogenic model for  $R$ , as discussed in Sect. 3.3, to find

$$\gamma = 1 + \frac{\tilde{p}}{\rho\tilde{\epsilon}} = \frac{(1+Z)[15 + (E_H/k_B T)Z(1+2Z)] + 32\alpha}{(1+Z)[9 + (E_H/k_B T)Z(1+2Z)] + 24\alpha}, \quad (7.8)$$

where

$$\alpha = \frac{\sigma T^4/c}{n_i k_B T} \quad (7.9)$$

is proportional to the ratio of radiation energy density to material energy density and  $n_i$  is the number density of the ions. Recall that this is the value of  $\gamma$  that is relevant to shock transitions. To find the value of  $\gamma_3$ , required for heat transport calculations, we need the specific heat at constant volume,

$$c_V = \left( \frac{\partial \tilde{\epsilon}}{\partial T} \right)_\rho = \frac{3}{2} \frac{n_i [1 + Z + T(\partial Z/\partial T)_\rho] k_B}{\rho} + \frac{n_i R'(T)}{\rho} + \frac{16\sigma}{\rho c} T^3, \quad (7.10)$$

in which  $R'(T)$  is the derivative of  $R$  with respect to  $T$ , and the partial derivative of  $\tilde{p}$  with respect to  $T$  at constant density, is

$$\left(\frac{\partial \tilde{p}}{\partial T}\right)_\rho = n_i [1 + Z + T(\partial Z/\partial T)_\rho] k_B + \frac{16\sigma}{3c} T^3. \quad (7.11)$$

This then gives

$$\frac{1}{(\gamma_3 - 1)} = \frac{\rho c_V}{(\partial \tilde{p}/\partial T)_\rho} = \frac{3[2(R'(T)/k_B) + 3[1 + Z + T(\partial Z/\partial T)_\rho]] + 96\alpha}{6[1 + Z + T(\partial Z/\partial T)_\rho] + 32\alpha} \quad (7.12)$$

for  $\gamma_3$ . Note that while  $\gamma$  depends on  $R$ , which increases to a maximum and then remains there as  $T$  increases,  $\gamma_3$  depends on  $R'$ , which drops abruptly to 0 when the plasma is fully ionized. Note also that for  $Z \propto \sqrt{T}$ ,  $T(\partial Z/\partial T)_\rho = Z/2$ . One can rearrange this last equation to have

$$c_V = \frac{1}{(\gamma_3 - 1)} \frac{1}{\rho} \left(\frac{\partial \tilde{p}}{\partial T}\right)_\rho, \quad (7.13)$$

a form that is useful for applications when one already has a sufficiently accurate value for  $\gamma_3$ .

We also need the sound speed, from (3.72), for calculations involving acoustic waves. To evaluate this, we need

$$\left(\frac{\partial \tilde{p}}{\partial \rho}\right)_T = \frac{(1 + Z)k_B T}{(Am_p)} \quad (7.14)$$

and

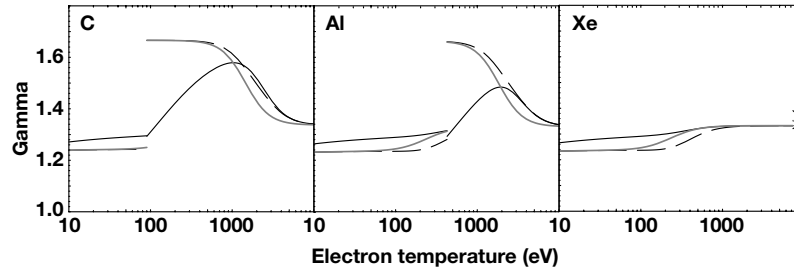
$$\left(\frac{\partial \tilde{\epsilon}}{\partial \rho}\right)_T = \frac{-4\sigma T^4}{c\rho^2}. \quad (7.15)$$

From these, one obtains a rather messy expression of for  $\gamma_s$ :

$$\gamma_s = \frac{45(1 + Z)^2 + 18(R'(T)/k_B)(1 + Z) + 480(1 + Z)\alpha + 512\alpha^2}{3(3(1 + Z) + 4\alpha)(3(1 + Z) + 2(R'(T)/k_B) + 32\alpha)}. \quad (7.16)$$

Figure 7.3 shows values of  $\gamma$ ,  $\gamma_s$ , and  $\gamma_3$  for near-equilibrium radiating plasmas composed of three different elements. Here we have assumed  $Z = 0.63\sqrt{T}$  with  $T$  in eV until  $T = 2.5Z_{\text{nuc}}^2$ , above which we have taken  $Z = Z_{\text{nuc}}$ , the nuclear charge. We have evaluated  $R$  as in Sect. 3.3, using a hydrogenic model of the ions and ignoring excited states. One can see in the figure the abrupt change in  $\gamma_s$  and  $\gamma_3$ , but not  $\gamma$ , when the plasma becomes fully ionized. It is not a terrible assumption to take  $\gamma_s = \gamma_3 = \gamma$ , except just above the temperature where an element becomes fully ionized. We ignore this regime in the applications. It is worth noting, though, that using a single value of  $\gamma$  does not permit one to use the simple equations of Sect. 3.1.1. One still must evaluate quantities such as the specific heat properly for the radiative regime.

To go beyond equilibrium models, one has the same sorts of choices we explored in Chap. 2 (and Sect. 6.2.4) regarding how to treat the entire physical



**Fig. 7.3.** Values of  $\gamma$  (solid curve),  $\gamma_s$  (dashed curve), and  $\gamma_3$  (thick gray curve) for carbon (top), aluminum (middle), and xenon (bottom)

system. Just as we saw there that there is a choice between a single-fluid and a multiple-fluid treatment of the particles, there is a similar choice regarding the radiation, especially with regard to the energy equation. A common type of model in simulations is the single-fluid, three-temperature model discussed in Chap. 2. Such models use a single continuity equation and a single momentum equation, but make the pressure the sum of the electron, ion, and radiation pressures. Then one works with separate energy equations for the electrons, the ions, and the radiation, keeping track of the energy exchange between these species. Such models often employ a generalized nonequilibrium diffusion treatment of the radiative energy transport, along the lines discussed in Sect. 6.2.4. Alternatively, they may employ more-sophisticated radiation transport methods in connection with a single-fluid, two-temperature (electron and ion) treatment of the matter. Such models, even with a diffusive treatment of radiation heat transport, are comparatively tractable and often give qualitatively correct results.

## 7.2 Radiation and Fluctuations

Strong radiation, when present, affects every hydrodynamic process that occurs in a medium. Much of the material in this chapter involves the examination of how radiation can alter the behavior of phenomena we have already explored, such as acoustic waves and shocks. Some additional phenomena, including thermal instabilities and diffusive heat waves, have analogues in systems with heat conduction.

### 7.2.1 Radiative Acoustic Waves; Optically Thick Case

To see the effect of radiation on acoustic waves, we will begin by examining what would happen deep within an optically thick, near-equilibrium, radiating medium such as a stellar interior or an experimental volume that is hot enough and dense enough. We begin with the limit in which the system is

so near equilibrium and so optically thick to the radiation that  $B \sim J_R$ . In this limit the radiative flux is given by (6.54). Accordingly, we use the results from the previous section, along with (3.71) through (3.73), with a diffusive model for the heat transport so that

$$\rho \frac{Dq}{Dt} = -\nabla \cdot \mathbf{F}_R = \nabla \cdot (\tilde{\kappa} \nabla T), \text{ to obtain} \quad (7.17)$$

$$\frac{D\tilde{p}}{Dt} - \frac{\gamma_s \tilde{p}}{\rho} \frac{D\rho}{Dt} = (\gamma_3 - 1) \nabla \cdot (\tilde{\kappa} \nabla T), \quad (7.18)$$

in which  $\tilde{\kappa} = \kappa_{\text{th}} + \kappa_{\text{rad}}$ , the sum of the thermal and radiative coefficients of heat conduction. We linearize this equation, taking  $\tilde{p} = \tilde{p}_o + \tilde{p}_1$ ,  $\rho = \rho_o + \rho_1$ , and  $T = T_o + T_1$ , working in the comoving frame and assuming that the zeroth-order gradients of temperature, velocity, and density are zero, to obtain

$$\frac{\partial \tilde{p}_1}{\partial t} - \frac{\gamma_s \tilde{p}_o}{\rho_o} \frac{\partial \rho_1}{\partial t} = (\gamma_3 - 1) \tilde{\kappa} \nabla^2 T_1. \quad (7.19)$$

We then need to eliminate one variable by linearizing the equation of state (7.6). We choose to eliminate  $T_1$ , but to do so we have to evaluate the variation in  $Z$ . Here we choose  $Z \propto \sqrt{T}$  in the ionizing regime. Then we find

$$\tilde{p}_1 - p_o \frac{\rho_1}{\rho_o} = \alpha \frac{T_1}{T_o}, \text{ where} \quad (7.20)$$

$$\alpha = \frac{16\sigma T_o^4}{3c} + \frac{\rho_o(1 + \beta Z_o)k_B T_o}{Am_p}, \quad (7.21)$$

with  $\beta = 1$  in a fully ionized plasma or  $3/2$  in an ionizing plasma. We note that  $p_o$  in (7.20) is the particle pressure and that  $\alpha$  has units of pressure. Substituting for  $T_1$  in (7.19), we have

$$\frac{\partial \tilde{p}_1}{\partial t} - \frac{(\gamma_3 - 1)\tilde{\kappa}T_o}{\alpha} \nabla^2 \tilde{p}_1 = \frac{\gamma_s \tilde{p}_o}{\rho_o} \frac{\partial \rho_1}{\partial t} - \frac{(\gamma_3 - 1)\tilde{\kappa}T_o}{\alpha} \frac{p_o}{\rho_o} \nabla^2 \rho_1. \quad (7.22)$$

Note that the coefficient in the second term on the left-hand side has units of  $\text{cm}^2/\text{s}$ , making it some sort of generalized kinematic diffusion coefficient (see Chap. 2). We can arrive at an equation that includes acoustic waves by noting that the linearized versions of the momentum equation (7.1) and the continuity equation imply that  $\partial^2 \rho_1 / \partial t^2 = \nabla^2 \tilde{p}_1$ , which enables us to differentiate (7.22) twice with respect to  $t$  to find

$$\left[ \frac{\partial^2}{\partial t^2} - \frac{\gamma_s \tilde{p}_o}{\rho_o} \nabla^2 \right] \frac{\partial \tilde{p}_1}{\partial t} = \frac{(\gamma_3 - 1)\tilde{\kappa}T_o}{\alpha} \left[ \frac{\partial^2}{\partial t^2} - \frac{p_o}{\rho_o} \nabla^2 \right] \nabla^2 \tilde{p}_1. \quad (7.23)$$

This is a fourth-order wave equation describing radiation-modified acoustic waves and related waves. We perform a plane-wave analysis as in Chap. 2 to find the dispersion relation, which is

$$\left[1 - \frac{\gamma_s \tilde{p}_o}{\rho_o} \frac{k^2}{\omega^2}\right] = -i \left( \frac{(\gamma_3 - 1) \tilde{\kappa} T_o}{\alpha} \omega \frac{k^2}{\omega^2} \right) \left[1 - \frac{p_o}{\rho_o} \frac{k^2}{\omega^2}\right]. \quad (7.24)$$

One can simplify this by defining  $\eta = 4\sigma T^4 / (3cp_o)$  to capture the relative importance of radiation and  $\gamma_s v_n^2 = (\omega/k)^2 (\rho_o/p_o)$  giving  $v_n = 1$  when the phase velocity equals the usual, isentropic sound speed, so that (7.24) can be written

$$\left[1 - \frac{(1 + \eta)}{v_n^2}\right] = -i \left[ \frac{\omega_n}{v_n^2} \right] \left[1 - \frac{1}{\gamma_s v_n^2}\right], \quad (7.25)$$

in which the normalized frequency,  $\omega_n$ , is given, assuming  $\tilde{\kappa} = \kappa_{\text{rad}}$ , by

$$\omega_n = \left( \frac{(\gamma_3 - 1)}{\left[1 + \frac{1}{4\eta} \left(\frac{1 + \beta Z_o}{1 + Z_o}\right)\right]} \frac{\omega}{\nu_e (c_s^2/c^2)} \right), \quad (7.26)$$

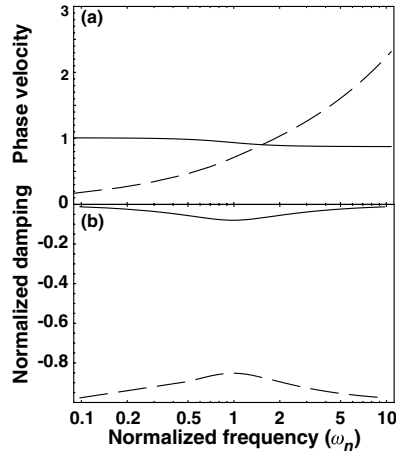
where we have introduced an extinction rate  $\nu_e = c\chi_R$  to clarify the normalization. It is evident that as radiation becomes negligible so  $\eta \rightarrow 0$ ,  $\omega_n \rightarrow 0$  and the phase velocity assumes its usual value ( $v_n = 1$ ). When  $\eta$  is not small, the behavior depends on  $\omega_n$ . At any given  $\eta$ ,  $\omega_n$  defines a frequency scale, so that “low” frequencies make this term small and “high” frequencies make it large. The high-frequency regime is the one in which thermal conductivity is very effective on the scale of the wavelength, smoothing the temperature fluctuations that acoustic waves otherwise produce. The waves in this regime have phase velocity ( $c_s/\sqrt{\gamma}$ ). Such waves are known as *isothermal acoustic waves*, as this is the phase velocity found from the Euler equations by assuming the plasma to be isothermal.

## Homework 7.2

Derive the dispersion relation for isothermal acoustic waves from the Euler equations. That is, demand constant temperature and see what happens.

In general, (7.25) has four roots, corresponding to two pairs of oppositely propagating waves. The roots are complex, giving the phase velocity and the damping, both normalized to the nonradiative, isentropic sound speed. [The spatial damping rate, normalized by the wavenumber, is the ratio of the imaginary root of (7.25) to the real root.] Figure 7.4 shows the normalized phase velocity and spatial damping rate, as a function of  $\omega_n$ , for values of the parameters shown in the caption. The weakly damped root with a normalized phase velocity near 1 is the acoustic wave. The other root, often described as a *thermal wave*, is very strongly damped except at very low frequency, where its phase velocity becomes negligible. It corresponds to a strongly damped perturbation in temperature and the other quantities.

For typical laboratory values (at approximately 1 g/cm<sup>3</sup>) of  $\chi_R \sim 10^4$  cm<sup>-1</sup>,  $c_s \sim 30$  km/s, and  $\gamma_3 \sim \gamma_s \sim 1.3$ ,  $\omega_n = 1$  when  $\omega \sim (3/\eta) \times 10^6$  rad/s, which corresponds to wavelengths of order  $2\pi\eta$  cm. Thus, once  $\eta$  decreases



**Fig. 7.4.** Acoustic waves in the radiation diffusion limit. (a) The phase velocity normalized by the nonradiative, isentropic sound speed. (b) The spatial damping length normalized by the wavenumber. Parameters were  $\gamma_3 \sim \gamma_s \sim 1.3$ ,  $\beta = 1$ , and  $\eta = 0.01$

below about  $10^{-5}$  (at approximately 100 eV), so the wavelength becomes shorter than  $1/\chi_R$ , the acoustic waves will be in the “high-frequency regime.” For stellar interiors, one might have  $\chi_R \sim 10 \text{ cm}^{-1}$ ,  $c_s \sim 100 \text{ km/s}$ , and  $\gamma_3 \sim \gamma_s \sim 1.3$ , so  $\omega_n = 1$  when  $\omega \sim (3/\eta) \times 10^4 \text{ rad/s}$ , which corresponds to wavelengths of order  $600 \eta \text{ cm}$ . Thus, only very short wavelengths, by comparison to the stellar radius, will be in the “high-frequency” regime.

### Homework 7.3

Figure 7.4 shows the wave properties as  $\omega$  varies for fixed  $\eta$ . Consider how the wave properties vary with  $\eta$  for  $\beta = 1$  and fixed  $\omega/(\nu_e c_s^2/c^2)$ . Plot the normalized phase velocity and damping length for  $0.01 \leq \eta \leq 10$  and discuss the results.

#### 7.2.2 Cooling When Transport Matters

We next turn to systems that are not quite so optically thick, so that  $B \neq J_R$  and radiation transport is significant. It is not immediately clear what may happen in this case. The question is where the radiation goes, how is it absorbed, and what are the consequences. We assume that the radiation is in steady state, as it equilibrates rapidly on the timescale of material motion. We further assume that the plasma particles are in near-equilibrium distributions, so they emit at the equilibrium rate. Under these assumptions, the divergence of the radiative flux is  $4\pi\kappa(B - J_R)$  as given by (6.38) for steady state, and in applications we will approximate  $\kappa \sim \kappa_p$ , the Planck mean opacity. Before

we consider the dynamics of fluctuations in this context, we need to know the rate of cooling produced by a plane-wave fluctuation in temperature and thus thermal emission. In the next section, we calculate this.

### Cooling of Temperature Fluctuations

We consider a system with no zeroth-order gradients. Such a system will be in a steady-state (or an equilibrium) in which the sources and losses of thermal radiation are in balance. Our goal is to determine the radiative cooling produced by a plane-wave fluctuation in the emission. Formally we write  $B = B_o + B_1$  and  $B_1 = \hat{B} \exp[ik(z - z_1)]$ , which defines the  $z$ -axis to lie along the wave vector  $\mathbf{k}$  of the fluctuation. Our convention will be that the physical quantity represented by any variable,  $a$ , is  $(a + a^*)/2$ . We then seek the radiation intensity due to  $B_1$ , ignoring the steady-state contribution due to  $B_o$ . Equation (6.44) then describes the incremental radiation intensity,  $I_1$ . We assume the medium is uniform and refraction is negligible, and we evaluate the intensity by integrating in the  $+z$  direction along some ray, so we have

$$I_1(z) = \int_{z_o}^z \frac{dz'}{\mu} \kappa \hat{B} e^{ik(z'-z_1)} e^{-\kappa(z-z')/\mu} + I(z_o) e^{-\kappa(z-z_o)/\mu}, \quad (7.27)$$

where  $\mu$  is the cosine of the angle of the ray relative to the  $z$ -axis. It is consistent with our context to take  $z_o$  to be a large negative number, so that  $|\kappa z_o| \gg 10$ . Then, after integrating, (7.27) becomes

$$I_1(z, \mu) = \frac{\kappa \hat{B}}{(i\mu k + \kappa)} e^{ik(z-z_1)} = \left[ \frac{(\kappa - i\mu k)}{(\mu^2 k^2 + \kappa^2)} \right] \kappa \hat{B} e^{ik(z-z_1)}. \quad (7.28)$$

When we integrate  $I_1$  to find  $J_1$  the imaginary term, which is odd in  $\mu$ , integrates to zero, and we are left with

$$J_1(z) = \frac{1}{2} \int_{-1}^1 d\mu I_1(z, \mu) = \hat{B} e^{ik(z-z_1)} \frac{\kappa}{k} \text{Cot}^{-1} \left( \frac{\kappa}{k} \right). \quad (7.29)$$

This is the result we need to evaluate the heat input to the matter in the plasma, from the steady-state limit of (6.38), as the negative of the input to the radiation.

For systems in which the radiation pressure is negligible, the impact of the radiation is through the heat it transports. It is then useful to express this result in terms of the fluctuation of the temperature,  $T_1$ . We treat the incremental energy loss rate, which is  $-\nabla \cdot \mathbf{F}_R$ , as a damping rate on the incremental local energy density,  $\rho c_V T_1$ . Thus we have, again in the comoving frame of a uniform medium,

$$\frac{\partial T_1}{\partial t} = -\nu_1 T_1 = \frac{4\pi\kappa}{\rho c_V} (J_R - B). \quad (7.30)$$



Just as we have expanded the other physical parameters, we expand  $J_R$  as  $J_R = J_o + J_1$ , where  $J_1$  is the deviation in  $J_R$  due to the temperature fluctuation from (7.29). Since  $J_o = B_o$  the first-order expression for the right-hand side is  $\kappa(J_1 - B_1)$ , where  $\kappa$  is the zeroth-order value of  $\kappa$ . In addition, it is clear from (7.30) that  $T_1$  must have the same plane-wave dependence as  $J_1$  and  $B_1$ , so we have

$$\nu_1 = \frac{4\pi\kappa\hat{B}_1}{\rho c_V \hat{T}_1} \left[ 1 - \frac{\kappa}{k} \text{Cot}^{-1} \left( \frac{\kappa}{k} \right) \right] = \frac{16\sigma\kappa T_o^3}{\rho c_V} \left[ 1 - \frac{\kappa}{k} \text{Cot}^{-1} \left( \frac{\kappa}{k} \right) \right]. \quad (7.31)$$

This result is plotted in Fig. 7.5. At small  $\kappa/k$ ,  $\nu_1$  becomes independent of  $k$  and can be designated as

$$\nu = \frac{16\sigma\kappa T_o^3}{\rho c_V}. \quad (7.32)$$

As  $\kappa/k$  increases, the damping is smaller. Consider this further. At small  $\kappa/k$ , which is the optically thin limit for the perturbation, the radiation travels many wavelengths before it is absorbed. The emission is small and the absorption is spatially uniform. (Thus, if one writes the equation for  $T_o$ , it will have a higher-order heating term.) As  $\kappa/k$  becomes larger, the absorption becomes increasingly local so the energy from any given temperature maximum stays increasingly near that maximum, slowing the cooling. At very large  $\kappa/k$ , there is a net flow of energy from hot regions to cool regions within each wavelength. However, in this limit  $J_R$  also approaches  $B$  and the transition to a diffusive regime. The frequency of a fluctuating perturbation also must be considered. At high enough frequency, ignoring the first term in (6.38) is no longer justified. One can estimate when this might be important by taking  $E_R \sim F_R/c$ ,  $\nabla \rightarrow k$ , and  $\partial/\partial t \rightarrow \omega$  in the two terms on the left-hand side of (6.38). They become comparable when  $\omega/k \sim c$ . This is not a surprising result. We are now ready to apply the cooling rate of (7.31) to two cases of interest.

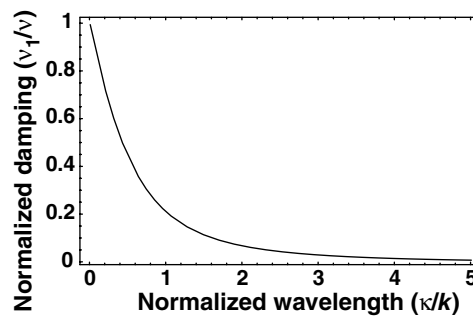


Fig. 7.5. Radiative cooling of optically thin fluctuations

### Cooling of Thin Layers

We have seen that faster shock waves produce higher postshock temperatures, and also that the opacity decreases as the temperature increases. As a result, experiments may produce optically thin shocked layers that endure for some time. The shocked layers produced by interstellar shocks in astrophysics are optically thin as well. Such layers may radiate so strongly that they cool substantially, their density profiles change, and more complicated dynamics also become possible. We will explore the profiles of the resulting cooling layers in our discussion of radiative shocks. It is often helpful, though, to evaluate the cooling rate of a hot, thin layer. If the corresponding cooling time is long compared to the duration of the system, then changes in layer structure due to the radiation will be minimal.

The cooling of a thin, planar layer of infinite lateral extent also provides a nice application of the radiative transfer equation. For a planar layer of thickness  $d$ , we use (6.45), writing

$$I_{\text{R}} = \int_0^d \rho \kappa_m B e^{-(z\rho\kappa_m/\mu)} \frac{dz}{\mu} = B \left( 1 - e^{-(d\rho\kappa_m/\mu)} \right). \quad (7.33)$$

We use this to find the radiation flux emerging from one surface of the layer, as it was defined in (6.9), but integrating over only the one hemisphere of outgoing radiation. This gives

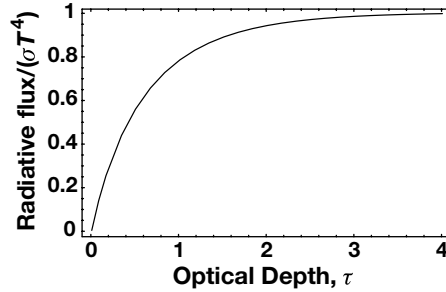
$$F_{\text{R}} = 2\pi \int_0^1 I_{\text{R}} \mu d\mu = \pi B \left[ 1 + e^{-\tau} (\tau - 1) - \tau^2 E_1(\tau) \right], \quad (7.34)$$

in which  $\tau = d\rho\kappa_m$  is the optical depth of the shocked layer based on its thickness and  $E_1(\tau)$  is the exponential integral function with  $n = 1$ . The total energy flux removed from the layer is twice this value, as radiation leaves from both sides. Figure 7.6 shows the radiative flux, normalized to  $\sigma T^4$ , as a function of optical depth. One sees that the flux increases linearly from zero with optical depth at small optical depth but soon saturates and approaches  $\sigma T^4$ .

From Fig. 7.6, one might expect cooling to be fastest when a layer is of order one optical depth thick, because one then extracts radiation from the entire volume but at a fairly high rate. This expectation would be false, however, because the amount of material to be cooled increases faster than the radiation flux. We can see this as follows. The radiation cooling rate  $\nu_{\text{rad}}$  is the ratio of twice the radiative flux from a single surface to the energy content per unit area of the layer

$$\nu_{\text{rad}} = \frac{2F_{\text{R}}}{\rho dc_V T} = \nu_{\text{rad}}^* \frac{[1 + e^{-\tau} (\tau - 1) - \tau^2 E_1(\tau)]}{\tau}, \quad (7.35)$$

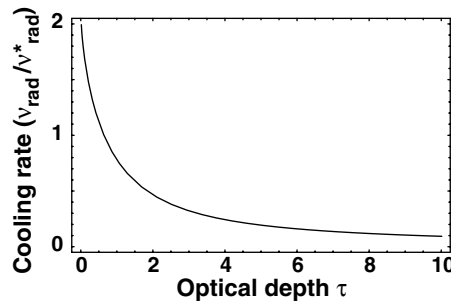
in which the normalizing factor  $\nu_{\text{rad}}^*$  is



**Fig. 7.6.** Radiative flux from optically thin layer. The ordinate shows the radiative flux leaving one surface of the layer, normalized by  $\pi B = \sigma T^4$ . The flux increases as the optical depth increases

$$\nu_{\text{rad}}^* = \frac{2Am_p}{(Z+1)k_B} \sigma T^3 \kappa_m = 2.2 \frac{A}{(Z+1)} T^3 \kappa_m, \quad (7.36)$$

in which the second equality gives  $\nu_{\text{rad}}^*$  in  $\text{s}^{-1}$  for  $T$  in eV and  $\kappa_m$  in  $\text{cm}^{-1}$ . Figure 7.7 shows the corresponding normalized cooling rate. Its limiting value as the optical depth becomes very small is 2. It decreases as optical depth increases, reaching about 0.1 for an optical depth of 10. Beyond that point the normalized cooling rate changes slowly, scaling as  $1/\tau$  at large  $\tau$ . Thus, if  $T \sim 10$  eV and  $\kappa_m \sim 10^6 \text{ cm}^2 \text{ g}^{-1}$ , one has  $\nu_{\text{rad}}^* \sim 10^9 \text{ s}^{-1}$  or ns cooling times for  $\tau \sim 0.5$ . The cooling times will be shorter for smaller optical depth, but only by up to a factor of 2, and longer for larger optical depth.



**Fig. 7.7.** The normalized cooling rate,  $\nu_{\text{rad}}/\nu_{\text{rad}}^*$  decreases as the optical depth increases

#### Homework 7.4

We did not explore the angular variation in the contributions to (7.34). One might imagine that the largest contributions could come at grazing angles,

where  $\mu$  is very small and the optical depth along a line of sight becomes large. The model used here would be less realistic if most of the emission came at grazing angles, because real systems will have layers that are not truly planar and certainly are not infinite in extent. Use a computational mathematics program to derive (7.34). Then modify the calculation to explore how large the contribution is from such grazing angles. Conclude whether or not the results above might be reasonable estimates for real layers.

It may be useful to develop a comparison of the cooling rate just found and the standard expression for the astrophysical cooling function,  $\Lambda$ . The cooling function  $\Lambda$  is the power loss per unit volume per unit electron density per unit ion density, in ergs-cm<sup>3</sup>/s or equivalent. Thus, the power loss per unit volume is  $n_e n_i \Lambda$ . The corresponding cooling rate,  $(\nu)_{\text{astro}}$ , is  $n_e n_i \Lambda / (\rho c_V T)$ . Setting this equal to  $\nu_{\text{rad}}^*$ , we find the opacity  $\kappa_{\text{astro}} = \rho \kappa_m$  corresponding to the optically thin astrophysical case, as  $\kappa_{\text{astro}} = n_e n_i \Lambda / (2\sigma T^4)$ . Note also that one can identify the emissivity of the thin layer as  $\kappa_{\text{astro}} d$ .

### 7.2.3 Optically Thin Acoustic Waves

The diffusive, near-equilibrium regime of Sect. 7.2.1 is not easily achieved in the laboratory or often encountered in astrophysics except within stars. Much more common are systems hot enough that radiative cooling matters, but optically thin, or at least not very thick, so that (7.30) and (7.31) describe the cooling. Let us consider how acoustic waves behave in this regime, in the limit that the system is so optically thin that  $\nu_1 = \nu$  from (7.32). Then from (3.71), and with the specific infinitesimal heat flow  $dq = c_V \partial T_1 / \partial t = -\nu c_V T_1$ , we obtain

$$\frac{Dp}{Dt} - \frac{\gamma_s p}{\rho} \frac{D\rho}{Dt} = -\nu \frac{\rho(1 + \beta Z) k_B T_1}{Am_p}, \quad (7.37)$$

in which we have used (7.10) for negligible radiation to evaluate  $\partial p / \partial T)_\rho$  and have evaluated the isentropic sound speed for a polytropic gas. Here again, as in (7.21),  $\beta = 1$  for an ionized plasma,  $3/2$  for an ionizing plasma with  $Z \propto \sqrt{T}$ , and some other value or function in more general cases. We once again linearize for an initially uniform plasma, using (7.20) and (7.21) for negligible radiation, to find

$$\frac{T_1}{T_o} = \frac{(1 + Z)}{(1 + \beta Z)} \left[ \frac{p_1}{p_o} - \frac{\rho_1}{\rho_o} \right], \quad (7.38)$$

from which we obtain from (7.37) in the comoving frame and linearizing

$$\frac{\partial p_1}{\partial t} - \frac{\gamma_s p_o}{\rho_o} \frac{\partial \rho_1}{\partial t} = -\nu \left( p_1 - \frac{p_o}{\rho_o} \rho_1 \right). \quad (7.39)$$

Here we again use the result from the continuity and fluid momentum equations that  $\partial^2 \rho_1 / \partial t^2 = \nabla^2 p_1$ , to find

$$\left[ \frac{\partial^2}{\partial t^2} - \frac{\gamma_s p_o}{\rho_o} \nabla^2 \right] \frac{\partial p_1}{\partial t} = -\nu \left[ \frac{\partial^2}{\partial t^2} - \frac{p_o}{\rho_o} \nabla^2 \right] p_1. \quad (7.40)$$

The corresponding dispersion relation is

$$-i\omega \left[ -\omega^2 + \frac{\gamma_s p_o}{\rho_o} k^2 \right] = -\nu \left[ -\omega^2 + \frac{p_o}{\rho_o} k^2 \right], \quad (7.41)$$

which can be solved for the normalized inverse phase velocity,  $c_s k / \omega$ , to obtain

$$\frac{c_s^2 k^2}{\omega^2} = \frac{1}{1 + \nu^2 / (\gamma_s^2 \omega^2)} \left[ 1 + \gamma_s \left( \frac{\nu}{\gamma_s \omega} \right)^2 + i(\gamma_s - 1) \frac{\nu}{\gamma_s \omega} \right]. \quad (7.42)$$

### Homework 7.5

It is curious that (7.39) and (7.41) do not depend on  $\beta$ , so that these waves seem not to care whether the system is fully ionized. Beginning with (7.37), derive (7.41) and discuss why there is no  $\beta$  dependence.

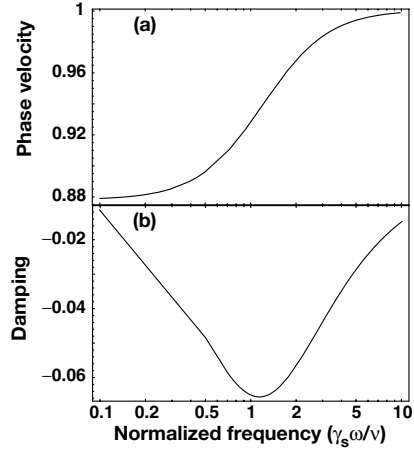
One sees in (7.42) that one recovers ordinary isentropic acoustic waves as the cooling rate goes to zero, and damped, isothermal acoustic waves as the cooling rate becomes large. Figure 7.8 shows how the phase velocity and damping rate implied by this dispersion relation depends on the natural normalized frequency for (7.42),  $\gamma_s \omega / \nu$ . The phase velocity increases from the isothermal sound speed at low normalized frequency to the isentropic sound speed at high normalized frequency. In both these limits, the spatial damping rate is small. The spatial damping rate increases somewhat during the transition.

Returning to (7.42), the limiting behavior in this regime is easy to recover and merits discussion. In the limit of very high frequency or very small damping, we evidently recover ordinary acoustic waves. For a high frequency with  $\nu / \omega \ll 1$ , we find

$$k = \frac{\omega}{c_s} \left[ 1 + i \frac{1}{2} \frac{\nu}{\omega} \frac{(\gamma_s - 1)}{\gamma_s} \right], \quad (7.43)$$

corresponding to damped sound waves with a characteristic, very-long damping length of  $(2\gamma_s c_s / \nu) / (\gamma_s - 1)$ . The opposite limit, in which  $\nu / \omega \gg 1$ , is more complicated because at a low enough frequency the waves will experience an optically thick medium and the present calculation will not apply. Assuming that over some range of frequencies this limit does make sense, one can see that  $k$  will be

$$k = \frac{\sqrt{\gamma_s} \omega}{c_s} \left[ 1 + i \frac{1}{2} \frac{\omega}{\nu} (\gamma_s - 1) \right], \quad (7.44)$$



**Fig. 7.8.** Acoustic waves in the optically thin limit. The phase velocity, normalized to the isentropic sound speed,  $c_s$ , and the spatial damping rate, normalized to the wavenumber  $k$  vs. normalized frequency

and that the damping length is  $[2c_s/(\nu\sqrt{\gamma_s})](\nu^2/\omega^2)/(\gamma_s - 1)$ . This damping length is also quite long, compared to the wavelength of the fluctuations. In this limit we have weakly damped, isothermal acoustic waves with phase velocity  $(c_s/\sqrt{\gamma})$ . Physically, in this case the radiation damps out the temperature fluctuations at a rate much faster than the wave frequency.

To see the implications of this, consider acoustic waves in a CH plasma at  $T = 100$  eV. With  $\kappa_P = 2 \times 10^5 \rho/T_{eV}$ ,  $\rho = 1$  g/cm<sup>3</sup>, the radiative damping rate from (7.32) is just over  $10^9$  s<sup>-1</sup>. If the sound speed is about  $10^6$  cm/s (10 km/s), then acoustic waves with wavenumbers of  $0.001$  cm<sup>-1</sup> will be the most damped. The corresponding wavelength of order 100 m is large compared to experiments, so this result is most relevant to the gradual damping of large-scale structures in the plasma. The waves inside the plasma, having larger wavenumbers, will be isentropic acoustic waves.

Our two treatments of radiative acoustic waves, in this section and Sect. 7.2.1, show rather different behavior, because they apply to different regimes. The lowest frequencies, in any medium, are optically thick, in the sense that the absorption will occur in a very small fraction of a wavelength, so the description of Sect. 7.2.1 will apply. The highest frequencies are optically thin, so the description of the present section will apply. As a result, Fig. 7.8 connects naturally to Fig. 7.4, because increasing the frequency also takes one from an optically thick to an optically thin regime. Overall, acoustic waves progress from isentropic to isothermal and back to isentropic as frequency increases. The transition between the two regimes of optical depth, and the even-more-complicated case of frequencies so high that the

propagation time of the radiation matters, are discussed in detail in Mihalas and Mihalas.

#### 7.2.4 Radiative Thermal Instability

An important application of cooling by radiation is the *radiative thermal instability*. This regulates the pulsations of Cepheid variable stars and creates structures within high-energy-density plasmas.

As we shall see, this instability occurs when a finite region of material cools by radiating, and when the derivative of the opacity with temperature has a certain relation to the other quantities. The radiative heat input to the matter is the negative of the input to the radiation from (6.38). For simplicity, we will assume that any adiabatic work is negligible, so that

$$\frac{\partial \epsilon}{\partial t} = \frac{dq}{dt} = \frac{4\pi\kappa}{\rho} (J_R - B) + \dot{q}_{\text{ext}}, \quad (7.45)$$

where  $\dot{q}_{\text{ext}}$  is the heating rate from external sources such as thermal heat conduction and one would have  $dq/dt < 0$  for net radiative cooling. Any adiabatic energy input would be added to both the expressions to the right of the first equal sign. We will then explore the evolution of temperature fluctuations under such conditions, also assuming the medium to be spatially uniform. We then Taylor-expand the equation for the temperature to find

$$\begin{aligned} \frac{\rho c_V}{4\pi} \frac{\partial T}{\partial t} &= \kappa_o (J_{R_o} - B_o) + \frac{\dot{q}_{\text{ext}}}{4\pi} + \\ \frac{\partial \kappa}{\partial T} (J_{R_o} - B_o) (T - T_o) &+ \kappa_o \frac{\partial (J_R - B)}{\partial T} (T - T_o) + \dots, \end{aligned} \quad (7.46)$$

where the derivatives are as usual evaluated at  $T_o$ . Here  $T = T_o + T_1$ , and (7.46) shows that the evolution of  $T_o$  is from the first two terms on the right-hand side so that  $T_o$  will tend to some steady-state value. The corresponding equation for  $T_1$ , with a damping rate  $\nu_2$ , is

$$\frac{\partial T_1}{\partial t} = -\nu_2 T_1 = \frac{4\pi}{\rho c_V} \left[ \frac{\partial \kappa}{\partial T} (J_{R_o} - B_o) + \kappa_o \frac{\partial (J_R - B)}{\partial T} \right] T_1, \quad (7.47)$$

in which we can obtain  $(J_{R_o} - B_o)$  from the energy input and  $\partial(J_R - B)/\partial T$  from (7.30) and (7.31), to find

$$\nu_2 = \frac{\dot{q}_{\text{ext}}}{c_V T} \frac{\partial \ln \kappa_o}{\partial \ln T} + \nu \left( 1 - \frac{\kappa}{k} \text{Cot}^{-1} \left( \frac{\kappa}{k} \right) \right). \quad (7.48)$$

Note that the “damping rate,”  $\nu_2$ , can be positive or negative depending on the sign and magnitude of  $\partial \kappa / \partial T$ . If  $\nu_2$  is negative then one has a *thermal instability*, meaning that small temperature fluctuations grow with time. For positive  $\dot{q}_{\text{ext}}$ , this occurs for  $\partial \kappa_o / \partial T < 0$ . This is sensible because, while the

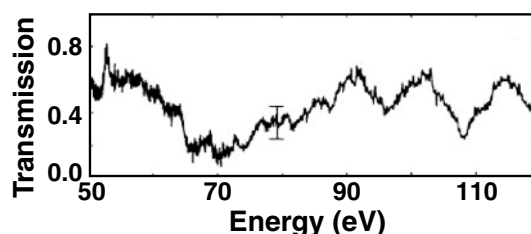
heat input does not change, a temperature increase leads to less radiative cooling and thus a further temperature increase (and vice versa). Of course, this will soon cause the assumption that the medium is uniform to become invalid, as the hotter regions will compress the less hot ones.

The general structure of  $\kappa_o$  is to decrease with  $T$  out to some small number of keV, and then to increase as bremsstrahlung and perhaps synchrotron radiation become significant. In addition, when an important ion in the plasma just begins to open up a closed shell of electrons, there can be an increase in  $\kappa_o$  with  $T$  over some range of temperatures. Curiously, this means that many high-energy-density plasmas are in the unstable regime, as are astrophysical plasmas with temperatures below 1 keV. In addition, even without an external heat source, one finds that temperature fluctuations relative to the steadily decreasing temperature will grow with time. However, the growth rate is relatively slow, as the derivative in (7.48) is typically  $-1$  to  $-2$ , so that any unstable structure will typically evolve on the same timescale as the plasma evolves. As a result, the thermal instability only occasionally becomes important. Structure in the long-term evolution of conductively heated, radiatively cooled high- $Z$  plasmas has been attributed to this instability. Also, one can sometimes see a variant of it in the simulated evolution of initially uniform plasmas that are cooling after being heated by radiation. Since the radiative damping is smallest at the largest  $k$ , this will tend to create fluctuations on the scale of the zones in a simulation.

Beyond the thermal instability, structure in  $\kappa_o$  can lead to other effects in the plasma evolution. For example, nonlinear oscillations in temperature can arise in a system that is steadily heated. This occurs in the visible layers in Cepheid-variable stars, steadily heated from within, which operate in a range of temperatures where the opacity of Fe has maxima and minima. When the visible layer of the star is at a temperature where  $\partial\kappa_o/\partial T > 0$ , the stronger radiative cooling can cause the temperature to decrease, overshooting the minimum in  $\kappa_o$ , so that  $\partial\kappa_o/\partial T < 0$ , in response to which the temperature increases, again causing  $\kappa_o$  to overshoot the minimum. The result is a steady oscillation in temperature and luminosity. Figure 7.9 shows the experimentally measured transmission through a sample of Fe at a temperature near 100 eV. High transmission corresponds to low opacity. One can see that the opacity fluctuates with temperature.

As another example, the presence of a region where  $\partial\kappa_o/\partial T > 0$  over some narrow range of temperatures in the plasma expanding from a laser-heated surface can lead to a local density maximum. The pressure of the adjacent regions compresses the region where the radiation losses are larger. In the context of laser fusion (see Hazak et al.), such structures have been designated *radiative plasma structures*.





**Fig. 7.9.** Opacity of Fe. The plot shows transmission through a sample reported by DaSilva et al., *Phys. Rev. Lett.* **69**, 493 (1992)

## 7.3 Radiation Diffusion and Marshak Waves

When a cool region warms by the transport of radiative heat from a hot region, this transport is often diffusive. The mean-free path of the radiation in the cool material can be quite short. However, the mean-free path often increases rapidly with temperature, so that diffusive heat transport is more complicated than simple diffusion. Diffusion in the presence of a variable diffusion coefficient is often referred to as *nonlinear diffusion*. Nonlinear diffusion is fundamental to high-energy-density plasmas, because they are ionizing. The opacity of each ionization state is different. In general, as material ionizes the spectral regions of largest opacity shift to higher energy, because more energy is needed to access the bound-free and bound-bound transitions of the more highly ionized state. In addition, even ordinary bremsstrahlung absorption is nonlinear. Its opacity decreases with increasing temperature as  $1/T_e^{3/2}$  for otherwise fixed conditions. Two kinds of nonlinear-diffusion problems merit our attention here. In the first, a constant-temperature source drives a radiative heat wave, known as a Marshak wave, into a cooler material. In the second, a finite amount of energy is spread through the material by radiative diffusion. We consider these in turn.

### 7.3.1 Marshak Waves

The Marshak wave describes the solution to a simple problem that nonetheless has great relevance to many real situations. The simple problem is the near-equilibrium diffusion of radiative energy into an initially cold material, through an initially sharp boundary, from a constant-temperature energy source. Marshak was the first to show that this problem admits self-similar solutions. The medium is assumed to be at rest and to remain at rest. We will revisit this assumption later, but note here that it would be a poor assumption if the temperature were high enough that the radiation pressure was the dominant pressure. Thus, Marshak waves are relevant to the common situation that the radiative heat transport is essential but the radiative pressure is small.

An essential example of this is the heating of a high-Z material wall by a sustained radiation source, such as the emission of thermal x-rays from laser-heated regions on the wall. An enclosed structure within which this occurs is known as a hohlraum. We will discuss hohlraums further in Chap. 8. They are of real importance for inertial fusion and for other experiments that require a sustained radiation environment.

To obtain a solvable description of the Marshak-wave problem, we assume that the radiative coefficient of heat conductivity scales as  $\kappa_{\text{rad}} \propto T^n$ . This is reasonable, with  $n \sim 4$  to 5 for typical materials in high-energy-density systems (in contrast,  $n \sim 6$  to 7 in typical astrophysical systems). We further assume constant density and specific heat. Given the assumptions, we want to examine the behavior of a system for which the first law of thermodynamics becomes

$$\rho \frac{\partial \epsilon}{\partial t} = \rho c_V \frac{\partial T}{\partial t} = \nabla \cdot \kappa_{\text{rad}} \nabla T = \frac{T_o}{n+1} \kappa_{\text{rad}})_{T_o} \nabla^2 \left( \frac{T}{T_o} \right)^{n+1}. \quad (7.49)$$

With  $f = T/T_o$  and  $W = \kappa_{\text{rad}})_{T_o} / [\rho c_V (n+1)]$ , this can be written as

$$\frac{\partial f}{\partial t} = W \nabla^2 f^{n+1}. \quad (7.50)$$

This equation has only the one-dimensional parameter,  $W$ , so recalling Chap. 4 we can expect to find a planar similarity solution with similarity variable  $\xi = x/\sqrt{Wt}$ . This gives

$$-\frac{\xi}{2} \frac{df}{d\xi} = \frac{d^2 f^{n+1}}{d\xi^2}. \quad (7.51)$$

### Homework 7.6

Beginning with  $\rho(\partial \epsilon / \partial t) = \nabla \cdot (\kappa_{\text{rad}} \nabla T)$  derive (7.51).

Note that  $f = 1$  at  $\xi = 0$ . One can show that the second derivative of  $f$  remains negative so that  $f$  eventually reaches zero at some  $\xi = \xi_o$ . This makes possible a simple, approximate calculation. One can assume that the radiative heat flux must be constant from the source location to the end of the heat wave. This must be approximately true; otherwise, the temperature somewhere would increase above the source temperature or decrease below that of its surroundings. The radiative flux, in terms of the variables just defined, is

$$F_{\text{R}} = -\kappa_{\text{rad}} \frac{\partial T}{\partial x} = -\frac{T_o}{n+1} \kappa_{\text{rad}})_{T_o} \frac{\partial f^{n+1}}{\partial x} = -\rho c_V T_o \sqrt{\frac{W}{t}} \frac{df^{n+1}}{d\xi}. \quad (7.52)$$

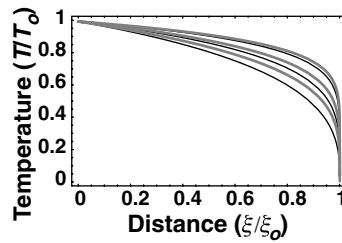
Even though  $f(\xi)$  retains the same shape, the flux decreases with time as the physical temperature gradient at the boundary decreases. If we assume that the flux at any given time is constant throughout the wave, then recalling that  $f = 0$  at  $\xi = \xi_o$  we find

$$f(\xi)^{n+1} = C(\xi_o - \xi) = (1 - \xi/\xi_o), \quad (7.53)$$

with  $C$  a constant equal to  $1/\xi_o$  because  $f = 1$  at  $\xi = 0$ . This is equivalent to

$$T = T_o(1 - \xi/\xi_o)^{1/(n+1)}. \quad (7.54)$$

Figure 7.10 shows the temperature profile from the constant-flux model and from the solution of the more exact equation (7.51), for some values of  $n$ . The constant-flux model is sufficiently accurate for nearly all purposes, as the other assumptions in the Marshak-wave model are certainly not exact (e.g., see the next section for a discussion of ionization).



**Fig. 7.10.** Marshak-wave temperature profiles. The normalized temperature,  $f = T/T_o$ , decreases nonlinearly as  $\xi/\xi_o$  approaches 1. The gray curves give the temperature profiles from the constant-flux model for  $n = 3, 4.5,$  and  $7$  from bottom to top. The black curves give the corresponding numerical solutions

Continuing with the constant-flux calculation, we can find  $\xi_o$  by realizing that the flux through the initial boundary must equal the rate of increase of energy,  $E_w$ , in the wave, or

$$\frac{\partial E_w}{\partial t} = \frac{\partial}{\partial t} \left[ \frac{dx}{d\xi} \int_o^{\xi_o} \rho c_V T d\xi \right] = \frac{\rho c_V T_o}{2} \sqrt{\frac{W}{t}} \left( \frac{n+1}{n+2} \right) \xi_o. \quad (7.55)$$

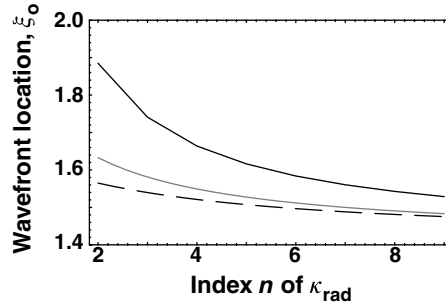
Setting this equal to the flux at  $x = 0$ , and knowing from (7.53) that  $\partial f^{(n+1)}/\partial \xi = -1/\xi_o$ , we find

$$\xi_o = \sqrt{2} \sqrt{(n+2)/(n+1)}. \quad (7.56)$$

Figure 7.11 compares this value of  $\xi_o$  with a more exact solution. One sees that the constant-flux model underestimates the extent of the heat front by roughly 10%. Here again, this is a small effect compared with other probable differences between a real situation and a Marshak-wave model.

The position of the radiation wavefront,  $x_o$ , from the constant flux model, is

$$x_o = \frac{\sqrt{(n+2)}}{(n+1)} \sqrt{\frac{\kappa_{\text{rad}} T_o}{\rho c_V}} \sqrt{2t}, \quad (7.57)$$



**Fig. 7.11.** The lines show the value of  $\xi_o$  from the constant-flux model (grey curve) or a numerical solution (black curve) for traditional Marshak waves. The dashed line shows the result for an ionizing radiation wave

and the front velocity  $u_o$  is

$$u_o = \frac{\sqrt{(n+2)}}{(n+1)} \sqrt{\frac{\kappa_{\text{rad}})_{T_o}}{\rho c_V}} \frac{1}{\sqrt{2t}}. \quad (7.58)$$

Note that this velocity decreases from infinity to very small values as time increases. Of course, the physical velocity is never infinite, because the assumptions of the model break down as  $t$  approaches zero. However, the behavior of the velocity has important consequences for real systems.

At first, the velocity of the radiation front far exceeds any other velocity in the system. In this regime, and formally until the velocity drops to equal the sound speed in the radiation-heated medium, the wave is known as a *supersonic radiation wave*. During this period, the radiation wave reaches any location in the medium first and is affected only by changes in the radiation source.

The velocity of the radiation wave eventually drops below the sound speed in the radiation-heated medium, at which time it becomes a *subsonic radiation wave*. During this period, the wave can be overtaken first by shock waves and later by sonic disturbances. In a real system, the advent of a radiation flux is rarely if ever the only process to occur at the boundary. Whether one considers the birth of a star or any other release of energy within an optically thick environment, or the initiation of an x-ray source within a high- $Z$  container, the inner boundary of the affected material is also disturbed. Very often, the absorption of radiation produces ablation at this boundary, launching a shock wave into the material. The location of the shock wave is initially proportional to  $t$ , and it slows very gradually. The shock wave initially compresses the material between the radiation source and the front, decreasing the thickness of the heated layer but increasing its density. The resulting change to the shape of the radiation wave will depend on the density dependence of the opacity. After the shock wave overtakes the radiation wave, the

shock wave will strengthen, because the preshock temperature will be lower, and the radiation wave will very likely slow even further as it penetrates higher density matter.

### Homework 7.7

Work through the constant-flux model, providing all the missing mathematical steps. Then plot the positions vs. time of the radiation wave and of a disturbance (in the radiation-heated material) moving at Mach 1 or Mach 10. Discuss the results.

Equation (7.51) can be solved numerically but to do so one must find workable boundary conditions. One knows that  $f = 1$  at  $\xi = 0$  and that  $f = 0$  at  $\xi = \xi_o$ , but one does not know  $\xi_o$ , the value of  $\xi$  corresponding to the head of the wave. One does know that the flux through  $\xi = 0$ , from (7.52) evaluated at  $\xi = 0$ , must equal the time rate of increase in the energy content of the radiation wave,  $E_w$ . Generalizing (7.55) and using (7.52), one has

$$\left(\frac{df}{du}\right)_{u=0} = \frac{-\xi_o^2}{2(n+1)} \int_0^1 f(u) du, \quad (7.59)$$

in which  $u = \xi/\xi_o$ . One can then solve (7.51) and (7.59), with independent variable  $u$ , iteratively for  $\xi_o$  and for the value of the integral in (7.59), by seeking conditions such that  $f = 0$  at  $\xi = \xi_o$ . This procedure produced the numerical curves shown in Figs. 7.10 and 7.11.

### 7.3.2 Ionizing Radiation Wave

The largest error in the Marshak-wave model, especially in a laboratory environment, is the assumption that the specific heat at constant volume,  $c_V$ , is constant. This is very much not true, as  $c_V$  depends on  $Z$ , through both the thermal energy and the ionization energy, and  $Z$  is not constant. We can describe a wave in which  $c_V$  and  $Z$  change through ionization as an *ionizing radiation wave*. (This should not be confused with an *ionization front*, discussed Sect. 7.5.) To obtain an evaluation of the difference between such a wave and a Marshak wave, we can revisit the analysis of the previous section. Assuming  $Z \propto \sqrt{T}$  and a hydrogenic model of the ion, we have

$$c_V = \frac{3}{2} \frac{(1 + (3/2)Z)k_B}{Am_p} + \frac{k_B E_H}{12T} \frac{(Z + 6Z^2 + 12Z^3)}{Am_p}. \quad (7.60)$$

This would not admit a self-similar solution if all the terms in  $c_V$  were important. However, for  $T_e > 10$  eV, the terms of highest order in  $Z$  dominate. In this regime we can take  $c_V = c_{V_o} \sqrt{T/T_o}$ , where  $c_{V_o}$  is the value of  $c_V$  when  $T = T_o$ . One then can show, just as in (7.49)–(7.51), that

$$-\frac{\xi}{2} \frac{df}{d\xi} = \frac{1}{\sqrt{f}} \frac{d^2 f^{n+1}}{d\xi^2}, \quad (7.61)$$

with the same definitions of  $f$  and  $\xi$ . Once again, this can be integrated numerically. Alternatively, one can develop a constant-flux description of this system. The flux becomes

$$F_R = -\rho c_{V_o} T_o \sqrt{\frac{W}{t}} \sqrt{f} \frac{df^{n+1}}{d\xi}. \quad (7.62)$$

Holding this constant and integrating with  $f = 1$  at  $\xi = 0$  gives

$$T = T_o (1 - \xi/\xi_o)^{2/(2n+3)}, \quad (7.63)$$

from which we integrate to get the increase in energy

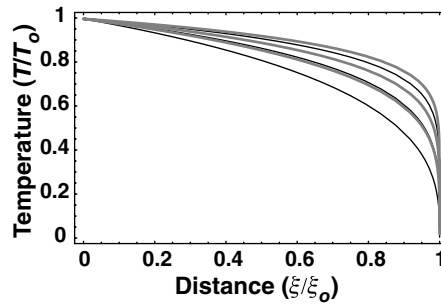
$$\frac{\partial E_w}{\partial t} = \frac{\partial}{\partial t} \left[ \frac{dx}{d\xi} \int_o^{\xi_o} \rho c_V T d\xi \right] = \frac{\rho c_{V_o} T_o}{2} \sqrt{\frac{W}{t}} \left( \frac{2n+3}{2n+6} \right) \xi_o. \quad (7.64)$$

Setting this equal to the flux, we find

$$\xi_o = \frac{2\sqrt{(2n+6)(n+1)}}{2n+3} \text{ and} \quad (7.65)$$

$$x_o = \frac{2\sqrt{(2n+6)}}{(2n+3)} \sqrt{\frac{\kappa_{\text{rad}} T_o}{\rho c_{V_o}}} \sqrt{t}. \quad (7.66)$$

Figure 7.11 shows the value of  $\xi_o$  at the front from (7.65). The Marshak wave will be shorter in an ionizing system than in a system with constant specific heat. Figure 7.12 compares the shape of this wave (from 7.63) with the shape of a traditional Marshak wave. One sees no dramatic differences.



**Fig. 7.12.** Ionizing radiation wave profiles. The normalized temperature  $f = T/T_o$  is somewhat flatter in an ionizing medium than it is in the fixed- $Z$  case of the traditional Marshak wave. Here the gray curves give the temperature profiles from the ionizing model for  $n = 3, 4.5,$  and  $7$  from bottom to top. The black curves give the corresponding numerical solutions for the traditional Marshak wave

### 7.3.3 Constant-Energy Radiation Diffusion Wave

Now we turn to the second case of common interest. A finite event, such as a laser pulse, a  $Z$ -pinch implosion, or an astrophysical burst, may produce a definite amount of radiation. If the radiation is released into a uniform near-equilibrium medium in which the radiation pressure is negligible and the radiation transport is diffusive, then the same fundamental equations apply as in the Marshak-wave case. We consider the planar case here. (Zeldovich and Raizer discuss the spherical case.) One still has, from (7.49),

$$\frac{\partial T}{\partial t} = \frac{\kappa_{\text{rad}})_{T_0}}{\rho c_V(n+1)T_0^n} \nabla^2 T^{n+1} = Y \nabla^2 T^{n+1}, \quad (7.67)$$

thus defining  $Y$  where, if  $T$  is in energy units, then  $Y$  has units of  $\text{cm}^2 \text{s}^{-1} \text{energy}^{-n}$ . Note that  $\rho c_V T$  is the energy per unit volume so  $T$  is the energy per unit volume per unit  $\rho c_V$ . However, the total energy per unit area is also fixed so this is a problem with two independent dimensional parameters. Defining the energy per unit area per unit  $\rho c_V$  as  $Q = \int T dx$ , which has units of  $\text{cm}^1 \text{energy}^1$ , the quantity  $Q^n Y t$  has units of  $\text{cm}^{(n+2)}$ . Thus, an effective dimensionless similarity variable is

$$\xi = x / (Q^n Y t)^{1/(n+2)}. \quad (7.68)$$

As a result, we expect that the position of any point on the heat wave, where for example the temperature is some fraction of the maximum temperature, will be  $\propto t^{1/(n+2)}$ . Since  $n$  is typically 4 or 5, such diffusion waves propagate *much* more slowly than Marshak waves. However, in contrast to the Marshak case there is no fixed temperature that makes a natural normalization for  $T$ . Indeed, we need a time-dependent normalization for  $T$  since the maximum temperature must decrease with time as energy is carried outward. However, the normalization of  $T$  need not be spatially dependent, as all the spatial dependence can be in the evolution with  $\xi$ . To see what normalization makes sense, we consider the spatial derivative of a normalized function,  $f$ , finding

$$\frac{\partial f}{\partial x} = \frac{1}{(Q^n Y t)^{1/(n+2)}} \frac{df}{d\xi}. \quad (7.69)$$

We can also see that the right-hand side (RHS) of (7.67), in terms of  $\xi$ , becomes

$$\text{RHS} = \frac{Y}{(Q^n Y t)^{2/(n+2)}} \frac{d^2}{d\xi^2} T^{n+1}. \quad (7.70)$$

Since the time derivative on the left-hand side of (7.67) will introduce a factor of  $1/t$ , it makes sense to multiply (7.70) by  $t$ , from which we can find that an effective normalization for  $T$ , with consistent units, is  $[Q^2/(Yt)]^{1/(n+2)}$ , so  $f = T/[Q^2/(Yt)]^{1/(n+2)}$ . It is important to note that the denominator in

the definition of  $f$  is not some initial temperature  $T_o$ . Rather, the value of  $f$  that we will find at  $\xi = 0$  will give  $T_o$  as a function of time when multiplied by this denominator. Because  $f$  is *not* time independent, to develop the equation for  $f$  we must observe that

$$\frac{1}{[Q^2/(Yt)]^{1/(n+2)}} \frac{\partial T}{\partial t} = \frac{\partial f}{\partial t} - \frac{f}{(n+2)t}. \quad (7.71)$$

We also have the usual type of relation between derivatives in  $t$  and  $\xi$ ,

$$\frac{\partial f}{\partial t} = \frac{-1}{n+2} \frac{\xi}{t} \frac{df}{d\xi}, \quad (7.72)$$

so (7.67) becomes

$$f + \xi \frac{df}{d\xi} + (n+2) \frac{d^2}{d\xi^2} f^{n+1} = 0. \quad (7.73)$$

The solution to this equation is

$$f(\xi) = \left[ \frac{n\xi_o^2}{2(n+2)(n+1)} \right]^{1/n} \left[ 1 - \left( \frac{\xi}{\xi_o} \right)^2 \right]^{1/n}. \quad (7.74)$$

This has a fixed shape in  $\xi$ , as expected. The time dependence is entirely included in the normalization of  $T$ , as it should be, so  $T(0) \propto t^{-1/(n+2)}$ .

### Homework 7.8

Show that (7.74) is a solution to (7.73). Clearly annotated work with a computational mathematics program is preferred.

Figure 7.13 shows the shape of the constant-energy radiation diffusion wave and compares it to that of the Marshak wave. Note that the absolute value of  $T_o$  decreases in time for the constant-energy wave though not for the Marshak wave. The constant-energy wave produces a much flatter temperature profile. Both waves have the very steep front that is characteristic of nonlinear diffusion waves.

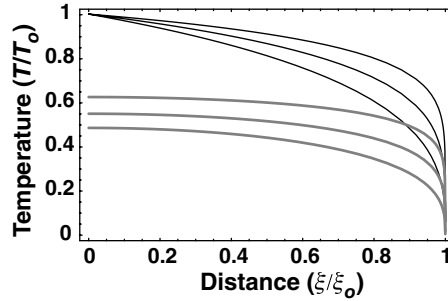
The value of  $\xi_o$  must be determined from an integral of the total energy. The definitions of  $Q$  and  $f$  imply that the correct normalization is

$$1 = \int_{-\infty}^{\infty} f(\xi) d\xi = \left[ \frac{n}{2(n+2)(n+1)} \right]^{1/n} \xi_o^{(2+n)/n} \sqrt{\pi} \frac{\Gamma(1 + \frac{1}{n})}{\Gamma(\frac{3}{2} + \frac{1}{n})}. \quad (7.75)$$

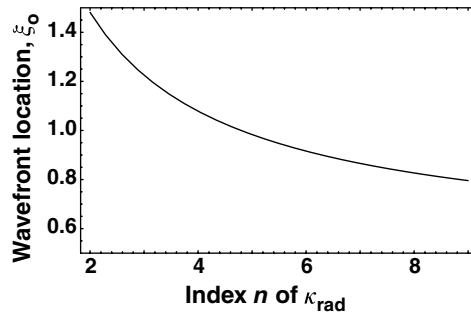
Figure 7.14 shows the value of  $\xi_o$  as a function of  $n$ . One sees that  $\xi_o \sim 1$  to within 10% over the range of interest for radiation waves in high-energy-density plasmas. We can use this value to evaluate the location of the heat front, obtaining (7.76). Note that in a real problem one probably knows  $Q$  and does not know  $T_o$ , but does know  $\kappa_{\text{rad}})_{T_o}/T_o^n$ . The heat front location is

$$x_o = \xi_o (Q^n Y t)^{1/(n+2)} \approx \left[ \left( \frac{Q}{T_o} \right)^n \frac{\kappa_{\text{rad}})_{T_o}}{\rho c_V (n+1)} \right]^{\frac{1}{(n+2)}} t^{1/(n+2)}. \quad (7.76)$$





**Fig. 7.13.** Radiation diffusion wave profiles. The normalized amplitude,  $T/T_0$ , is shown against the normalized similarity variable,  $\xi/\xi_0$ , for both constant-energy radiation diffusion waves (gray) and Marshak waves (black). In each case, from bottom to top, the curves correspond to  $n = 3, 4.5,$  and  $7$



**Fig. 7.14.** Wavefront location for a constant-energy diffusion wave

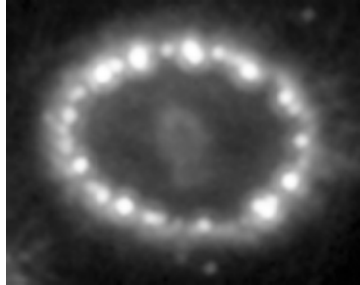
**Homework 7.9**

Consider a gold container shaped so that a planar approximation is reasonable, having planar walls spaced 1 mm apart in vacuum. Assume  $\rho = 20 \text{ g/cm}^3$  and treat  $c_V = 10^{12} \text{ ergs/(g eV)}$  as constant. Use other parameters from Chap. 6 as appropriate. Suppose  $100 \text{ kJ/cm}^2$  is the initial energy content of the vacuum between the walls and that the initial wall temperature is negligible. Approximate the heat front in the walls as a square wave. From zero to 10 ns, find the position of the heat front and the temperature of the surface as a function of time. Plot the ratio of the energy content of the walls to the energy content of the vacuum. Discuss the result.

**Homework 7.10**

Develop the equivalent of (7.76) for a spherically symmetric system.

This concludes our discussion of radiation diffusion waves. We discuss the related topic of ionization fronts at the end of this chapter.



**Fig. 7.15.** An image of the supernova remnant SNR 1987A, taken in November 2003. Credit: NASA, P. Challis, R. Kirshner (Harvard-Smithsonian Center for Astrophysics) and B. Sugerman (STScI)

## 7.4 Radiative Shocks

A *radiative shock* is one in which the structure of the density and temperature is affected by radiation from the shock-heated matter. This simple definition covers an enormous range of phenomena, all at a high enough temperature that we are fortunate not to encounter them in ordinary life. Yet radiative shocks can readily be produced in high-energy-density experiments, and they are frequently encountered in astrophysics. An astrophysical example is found in the supernova remnant developing from SN 1987A, shown in Fig. 7.15. The bright spots in this image are produced by the collision between the ejecta from the star and matter at the edges of the inner ring that encircled the star. Analysis of spectra has shown that the shock waves being driven into the ring are radiative, at least in places. The presence of bright spots, rather than a continuous ring of emission, indicates that there are spikes of dense material at the inner edge of the ring. These might be a result of the Rayleigh–Taylor instability during ring formation.

### 7.4.1 Regimes of Radiative Shocks

Here we discuss the conditions under which a radiative shock occurs, and the physical conditions that determine its structure. In the introduction to this chapter (see Fig. 7.1) we found, depending in detail on density and material, that the radiative flux and pressure became important at temperatures of tens of eV and hundreds of eV, respectively. Shock waves can provide the heating that pushes a plasma into a radiative regime, or they can occur within a plasma that is already in a radiative regime. In order for a shock to push a plasma into a radiative regime, it must at minimum be fast enough that the radiative fluxes, which scale as the fourth power of the temperature and thus the eighth power of shock velocity, exceed the material energy fluxes, which scale as the third power of shock velocity.

In the nonradiative regime, the immediate postshock temperature  $T_s$  is given by (4.20), which we rewrite here as

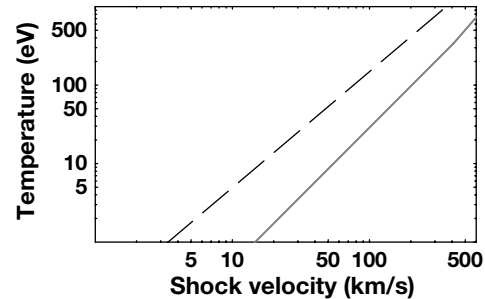
$$RT_s = \frac{2(\gamma - 1)}{(\gamma + 1)^2} u_s^2, \quad (7.77)$$

in which  $u_s$  is the shock velocity and it will be useful at times below to work with the gas “constant”  $R = k_B(Z + 1)/(Am_p)$ , which in general is temperature dependent. Note that  $RT_s$  has units of energy per unit mass. For  $\gamma = 4/3$  and  $Z + 1 = A/2$ ,  $T_s$  is 6.4 eV at  $u_s = 100$  km/s, which is one reason why radiative effects are rarely important for shock velocities much below 100 km/s.

The average number of electrons that share energy with each ion is  $Z$ , but this can be a source of difficulty in shock waves. We have already discussed how  $Z$  can vary with temperature, in Chap. 3. In addition, the shock heats the ions and then the electrons and ions equilibrate, so that in sufficiently low-density matter  $Z$  would be zero immediately following the density jump. Thus, in general, one may need to allow separate temperatures for ions and electrons, a point we return to in Sect. 7.4.9 (which was also discussed previously in Sect. 2.3.3 with reference to Fig. 2.3). It is the electrons, though, that couple significantly to the radiation. Here for simplicity we assume immediate equilibration of ions and electrons. In practice, this means that the equilibration zone just behind the shock (the jump in density and ion temperature) where ions and electrons equilibrate is ignored. The radiation from this equilibration zone increases as the fourth power of the electron temperature, so that most of the equilibration zone is not a significant contributor to the radiation dynamics. In addition, as we will see in Sect. 7.4.9, the equilibration zone is quite small.

Figure 7.16 shows the temperature implied by (7.77), for Xenon and  $C_1H_1$ . For a plasma of C and H, one replaces  $A$  by  $(1+12) = 13$  and takes  $Z = 1 + Z_C$ , where H is assumed to be ionized and  $Z_C$  is the average ionization of the carbon. This modifies the result at low temperature, but makes little difference on a log-log plot. This equation only applies while the radiation pressure remains negligible. When radiation pressure matters, a more careful calculation based on the fundamental equations would be needed. But the figure suffices to indicate the conditions required to reach the radiative regime. In round numbers, one needs shock velocities of tens of km/s to reach temperatures of tens of eV where radiative fluxes matter, and of hundreds of km/s to reach temperatures of hundreds of eV where radiative pressure matters. The velocities required with xenon are smaller than those required with CH, by a factor of a few.

By the time that radiative fluxes exceed material energy fluxes, the radiation will have affected the medium ahead of the density increase produced by the shock. The affected region is a *radiative precursor*, which we discuss as a separate topic in Sect. 7.4.3. To connect our discussion with other usage, we should begin by identifying two possible types of “radiative precursors.” The first we will call a *transmissive precursor*. The most familiar example is lightning. One sees a precursor – the lightning flash – before the resulting thunder,



**Fig. 7.16.** Postshock temperatures, for Xenon (dashed) and  $C_1H_1$  (gray). This figure ignores the role of radiation pressure at high velocity

which has evolved from the shock wave, arrives. In this case the precursor is created by the explosion that drives the shock and not by the shock itself. A second example would be an explosion in the atmosphere strong enough to drive a radiative shock. In this case, some of the radiation from the shock itself could be seen far beyond the volume directly affected by the radiative shock. Thus, one would say that a transmissive precursor is radiation from a shock front or its source that is weakly absorbed while propagating. Thus, it can be seen at a long distance. This type of precursor is not of much interest to us, although we will discuss it briefly below.

The second type of precursor is of much more interest to us and we call it an *absorptive precursor*. In this type of precursor, the radiation is absorbed and is intense enough to affect the upstream medium, principally by increasing its temperature. Unless we specify otherwise, when we write of a “precursor” or “radiative precursor” in the following, we refer to this type of precursor. An important issue for precursors is that of geometry. In order for the precursor to remain planar, a real experiment would need a radiation source whose lateral size substantially exceeded the steady-state precursor length. This is a very demanding constraint. Spherical experiments can avoid this constraint but suffer severely from the dilution of their energy in the three-dimensional expansion.

The concept of a shock inherently involves some separation of physical scales, as we discussed in Chap. 4. In ordinary hydrodynamic shocks, the scale on which viscous diffusion matters must be much smaller than the global scale of the flow. Radiation introduces another scale into the problem, fundamentally related to optical depth. (Likewise, electron-ion energy exchange (Sect. 7.4.9), heat conduction, or magnetohydrodynamic effects also introduce additional scales under various circumstances.) Yet the physical scale over which the radiation matters is much larger than the scale of any viscous effects. As a result, there are two ways to think about the entire system including the effects of radiation both upstream and downstream of the density increase associated with the shock.

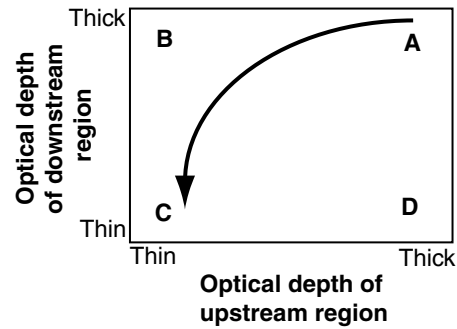
On the one hand, if one views the medium as infinite (measured in optical depths), then one may take the point of view that the radiation alters the structure of the shock transition, extending it in space over a (potentially large) number of radiation mean-free paths. In this case one will speak of the “shock” as the entire region between a distant, undisturbed upstream region and a distant, steady-state downstream region. One would then speak of the comparatively localized density increase as the “density jump” or the “viscous shock transition.” This is the viewpoint taken in much prior literature, including Zeldovich and Raizer and Mihalas and Mihalas.

On the other hand, and as is discussed in the Introduction to this chapter, the system may be optically thin. It may be thin in the upstream direction, the downstream direction, or both. What specifically this means is that the sum of radiation from distant sources and radiation returning to the shock from any matter it has heated is negligible. Whenever the entire region affected by radiation from the shock is not well isolated from other influences, it seems more natural to speak of the “shock” as the region across which the rapid density increase takes place. This use of “shock” is more common in discussions of optically thin astrophysical shocks, as for example in *Shu*. In this case, the interactions of the radiation and the surrounding medium may affect both the upstream and the downstream conditions.

Optical depth provides an effective way to classify radiative shocks. We saw in Chap. 6 that the treatment of radiation transport depends on the structure of the medium within which the transport occurs, and in particular on optical depth. The different regimes of radiation transport correspond to major differences in shock behavior. In one limit – that of very small optical depth, where the radiation serves only to cool the shocked layer – the shocked layer can evolve to become orders of magnitude denser than the preshock medium. In another limit – that of an optically thick and radiation-dominated plasma – the increase in density is limited to a total of a factor of 7. (Recall from Chap. 3 that the radiation-dominated plasma behaves like a polytropic gas with  $\gamma = 4/3$ .) An effective way to categorize radiative shocks and their behavior is to plot them in a space defined by the optical depth of the upstream and downstream regions. Figure 7.17 shows a qualitative depiction of this space. We next briefly discuss each of the four labeled regions.

### Thick-Thick Shocks

In regime A, both the downstream and the upstream regions are optically thick. This is the realm in which it makes the most sense to treat the viscous density increase and all the radiative effects as part of a single, extended, shock structure. Many of the features of this structure can be found from a theory that assumes the medium to be in LTE everywhere. We discuss this regime in Sect. 7.4.6. For typical ideal gases with  $\gamma \geq 4/3$ , the density ratio never exceeds 7. In addition, under some circumstances the density transition is continuous, with no localized jump. Astrophysical environments in which



**Fig. 7.17.** Radiative shock regimes, identified in a space based on optical depth. The four regimes corresponding to the corners of this plot are discussed in the text. The curve shows the qualitative trajectory of a supernova blast wave

such shocks exist are necessarily both hot and dense. Shocks in stellar interiors are of this type, as is the blast wave within the exploding star in a supernova. Such shocks may also exist within some astrophysical “compact objects,” such as pulsars, but their treatment would have to be relativistic. It is difficult, however, to imagine planar laboratory experiments in this regime other than transiently and in special cases. One difficulty is that the precursor length increases so strongly with shock velocity (see Sect. 7.4.4) that one could not produce a measurable precursor of finite length for realistic variations of the experimental parameters. There may be more potential for experiments in spherical geometry, but the challenge of producing a system many optical depths in scale will remain substantial.

### Thick-Thin Shocks

In regime B the downstream region is optically thick but the upstream region is thin. We discuss this regime in Sect. 7.4.6. There is a cooling layer downstream of the viscous shock transition, followed by a steady downstream final state. This regime is common in experiments, in which an optically thick piston (and in some cases optically thick shocked material) drives a radiative shock into a medium whose depth is small compared to the steady-state precursor length. The upstream medium is then quickly heated so that it becomes optically thin. Astrophysical examples of such systems include the blast wave in a supernova as it emerges from the star and the accretion shocks produced in some binary systems.

### Thin-Thin Shocks

In regime C, discussed in Sect. 7.4.5, both downstream and upstream regions are optically thin. Such shocks are the most-commonly observed in

astrophysics, in part because they are easy to see (as the radiation escapes). Supernova remnant (SNR) shocks in dense enough environments are of this type – it is thought that Type II supernovae from red-supergiant precursor stars produce such conditions. Many shock-cloud interactions, including some of those driven by SNR shocks, are also of this type. Shocks that propagate up jets (or are driven by clumps propagating up jets) may be of this type. In such shocks, the entire downstream region is a radiative cooling layer, and it ends (in large enough systems) when the downstream temperature reaches a value determined by local sources and losses of energy rather than by the shock. The density increase associated with such shocks is formally unbounded in the sense that it is limited only by external factors, such as the compression of an initially negligible magnetic field or the presence of a limiting temperature due to other energy sources. Some experiments, with shocks in sufficiently low-density gases, may produce these conditions.

### Thin-Thick Shocks

Regime D is not trivial to produce in steady state, as it would require that the shocked material become optically thin when it is shocked while simultaneously remaining optically thick in the upstream region, over a sufficient distance to sustain a steady precursor. Such a change in optical depth can be produced in ionization fronts driven by radiation, discussed in Sect. 7.5 and in Sect. 8.2.3. Obtaining this response in a shock involving flowing material is more difficult. It might occur, for example, if a very-high-velocity, low-density incoming flow impacted a comparatively dense material. If such a system could be produced, it would have a very dense shocked layer as energy continued to be lost in the downstream direction. Two transient examples are certain shock-cloud collisions and certain experiments. A shock-cloud collision in which the cloud was dense enough and large enough to be optically thick for some time would be of this type. The collision of SNR 1987A with its inner “ring” may be of this type. An experiment might be in this regime while a hot, thin layer of gas drives a shock through a much larger volume of gas. All these cases seem likely to transition to the thin–thin regime if driven harder or longer, and they may never develop a thick upstream region in the sense discussed above.

### 7.4.2 Fluid Dynamics of Radiative Shocks

It turns out that several important properties of radiative shocks are independent of the details of the radiation transport. In this section, for a polytropic gas, we consider how the fluid properties must vary within radiative precursors and radiative shocks. That is, we will consider what things must be truly independent of the details of the radiative transport. Our conclusions here will apply even to shocks whose transport is far more complex than are the

models we use later. One example of such a complex situation would be the transport of energy by line radiation in the presence of significant Doppler shifts. We will frame this discussion in the radiative flux regime, assuming the radiation pressure and energy density to be negligible. We analyze a planar system in steady state, working as usual in the shock frame. In this case the divergence of the flux terms in the conservative form of the mass, momentum, and energy equations must be zero. We will work in our usual shock frame as described in Chap. 4 so that the incoming fluid has a negative velocity. The mass flux  $\rho u$  must be constant everywhere and equal to its value in the region beyond the precursor,  $-\rho_o u_s$ , where the far upstream density is  $\rho_o$  and the shock velocity is  $u_s$ . The constancy of momentum flux gives

$$p + \rho u^2 = \rho_o u_s^2 + p_o, \quad (7.78)$$

in which the initial upstream pressure is  $p_o$ . The continuity and momentum relations thus give

$$\frac{p}{\rho_o u_s^2} = 1 - \frac{\rho_o}{\rho} + \frac{p_o}{\rho_o u_s^2} \quad \text{and} \quad (7.79)$$

$$\frac{RT}{u_s^2} = \frac{\rho_o}{\rho} \left( 1 + \frac{p_o}{\rho_o u_s^2} \right) - \left( \frac{\rho_o}{\rho} \right)^2, \quad (7.80)$$

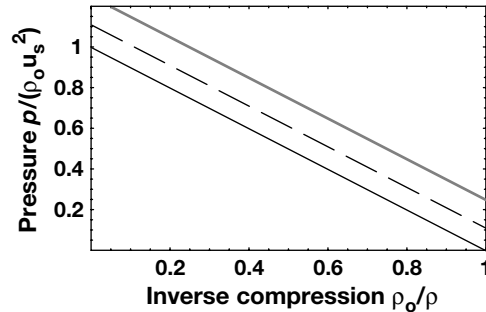
where we have used  $p = \rho RT$  so  $RT$  is proportional to the thermal energy per unit mass of the plasma at temperature  $T$  and  $R = (1 + Z)k_B/(Am_p)$ . These results are shown in Figs. 7.18 and 7.19. We use as the independent variable the *inverse compression*,  $\rho_o/\rho$ . It is worth noting that radiation does not enter into these relations (so long as  $p_R \ll p$ ), and that these figures apply both to nonradiative shocks and to shocks in the radiative flux regime. Whether in the precursor or across the shock jump, a change in compression corresponds to an increase in pressure and a change in temperature as shown. The pressure increases continuously as  $\rho_o/\rho$  decreases, and so places no constraints on the shock transition. In contrast, the competition between heating and  $pdV$  work of compression creates the maximum in the thermal energy in Fig. 7.19. A formal discontinuity in density occurs only if the initial and final states of the viscous density transition are on opposite sides of the temperature maximum seen in this figure. (The temperature cannot increase and then decrease across the density transition, without unphysical consequences for the radiation flux.) In sufficiently weak shocks, the inverse compression can remain to the right of this maximum, producing a continuous transition. This also can occur in radiation-dominated shocks under certain conditions.

The radiation flux enters into the energy flux equation, which for a polytropic gas gives

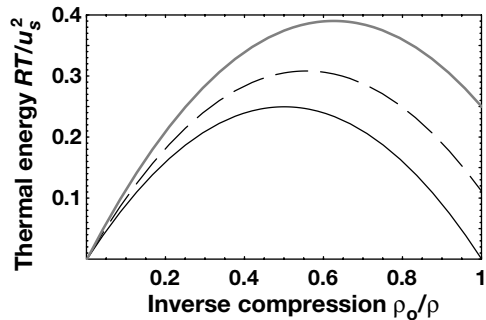
$$\left( \frac{\gamma p}{\gamma - 1} + \frac{\rho u^2}{2} \right) u + F_R = -\frac{\rho_o u_s^3}{2} \left( 1 + \frac{2\gamma}{\gamma - 1} \frac{p_o}{\rho_o u_s^2} \right) + F_o, \quad (7.81)$$

in which  $\rho_o$ ,  $p_o$ , and  $F_o$  are the density, pressure, and radiation flux, respectively, in some presumably steady upstream state. Note that the negative sign





**Fig. 7.18.** Postshock material pressure, normalized to  $\rho_0 u_s^2$ , against inverse compression, for zero upstream pressure (black), Mach  $3/\gamma$  (dashed), and Mach  $2/\gamma$  (gray)



**Fig. 7.19.** Postshock temperature. The thermal energy of the matter per unit mass,  $RT$ , normalized to  $u_s^2$  is shown against the inverse compression,  $\rho_0/\rho$ , for zero upstream pressure (black), Mach  $3/\gamma$  (dashed), and Mach  $2/\gamma$  (gray)

on the first term on the right-hand side is the consequence of the flow velocity being negative (according to our standard conventions for shocks throughout this book) and of taking the “shock velocity,”  $u_s$ , as a positive quantity.

One must understand the physical context in order to set  $F_o$ . There are three limiting cases in which  $F_o$  has some specific value for specific reasons. First, in a shock wave that is fully contained within an optically thick medium, it is sensible to consider the upstream state to be beyond the reach of any radiation, so one takes  $F_o = 0$ . In this case, the energy flux into the system is the (negative) value of the material energy influx. Second, in a planar shock of infinite lateral extent with an upstream region that has limited optical depth,  $F_o$  will correspond to the radiation lost upstream beyond some location that is identified as a reasonable, steady initial state. In this case, the values of density and temperature at this location may have evolved to this state (from some other values) during the initiation of the shock. In addition, the net energy influx to the rest of the system in this case is reduced

by comparison with the first case. It is the difference between the incoming material energy flux and the outgoing radiation flux. Third, one may have a (more or less) planar shock of finite lateral extent. This is relevant to various experiments that produce optically thin upstream regions. In this case  $F_o$  represents the sum of the energy lost beyond some designated axial position and the energy lost radially before reaching that position. This is the case because (7.81) keeps track of energy conservation, so that any energy removed from the radiation flux must be absorbed in the matter. As a result, when absorption decreases the upstream radiative energy flux, the returning material energy flux (from further upstream) must also decrease. In contrast, when lateral losses decrease the upstream radiative energy flux, the material energy flux does not decrease.

### Homework 7.11

Demonstrate this point explicitly by considering a system having a planar flow of material within a cylinder of some diameter and of finite length yet losing radiation both radially and axially, and integrating over the cylinder.

To further clarify what the radiation is doing, it is worthwhile to discuss the recycling of energy that occurs in radiative shocks. The flow of energy through the system involves the following sequence as the shock is established. One begins with incoming mechanical and internal energy from far upstream. In the shock and the shocked matter, the plasma converts the mechanical energy to additional internal energy and to radiation. Some of the radiation flows away downstream with the material. The rest of the radiation flows upstream. If absorbed there (reducing  $F_R$  where the absorption occurs) the radiation adds internal energy to the incoming material. In a steady state with any upstream heating by radiation, the mechanical and internal energy incoming to the shock is larger than its initial value before the shock was established. Thus, the elements of the shock as a system include incoming material energy, recycling of energy by upstream absorption of radiation, the escape of radiation upstream including perhaps radially, and the escape of radiation and material energy downstream.

Returning to (7.81), in the absence of a heat flux ( $F_R$  and  $F_o$  here, but this could be any heat flux), this equation provides a second, redundant condition for the pressure. The simultaneous solution of (7.79) and (7.81) then determines the only possible compression at the shock. The presence of the heat flux opens a larger range of possibilities. Equations (7.79) and (7.81) imply

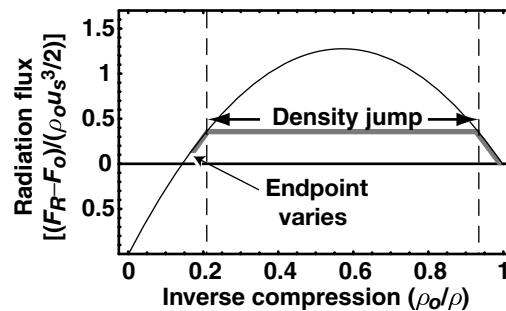
$$F_R - F_o = \frac{\rho_o u_s^3}{2} \left[ \frac{2\gamma}{\gamma-1} \frac{\rho_o}{\rho} - \frac{\gamma+1}{\gamma-1} \left( \frac{\rho_o}{\rho} \right)^2 - 1 - \frac{p_o}{\rho_o u_s^2} \frac{2\gamma}{\gamma-1} \left( 1 - \frac{\rho_o}{\rho} \right) \right]. \quad (7.82)$$

We will use this equation extensively in what follows, choosing  $F_o$  according to the discussion above.

**Homework 7.12**

Derive (7.82).

Figure 7.20 shows the dependence of the net radiation flux ( $F_R - F_o$ ) on the inverse compression, for  $p_o = 0$ . The flux is normalized to the incoming kinetic energy flux,  $\rho_o u_s^3/2$ . Note that this curve depends on  $\gamma$  while the previous figures for pressure and the specific thermal energy  $RT$  do not. The net, normalized radiation flux rises above zero only in the presence of energy recycling in the plasma, when shock-heated matter emits radiation that warms upstream matter which then carries the energy back to the shock. A rather magical physical system, with multiple independent recycling loops, would be needed to drive the flux above 1. The net, normalized radiation flux reaches  $-1$  if all the energy entering the system is finally radiated away downstream or otherwise lost, corresponding to a state of formally infinite compression and zero temperature.



**Fig. 7.20.** Radiative flux in radiative shocks, normalized to  $\rho_o u_s^3/2$ , against the inverse compression,  $\rho_o/\rho$ , for  $\gamma = 4/3$ . The gray line shows a characteristic shock trajectory

Figure 7.20 is tremendously important for understanding the properties of radiative shocks. We discuss some aspects of this now, but we will return to this type of figure repeatedly in what follows. The general behavior of a radiative shock is shown as a trajectory on this figure. There is some compression in the precursor region, as  $(F_R - F_o)$  increases from zero to a maximum that equals the value of  $(F_R - F_o)$  entering the precursor from the shocked region. Because  $(F_R - F_o)$  is continuous across the viscous shock transition (the density jump), this value of  $(F_R - F_o)$  then fixes the compression produced by this transition. Further evolution may also occur after the density jump. The nature of this evolution depends on the downstream boundary condition. This boundary condition could correspond to (a) a positive radiation flux if there is a bright source downstream of the shock, (b) a radiation flux of zero if the downstream region is optically thick, or (c) a negative radiation flux if the shocked matter also cools by emitting radiation that is lost

from the system. The value of the inverse compression where  $(F_R - F_o) = 0$  corresponds to the final state one would reach in a nonradiative shock, and is  $(\gamma + 1)/(\gamma - 1)$ . This may or may not correspond to the final density in an actual radiative shock. We will discuss some specific cases in later sections.

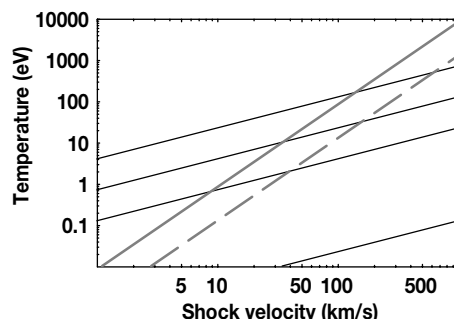
When we reach the point of discussing the structure within the shocked material in a radiative shock, we will need an equation describing the spatial evolution. We will limit our discussion to steady-state shocks. Our approach will be to assume that either the final state or the immediate postshock state can be determined on the basis of the fundamental conservation equations. From this starting point we will integrate the energy equation to determine the profiles. We again start with (7.2), but now we are interested in the spatial derivatives. We assume a polytropic gas, work in the radiative flux regime, and substitute from (7.78) and the continuity relation to obtain

$$\frac{\rho_o u_s^3}{2} \frac{\partial}{\partial x} \left[ -\frac{2\gamma}{\gamma - 1} \left( \frac{\rho_o}{\rho} \right) + \frac{\gamma + 1}{\gamma - 1} \left( \frac{\rho_o}{\rho} \right)^2 \right] = -\frac{\partial F_R}{\partial x}. \quad (7.83)$$

This is the equation we will use to explore the spatial profiles.

It may or may not be feasible for a shock to radiate away all of its energy. The upper limit on the radiative flux,  $F_R$ , emitted from the shocked layer equally in both upstream and downstream directions is  $\rho_o u_s^3/4$  in each direction, as this would radiate away all the energy. Thus, based on the following discussion of optically thin precursors, the upper limit on the temperature in the precursor is approximately  $T_{\text{eff}} = [\rho_o u_s^3/(4\sigma)]^{1/4}$ . This temperature increases with density, while the immediate postshock temperature from (7.77) depends only on  $u_s$ . As a result, at any velocity there is a density above which a steady shock cannot radiate away all the energy. Figure 7.21 shows the velocity dependence of this limiting  $T_{\text{eff}}$ , for several densities from  $10^{-21}$  g/cm<sup>3</sup> to 1 g/cm<sup>3</sup>. It also shows the nonradiative postshock temperature from (7.77), with  $R = k_B(1 + Z)/(Am_p) = 1.5 \times 10^{11}$  ergs/g<sup>-1</sup>/eV<sup>-1</sup> or  $1.5 \times 10^{12}$  ergs/g<sup>-1</sup>/eV<sup>-1</sup>, corresponding roughly to a Xenon plasma or a low- $Z$  plasma, respectively. When the upper limit on  $T_{\text{eff}}$  exceeds the nonradiative postshock temperature (i.e., when the black line is above the gray line in the figure), the implication is that radiation will be too weak to fully cool the shocked medium, which will then remain warm in the postshock state. In addition, as we discuss in the next section, the shocked layer in a laboratory experiment is unlikely to remain optically thin in steady state.

On the one hand, the effective temperature as just defined (and therefore the temperature in the precursor) will indeed be negligible in interstellar astrophysical cases, where the density is less than or of order  $10^{-21}$  g/cm<sup>3</sup>. Recall that if the downstream region is optically thin, of optical depth  $\kappa d$ , then its temperature  $T_f$  is larger than the effective temperature, as  $\sigma T_{\text{eff}}^4 = \kappa d \sigma T_f^4$ . On the other hand, we show below that the precursor temperature may approach  $T_{\text{eff}}$  in laboratory experiments. In this case, for  $\gamma \leq 5/3$ , the



**Fig. 7.21.** Temperatures vs. shock velocity (km/s). The solid curves show the effective temperature that would be necessary to radiate away the entire incoming energy flux. From bottom to top, the densities are  $10^{-21}$  g/cm<sup>3</sup>,  $10^{-6}$  g/cm<sup>3</sup>,  $10^{-3}$  g/cm<sup>3</sup>, and 1 g/cm<sup>3</sup>. The gray curves show the postshock temperature that would be present without radiative effects, for Xe (solid, with  $\gamma = 4/3$ ) and H<sub>2</sub>He<sub>1</sub> (dashed, with  $\gamma = 5/3$ )

initial normalized pressure,  $p_o/(\rho_o u_s^2)$ , will be  $\leq 0.25$ , which is small but may not always be negligible.

Let us return to the properties of the precursor region. Assuming zero initial upstream pressure  $p_o$ , (7.80) and (7.81) can be solved for the normalized temperature in the precursor,  $RT_p/u_s^2$ , which turns out to depend only on the normalized net radiation flux,  $F_{Rn} = 2(F_R - F_o)/(\rho_o u_s^3)$ . Note that  $F_{Rn}$  has value of 1/2 when half the incoming energy flux is escaping upstream but might be larger than this in the presence of recycling. One finds

$$\frac{RT_p}{u_s^2} = \frac{1}{(\gamma + 1)^2} \left( 1 - \sqrt{1 - (\gamma^2 - 1)F_{Rn}} \right) \left( \gamma + \sqrt{1 - (\gamma^2 - 1)F_{Rn}} \right). \quad (7.84)$$

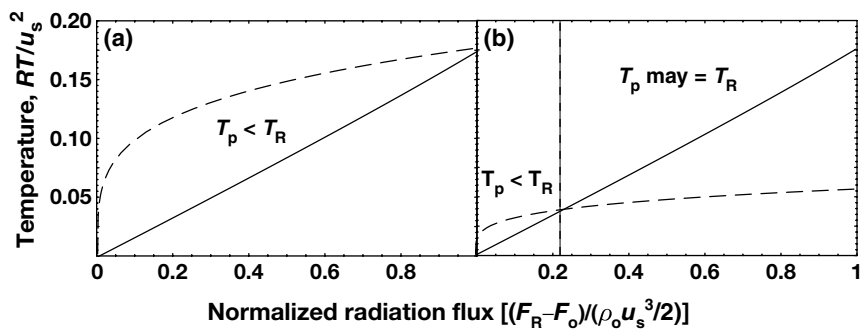
This temperature goes to zero as  $F_{Rn}$  goes to zero, as it should. As  $F_{Rn}$  increases,  $T_p$  can approach but cannot exceed the temperature of the radiation from the shocked matter. (We discuss this point further below.) This enables us to do some useful reasoning.

We will characterize the radiation fluxes emitted from optically thick layers of material as thermal radiation, and will construct radiation fluxes in thin layers by considering thermal emission rates as we discussed in Chap. 6. With specific regard to the precursor region, we can say that there is a minimum possible temperature of the radiation from the (downstream) shocked material,  $T_{\min}$ , given by  $\sigma T_{\min}^4 = F_{Rn}$ . The downstream temperature can, of course, be higher than this, for example, if the downstream region is optically thin or if there is a significant radiation flux returning from upstream ( $F_{Rn}$  is a net flux). But in any event  $T_{\min}$  is the smallest, this temperature can be. With this definition, the normalized radiation flux is

$$F_{Rn} = \frac{\sigma T_{\min}^4}{\rho_o u_s^3/2} = \frac{\sigma u_s^8 (RT_{\min}/u_s^2)^4}{R^4 \rho_o u_s^3/2} = Q \left( \frac{RT_{\min}}{u_s^2} \right)^4. \quad (7.85)$$

This defines a radiation strength parameter,  $Q = 2\sigma u_s^5/(R^4 \rho_o)$ , which has a typical value of  $10^4$  to  $10^5$  in laboratory radiative shocks. Here  $Q$  is nondimensional but must be evaluated using consistent units, such as cgs units with  $\sigma$  in  $\text{ergs/s}^{-1} \text{ cm}^2/\text{eV}^4$  and with  $R$  in  $\text{ergs/g}^1/\text{eV}^1$ .

Figure 7.22 shows the values of  $RT_p/u_s^2$  and  $RT_{\min}/u_s^2$  vs.  $F_{Rn}$ , for  $Q = 10^3$  and  $10^5$ . At  $Q = 10^3$  and for this  $\gamma$ , the precursor temperature remains always below the minimum radiation temperature. This guarantees that the radiation in the precursor will always be out of equilibrium, and that the absorption of this radiation will determine the precursor structure. *This conclusion is independent of any specific model regarding radiation transport in the precursor.* In this case, the precursor is always a transmissive precursor in the sense discussed above. In contrast, at  $Q = 10^5$ ,  $T_p$  exceeds  $T_{\min}$  when the radiation flux is large enough. When this occurs, the precursor in this region is an absorptive precursor if the downstream matter is optically thick and may or may not be otherwise. Wherever  $T_p$  exceeds  $T_{\min}$ , there must be offsetting radiation fluxes, increasing the actual radiation temperature, because the temperature in the precursor,  $T_p$ , cannot rise above the “temperature” of the radiation that provides the heating. Zeldovich and Raizer provide a thorough discussion of this point in the context of optically thick systems, for which they take the postshock temperature to be well defined. We will see below that the radiative flux from the shocked region may, in general, have components corresponding to a range of temperatures. Even so, the precursor temperature cannot become so large that the flow of energy from the shocked matter to the precursor and back would act to decrease the entropy, which would violate the second law of thermodynamics. (A parcel of energy  $\Delta\epsilon$  flows as radiation from the shocked matter to the precursor,



**Fig. 7.22.** Scaling of precursor and minimum radiation temperatures. The figure shows  $RT_p/u_s^2$  (solid) and  $RT_{\min}/u_s^2$  (dashed) vs.  $F_{Rn}$ , for  $\gamma = 4/3$ . The radiation strength parameter  $Q$  is  $10^3$  in (a) and  $10^5$  in (b)

carrying entropy  $\Delta\epsilon/T_R$  and is carried back to the shock as material energy, with entropy  $\Delta\epsilon/T_p$ . The initial entropy must not exceed the returning entropy, so  $T_p \leq T_R$ .) Thus, for the case of Fig. 7.22(b), if  $F_{Rn} \sim 0.6$ , then the normalized radiation temperature must be at minimum equal to  $RT_p/u_s^2 \sim 0.1$ .

In the prior literature of radiative shocks, the distinction between a *subcritical shock*, having  $T_p < T_f$ , and a critical or *supercritical shock*, having  $T_p = T_f$ , is emphasized. Here  $T_f$  is the steady-state temperature of the downstream region (assuming this exists). In the context of the previous paragraph,  $T_f \geq T_{\min}$ . The traditional viewpoint can be summarized as follows. The radiative flux must be continuous when one crosses the shock, which sets the immediate postshock density as can be seen in a figure like Fig. 7.20. Because the temperature curve (Fig. 7.19) is shifted to the left relative to the flux curve (Fig. 7.20), the immediate postshock temperature  $T_2$  is *always* higher than  $T_p$ .  $T_f$  is smaller than  $T_2$ , but how much smaller depends on the details of the radiation transport and on the radiation strength  $Q$ . If  $Q$  is small enough, then  $T_p$  can never equal  $T_f$ , as in Fig. 7.22(a). Such a shock is known as a subcritical shock. The traditional notion is that a strong enough shock will produce  $T_p = T_f$  and such a shock is known as a critical or supercritical shock. This is implied by an LTE analysis using equilibrium radiation diffusion. However, a radiation diffusion model is a poor vehicle for the study of the inherently abrupt structures involved in shocks. We will see below that  $T_p$  may approach  $T_f$  in various realistic circumstances, but that this is a limiting case rather than the threshold of a regime. One could self-consistently define a supercritical shock as one in which  $T_p > T_{\min}$  in the sense of Fig. 7.22, but this is not the traditional definition. In short, the present author is rather skeptical of the existence and usefulness of supercritical shocks as traditionally defined.

Finally, consider the density in the precursor. By comparing Figs. 7.19 and 7.20, one can see that there is a nonzero final temperature for any finite inverse compression and thus for any possible final value of  $F_R$ . For example, the final normalized temperature is approximately 0.12 for the specific value of the inverse compression (approximately 0.15) corresponding to  $(F_R - F_o) \sim 0$  and  $\gamma = 4/3$  in Fig. 7.20. One can see in Fig. 7.19 that if the precursor temperature reaches this final temperature then the density increase in the precursor will be between 10 and 15%. This is quite small in comparison with the total density increase, of order 10, which justifies somewhat the assumption in some following sections that the density is unchanged in the precursor.

### 7.4.3 Models of Radiative Precursors

Now we turn from the general fluid dynamics, which applies to every radiative shock, to specific models of the shock structure. We will consider this in two steps, analyzing first the structure of the precursor and second the structure

of the shocked layer itself. Any such analysis must be within the context of a chosen treatment of radiative transfer. We consider two examples for precursors, involving first the diffusion regime and then the transport regime.

Before turning to specific models, we first discuss one simple threshold for radiative effects. One can say that a radiative precursor will be present when the flux of ionizing photons radiated ahead of the shock equals the flux of neutral atoms incident on the shock. This point of view is that one will certainly see heating and a change of state of the upstream medium when all (or most) of the incoming atoms are ionized. To be precise, the threshold would be when the upstream flux of ionizing photons times the fraction that are absorbed in the upstream region equals the flux of incoming atoms. One way to express the flux of ionizing photons is as the flux of photons emitted by a black body at the postshock temperature, which is  $2.3 \times 10^{23} T_s^3$ , with  $T_s$  in eV, times the fraction of these photons that are emitted and are ionizing. This fraction is the product of the emissivity of the downstream region,  $\epsilon_d$ , and the fraction  $\alpha_i$  of all photons that are ionizing. Recall that  $\epsilon_d$  is equal to the optical depth if the downstream region is optically thin. The fraction  $\alpha_i$  is near unity for shock velocities above 50 km/s.

### Homework 7.13

Working with the Planck description of blackbody radiation, find and plot the fraction of photons that are ionizing as a function of temperature. You will need a computational mathematics program to generate the plot.

The fraction of ionizing photons that is absorbed in the upstream region is equal to the upstream emissivity,  $\epsilon_u$ . One can assemble the last few lines of material into an equation for the threshold:

$$2.3 \times 10^{23} \epsilon_u \epsilon_d \alpha_i T_s^3 > \rho u_s / (A m_p). \quad (7.86)$$

Using (7.77), one can convert this into a threshold for the shock velocity, given by

$$u_s > 270 [\rho / (\epsilon_u \epsilon_d \alpha_i)]^{1/5} \text{ km/s}. \quad (7.87)$$

In laboratory experiments with dense gases or foams, the quantities in square brackets may all be of order unity. For low-density astrophysical systems, (7.87) is correct but not very useful. With  $\rho$  of order  $10^{-24} \text{ g/cm}^3$ , obtaining a radiative precursor will require first of all that the postshock temperature be high enough to obtain a significant fraction of ionizing photons. Beyond that it will depend on the optical depth of the system. We will consider next two limiting cases of optical depth – very thick and very thin.

### Diffusive Radiative Precursors

In the diffusion regime, one is tempted to model the precursor in a supercritical shock as a Marshak wave, since it is a diffusive radiation wave emanating



from a constant-temperature source. In this case, one must deal heuristically with the fact that the source is moving. The Marshak wave has a length-dependent velocity, being very fast when it is short (early in time after its initiation) and slowing down monotonically as its length increases. One can argue that, in steady state, the precursor length ahead of a shock must be such that the diffusion wave velocity equals the shock velocity. In either (7.57) or (7.66) one has the length  $x_o = \xi_o \sqrt{Wt}$  with  $\xi_o$  a constant near 1.6 and  $W = \kappa_{\text{rad}} T_o / [\rho c_V (n+1)]$ . Matching the precursor velocity to the shock velocity gives  $u_s = (\xi_o/2) \sqrt{W/t}$ , which determines the “time” in the Marshak wave evolution at which the length is maintained. Combining these gives the steady-state precursor length as

$$x_o = \xi_o^2 W / (2u_s). \quad (7.88)$$

Note that  $W$ , being proportional to the coefficient of radiative heat conduction,  $\kappa_{\text{rad}}$ , scales as a large power of the temperature ( $T^4$  to  $T^7$ ). This is why the precursor length has a *very* strong dependence on shock velocity. However, in the context of real systems one would have difficulty observing such precursors. One can use the steady-state precursor length in the initial relation for  $x_o$  to find the time,  $t_o$ , required for the diffusion wave to reach this length. One obtains

$$t_o = (x_o / \xi_o \sqrt{W})^2 = \xi_o^2 W / (4u_s^2). \quad (7.89)$$

This time also increases very rapidly with shock velocity. Real experiments in the planar geometry of this analysis will achieve steady-state precursors only over a very narrow range in velocity.

The qualitative situation is better for experiments in spherical geometry, if in fact they can produce large enough systems that a diffusion model can meaningfully apply. Zeldovich and Raizer consider the case of nonlinear radiative heat diffusion from a point source in spherical geometry. They find that the diffusion wave moves with  $r \propto t^{1/(3n+2)}$ , so that the velocity is  $\propto t^{-(3n+1)/(3n+2)} \sim 1/t$ .

It is not too hard to improve the analysis above for a planar system. In the radiative flux regime and for a polytropic gas the energy flux equation, (7.2), becomes

$$\frac{\partial}{\partial t} \left( \frac{\rho u^2}{2} + \rho \epsilon \right) + \nabla \cdot \left[ \rho \mathbf{u} \left( \gamma \epsilon + \frac{u^2}{2} \right) \right] = -\nabla \cdot \mathbf{F}_R. \quad (7.90)$$

To improve the analysis at a first level, we can ignore the changes in density and velocity in the precursor, treating it as a plasma moving with a constant velocity of  $-u_s$ . Approximating  $\gamma$  as constant, and assuming a one-dimensional shock, this equation simplifies to

$$\rho \frac{\partial \epsilon}{\partial t} - u_s \gamma \rho \frac{\partial \epsilon}{\partial x} = -\frac{\partial}{\partial x} F_R. \quad (7.91)$$

Applying the analysis of Chap. 4, one can show that this equation does not admit a self-similar solution.

**Homework 7.14**

Determine whether (7.91) admits a self-similar solution, assuming a diffusive model for  $F_R$ .

But we can hypothesize that a steady precursor may develop, reflecting the balance of upstream diffusion and downstream flow, so we consider this problem in steady state. If we take  $\epsilon = c_V T$ , approximating  $c_V$  as constant, and assume  $F_R$  is produced by diffusive heat transport we find

$$-u_s \gamma \frac{\partial T}{\partial x} = \frac{\partial}{\partial x} \frac{\kappa_{\text{rad}}}{\rho c_V} \frac{\partial}{\partial x} T, \quad (7.92)$$

from which as in Sect. 7.3 we obtain

$$\frac{\partial f}{\partial x} = -\frac{W}{\gamma u_s} \frac{\partial^2 f^{n+1}}{\partial x^2} \quad (7.93)$$

in this case with  $f = T/T_{\text{eff}}$  (taking the radiation flux leaving the shocked matter to be  $\sigma T_{\text{eff}}^4$ ). Now we define a new variable  $\zeta = x\gamma u_s/W$ , so that (7.93) becomes

$$\frac{\partial f}{\partial \zeta} = -\frac{\partial f^{n+1}}{\partial \zeta^2}. \quad (7.94)$$

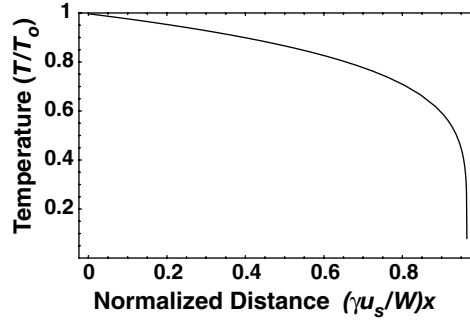
The specific solution here must be such that the upstream radiative heat flux at the precursor boundary balances the thermal energy brought back to the shock by the incoming flow. (This has some subtle aspects, because the diffusion treatment is not fully self-consistent.) Here we write the flux balance as

$$\kappa_{\text{rad}} \nabla T_R \Big|_{\text{shock}} = \gamma \rho \epsilon u_s. \quad (7.95)$$

Upon ignoring differences between  $T_{\text{eff}}$ ,  $T_R \Big|_{\text{shock}}$ , and the material temperature at the shock, which are caught up in the subtleties just mentioned, this becomes

$$\frac{\partial f}{\partial \zeta} \Big|_{\zeta=0} = \frac{1}{n+1}. \quad (7.96)$$

Figure 7.23 shows the precursor profile for  $n = 4$ . Solutions to (7.94) that produce a heat front place it where  $\zeta$  is some fraction of 1. Let us compare the size of this precursor with our simple estimate above. The ratio of the result of the improved calculation to that of the simple estimate is  $\sqrt{W}/(\xi_o \gamma u_s \sqrt{t}) = (u_o/u_s)[2/(\xi_o^2 \gamma)]$ . Here  $u_o$  is the velocity of the Marshak wave. Our estimate above took  $u_o = u_s$ . One can see that the precursor length from the diffusion model is a fraction  $2/(\xi_o^2 \gamma)$  of the precursor length from the Marshak-wave estimate, which is approximately one half. In other words, the effect of the incoming flow is to reduce the size of the precursor in addition to limiting its expansion.



**Fig. 7.23.** A diffusive precursor profile. A sample solution of (7.92), for  $n = 4$ , which gives  $f'(0) = -0.2$ . The temperature ratio  $f$  is shown on the ordinate, with the normalized distance  $x\gamma u_s/W$  on the abscissa

### Homework 7.15

Solve (7.94) numerically, for several relevant values of  $n$ . Comment on the results.

We can estimate the diffusive precursor length as follows. For typical parameters ( $\chi_R \sim 10^6 T_{\text{eV}}^{-4/3} \text{ cm}^2/\text{g}$ ,  $n = 4$ ,  $\gamma = 4/3$ ,  $\rho = 0.1 \text{ g/cm}^3$ ,  $c_V = 10^{12} \text{ ergs g}^{-1}/\text{eV}$ ,  $u_s = 2 \times 10^6 T_{\text{eV}}^{1/2} \text{ cm/s}$ ), one finds  $W = 10^{-4} T_{\text{eff}}^{13/3}$  and  $W/(\gamma u_s) = W/(8 \times 10^6 T_{\text{eff}}^{1/2})$ , with  $T_{\text{eff}}$  in eV. This is  $19 \mu\text{m}$  at  $100 \text{ eV}$  for this density but would be  $1.9 \text{ mm}$  for  $\rho = 0.01 \text{ g/cm}^3$ . Figure 7.24 shows the steady-state diffusive precursor lengths in a space of density and temperature. The boundary labeled “limit” in this figure is where the precursor length decreases to ten interparticle spacings for ionized Be. The model is certainly not valid beyond that point. One should also note that in many real circumstances the diffusive model may not apply because the opacity in the heated precursor is too small. The next section considers this situation.

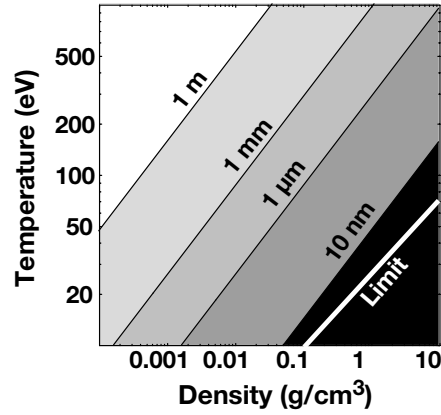
### Transmissive Radiative Precursors

For any transmissive precursor that extends over many absorption lengths, including a subcritical shock in the diffusive regime, one can model the precursor using the nonequilibrium diffusion theory of Sect. 6.2.4. One has from (6.57) and (6.59), again under the assumptions that the density does not change and that the system is in the radiative flux regime, that

$$\mathbf{F}_R = -\frac{4}{3\bar{\chi}} \nabla \sigma T_R^4, \text{ and} \quad (7.97)$$

$$\nabla \cdot \mathbf{F}_R = 4\kappa_P \sigma T_P^4 - 4\kappa_E \sigma T_R^4, \quad (7.98)$$

in which  $\bar{\chi}$  and  $\kappa_E$  are averaged opacities defined in that section and  $\kappa_P$  is the Planck mean opacity. Although there are three distinct opacities in



**Fig. 7.24.** Diffusive precursor lengths. The boundaries show the indicated values. Experiments with foams and gasses at densities below  $10 \text{ mg/cm}^3$  will tend to produce very large precursors

(7.98), it seems common in the literature to assume without comment that these are all equal. While this has the virtue of being consistent with the treatment in some computer codes employing nonequilibrium diffusion, it is not numerically correct, and may introduce significant errors. In the case of the subcritical shock with  $T_p$  less than some large fraction of  $T_R$ ,  $T_p$  can be ignored in (7.98). In this case the radiation flux is just attenuated. Solving (7.97) and (7.98), with the optical depth  $\tau$  defined here as the magnitude of the distance times the “opacity,”  $\tau = \sqrt{\chi \kappa_E z}$ , gives

$$F_R = F_o e^{-\sqrt{3}\tau}, \quad (7.99)$$

in which  $F_o$  is the radiation flux emerging from the shock. If the absorption is dominated by bremsstrahlung, this result may be accurate. However, only a fairly sophisticated computer code will treat the opacities here correctly and thus evaluate the exponential scale length accurately.

### Radiative Precursors in the Transport Regime

In many real situations the radiative transfer within the precursor (and the shocked matter too) may be in the transport regime. The transport regime is the relevant one in the case that the upstream plasma is optically thin or is limited in extent. Moreover, in this case the radiative flux may approach the full flux from the shocked region,  $\sigma T_{\text{eff}}^4$ , while in the diffusion regime the Rosseland flux is much smaller, being a blackbody flux  $\sigma T_R^4$  multiplied by the (small) fractional change in temperature per unit (reduced) optical depth. We assume first that the radiation pressure is negligible, that the radiation and matter temperatures are the same, and that the density and velocity in the

precursor region are not changed by the precursor (just as is the case for the Marshak wave). Under these assumptions and for a planar precursor (7.2) becomes

$$\rho \frac{\partial \epsilon}{\partial t} + \rho u \gamma \frac{\partial \epsilon}{\partial z} = - \frac{\partial F_R}{\partial z} = 4\pi \kappa (J_R - B). \quad (7.100)$$

This equation allows useful estimates of the steady-state plasma temperature and the time required to reach steady state. First consider  $(J_R - B)$ . The average intensity  $J_R$  has three components. These are the contribution from the shocked matter ( $J_1$ ), the contribution from the region between the shock and a given location ( $J_2$ ), and the contribution from the region upstream of the given location ( $J_3$ ). To make a simple analysis tractable, we suppose that the upstream plasma has a characteristic size  $D$  in the upstream direction and is infinite laterally. We then examine the plasma in a location a distance  $d$  from the shock.

In calculating the contribution from the shocked matter, we take the radiation intensity (power per unit area per unit solid angle) to be  $\sigma T_{\text{eff}}^4/\pi$ . Then we have

$$J_1 = \frac{\sigma T_{\text{eff}}^4}{4\pi^2} \int e^{-\kappa d/\mu} d\Omega = \frac{\sigma T_{\text{eff}}^4}{2\pi} \int_0^1 e^{-\kappa d/\mu} d\mu, \quad (7.101)$$

in which we integrate over the hemisphere facing upstream. This integral evaluates to

$$J_1 = \frac{\sigma T_{\text{eff}}^4}{2\pi} [e^{-\kappa d} - \kappa d \Gamma(0, \kappa d)] \approx \frac{\sigma T_{\text{eff}}^4}{2\pi}, \quad (7.102)$$

in which  $\Gamma$  is the incomplete Gamma function and the second approximate equality requires that we stay where the optical depth to the shock ( $\kappa d$ ) is small.

To find  $J_2$  and  $J_3$ , we will have to integrate  $B$  over space. We will designate the plasma temperature as  $T_p$  so that  $B = \sigma T_p^4/\pi$ . Our point of view here is that we are analyzing a very thin system, so that measured in optical depths, the precursor of interest is very near the shock. This allows us to assume that  $T_p$  is constant for the purpose of evaluating this integral. We then have

$$J_2 = \frac{1}{4\pi} \int d\Omega \int_0^d \kappa B e^{-\kappa z/\mu} dz/\mu, \quad (7.103)$$

in which the integral over distance evaluates the radiation intensity at a polar angle corresponding to  $\mu$  and the solid-angle integral is over the forward hemisphere. One finds

$$J_2 = \frac{B}{2} [1 - e^{-\kappa d} + \kappa d (\Gamma(0, \kappa d))]. \quad (7.104)$$

For small  $\kappa d$ ,  $J_2 \sim \kappa d [1 + \Gamma(0, \kappa d)]$ .

Proceeding to  $J_3$ , one has

$$J_3 = \frac{1}{4\pi} \int d\Omega \int_0^{D-d} \frac{\kappa}{\mu} B e^{-\kappa z/\mu} dz = \frac{B}{2} [1 - e^{-\kappa D} + \kappa D \Gamma(0, \kappa D)], \quad (7.105)$$

in which the solid-angle integral is now over the entire hemisphere in the downstream direction and in writing the rightmost expression we have assumed  $d \ll D$ . We can now rewrite (7.100), realizing that  $\epsilon = RT_p/(\gamma - 1)$ , as

$$2\kappa [\sigma T_{\text{eff}}^4 - \sigma T_p^4 (1 + e^{-\kappa D} - \kappa D \Gamma(0, \kappa D))] = -\frac{\rho_o u_s \gamma R T_p'}{\gamma - 1}. \quad (7.106)$$

The term on the right-hand side of this equation should be small, by our assumptions. One can check this, using (7.84) to evaluate the derivative of  $RT_p$ , taking the derivative of  $F_{Rn}$  to be  $-\kappa F_{Rn}$ . One finds this term to be small for any strength parameter  $Q > 10$ . Thus one finds for  $T_p$ :

$$T_p = \frac{T_{\text{eff}}}{[1 + e^{-\kappa D} - \kappa D \Gamma(0, \kappa D)]^{1/4}}. \quad (7.107)$$

For small  $\kappa D$ ,  $T_p$  is 84% of  $T_{\text{eff}}$ . This turns out to equal the final downstream temperature in a shock that is thick downstream but thin upstream (see Sect. 7.4.5). In this equation,  $T_p$  increases as  $\kappa D$  does, approaching  $T_{\text{eff}}$  at large  $\kappa D$ . However, the calculation of  $J_R$  becomes invalid as  $\kappa D$  increases. One knows from the flux balance equation that  $T_p$  will decrease as the net radiation flux decreases with increasing  $\kappa D$ . However, one may wonder how  $T_p$  near the shock can be a fixed fraction of  $T_{\text{eff}}$  in spite of the discussion relating to Fig. 7.22 above. The answer is that this system is optically thin by assumption and that much of the radiation flux crossing the shock is in the end lost from the system and so increases  $F_o$ . We do not know how much is lost a priori, as is indeed the case in real systems of this type. As we discussed in Sect. 7.4.2, the flux balance equation demands only a relation between the temperature profile in the precursor and the radiation flux that is actually absorbed there.

The relevance of (7.107) depends on how readily the steady state is achieved and whether there is in fact time for the precursor plasma to be heated as the shock approaches it. To evaluate this, we use (7.80) and take  $\partial\epsilon/\partial t \sim \epsilon(T_{\text{eff}})/t_{ss}$ , defining the time we seek as  $t_{ss}$ , and we take  $J_R \sim J_1$  and  $B \sim 0$ . This gives

$$t_{ss} = \frac{\rho\epsilon(T_{\text{eff}})}{2\kappa\sigma T_{\text{eff}}^4}. \quad (7.108)$$

This turns out to be remarkably fast. For a laboratory plasma, we can take  $\epsilon \sim 10^{12} T_{\text{eff}}$  ergs/g and  $\kappa \sim 10^6/T_{\text{eff}}$  cm<sup>-1</sup> to find  $t_{ss} \sim 10^{-6} \rho/T_{\text{eff}}^2$ , which is 1 ns for  $T_{\text{eff}} \sim 10$  eV and  $\rho \sim 0.1$  g/cm<sup>3</sup>. Radiation hydrodynamic experiments usually have timescales of at least several nanoseconds. Thus we conclude that

the precursor plasma may approach its steady-state temperature reasonably quickly in laboratory experiments. For an interstellar astrophysical plasma with  $\kappa \sim 5 \times 10^{-37} \text{ cm}^{-1}$ ,  $\rho \sim 10^{-23} \text{ g/cm}^{-3}$ , and  $T_{\text{eff}} \sim 10 \text{ eV}$  one finds  $t_{ss} \sim 10^{10} \text{ s} \sim 300 \text{ years}$ . This too is very fast, but it will turn out that  $T_{\text{eff}}$  is typically not large enough to be significant in such astrophysical plasmas. In both these examples, there will be an initial transition period, during which the shock processes precursor material that has not yet reached this steady state.

Thus, the precursor may approach steady state and the plasma temperature  $T_p$  will be close to  $T_{\text{eff}}$  in a shock with an optically thin upstream layer. Note that this result has no explicit dependence on the value of  $T_{\text{eff}}$ . Instead, the temperature in the precursor will depend primarily on the optical depth of the (downstream) shocked layer. The temperature near the shock will tend to increase as the optical depth of the precursor increases, but a more complete calculation would be needed to assess how much. Thus, *the traditional distinction between subcritical and supercritical shocks has no relevance when the upstream medium is optically thin*. The precursor will be determined by  $T_{\text{eff}}$ , which in turn is determined by details of the downstream layer discussed in the next two sections.

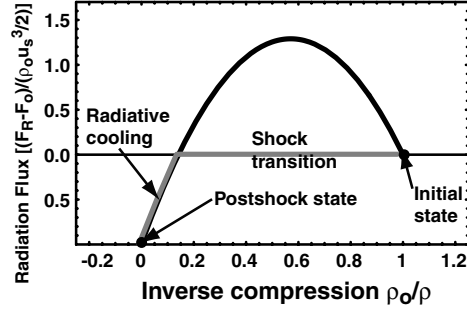
### Homework 7.16

Evaluate the net radiation flux ( $F_R - F_o$ ) for an optically thin precursor using a calculation similar to that done in (7.101) and (7.105).

#### 7.4.4 Optically Thin Radiative Shocks

In the present section we consider shocks that are optically thin throughout, so that radiation freely escapes in both directions. We also assume that nearby radiation sources are negligible, so that the shock exists in isolation. In this case all of the incoming energy eventually leaves the system as radiation. It is important that an optically thin system is *not* energy conserving. Some or all of the incoming material energy flux leaves the system as radiation. A limiting case is to assume that the pressure in the precursor is negligible compared to  $\rho_o u_s^2$ . In this case, the ordinate of Fig. 7.20,  $(F_R - F_o)/(\rho_o u_s^3/2)$ , is zero at the shock but  $F_o$  is finite and represents energy escaping in the upstream direction. The density jump at the shock will then be  $(\gamma+1)/(\gamma-1)$ , and the normalized radiation flux will decrease after the shock to  $-1$ , as all the incoming energy is converted to radiation. The immediate postshock temperature will equal that in the nonradiative case. The final density will be formally infinite, limited only by factors outside this analysis, such as increasing magnetic field, increasing optical depth of the downstream plasma, or external radiation sources.

For the remainder of this section, we consider the case that corresponds to interstellar astrophysical shocks, under conditions that allow the radiation



**Fig. 7.25.** Fluid dynamics of an optically thin shock. The gray curve shows the trajectory of the system, from an initial state with negligible pressure

to carry away all the incoming energy. Our evaluation of the initial conditions, from the previous paragraphs, applies. As a result, Fig. 7.25 shows the evolution of the shock. From the initial state we have discussed, the shock transition takes the inverse compression to  $(\gamma - 1)/(\gamma + 1)$ . After this, radiative cooling moves the inverse compression toward zero. The question we address here is what profiles develop during this cooling.

To work with (7.83), we consider that  $J_R$  is negligible compared to  $B$ , so in the transport regime  $\partial F_R / \partial x = -4\pi\kappa B$ . We now want to simplify (7.83), to find the essential parameters that control the behavior. We use the subscript  $i$  to designate the immediate postshock state. Then we can take  $\kappa = \kappa_i(\rho/\rho_i)^m(T/T_i)^{-n}$  so that

$$\frac{\partial F_R}{\partial x} = -4\pi\kappa_i B_i \left( \frac{\gamma + 1}{\gamma - 1} \frac{\rho_o}{\rho} \right)^{-m} \left( \frac{T}{T_i} \right)^{4-n}. \quad (7.109)$$

With this definition, the natural normalization of (7.83) is to create a radiation parameter  $R_r$ , defined by

$$R_r = \frac{\gamma + 1}{\gamma} \frac{4\pi B_i}{\rho_o u_s^3} = \frac{\gamma + 1}{\gamma} \frac{4\sigma T_i^4}{\rho_o u_s^3}, \quad (7.110)$$

which is approximately the ratio of the radiative flux from an optically thick shocked layer at temperature  $T_i$  to the incoming energy flux. When  $R_r > 1$ , the radiative fluxes in the postshock layer exceed the material energy fluxes. Thus, when  $R_r \geq 1$  a *cooling layer* will develop, in which the plasma temperature decreases until a sustainable value is reached. In the limit we are considering  $T_i \propto u_s^2$ , so  $R_r$  increases as  $u_s^5$ . We also define an optical depth variable  $\tau = -\kappa_i z$ . In addition, (7.80) (the equation of state) implies

$$\frac{T}{T_i} = \frac{(\gamma + 1)^2}{(\gamma - 1) \left( 2 + (\gamma + 1) \frac{p_o}{\rho_o u_s^2} \right)} \frac{\rho_o}{\rho} \left( 1 + \frac{p_o}{\rho_o u_s^2} - \frac{\rho_o}{\rho} \right), \quad (7.111)$$



where we keep the terms proportional to  $p_o$  (which will be needed later). With these definitions, (7.83) can be rewritten as

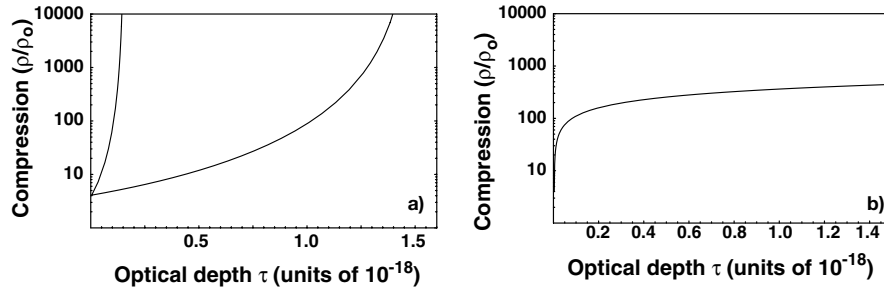
$$\begin{aligned} & \frac{\gamma + 1}{\gamma} \left[ -\frac{\gamma(1 + \frac{p_o}{\rho_o u_s^2})}{\gamma - 1} + \frac{\gamma + 1}{\gamma - 1} \left( \frac{\rho_o}{\rho} \right) \right] \frac{\partial}{\partial \tau} \left( \frac{\rho_o}{\rho} \right) \\ &= R_r \left( \frac{\gamma + 1}{\gamma - 1} \right)^{-m} \left( \frac{\rho_o}{\rho} \right)^{-m} \left[ \frac{(\gamma + 1)^2}{2(\gamma - 1)} \left( 1 - \frac{\rho_o}{\rho} \right) \frac{\rho_o}{\rho} \right]^{4-n}. \end{aligned} \quad (7.112)$$

Given boundary conditions at the shock, along with  $R_r$ ,  $p_o$ , and the parameters describing the material in the system ( $\gamma$ ,  $m$ ,  $n$ ), one can integrate this equation to find the density profile. Note that if there is absorption of radiation in the precursor, then  $p_o$  is not the pressure at the shock, and  $(F_R - F_o)$  is no longer zero at the shock. In this case  $p_o$  is the pressure at a point beyond which there is negligible heating by radiation. The pressure on the upstream side of the shock has been accounted for through both  $p_o$  and its relation to compression.

We will see below that  $R_r$  is within a few orders of magnitude of unity for typical laboratory experiments. Under interstellar astrophysical conditions, however,  $R_r$  is enormous. Using  $\gamma = 5/3$ ,  $k_B(1 + Z)/Am_p = 1.5 \times 10^{12}$  ergs/g/eV, and  $\rho_o = 10^{-22}$  g/cm<sup>3</sup>, one finds  $R_r \sim 10^{18}$  for  $u_s \sim 100$  km/s. Figure 7.26 shows the resulting density profiles for relevant parameters with  $R_r = 10^{18}$  or  $10^{19}$ . Note that the optical depth required for the evolution of the profile is approximately  $1/R_r$ . This remains true even as  $R_r$  becomes much smaller ( $\sim 10$ ). In the astrophysical case, recall (from Sect. 6.2.2) that we had  $\kappa \sim 10^{-38}$  cm. Thus the cooling distance  $1/(\kappa R_r)$  for these parameters is of order  $10^{20}$  cm or 100 light years. This distance becomes smaller as the shock velocity increases, producing more radiation.

As we remarked above, it will be some factor not in this model that stops the increase in density. The increase will slow or stop, for example, if the slope of  $\kappa$  with  $T$  changes or if the magnetic pressure becomes significant. Interstellar astrophysical shocks often cool by radiation until the temperature of the downstream, shocked material equals that of the nearby environment both beyond and ahead of the shock. These are sometimes known by the horribly unphysical designation “isothermal shocks.” (The term *isothermal shock* is also sometimes used to describe the very idealized limit in which a shock in the presence of heat conduction may have no jump in temperature where the density jump occurs. The presence of a lighter particle species, such as photons, that transports heat eliminates this solution except as a limiting case. One example is the “supercritical shock” as discussed in Sect. 7.4.6.)

One sees in Fig. 7.26a a density increase that becomes increasingly rapid with increasing optical depth. This type of cooling is sometimes known as *catastrophic cooling*. This occurs, for example, when *old supernova remnants* cool sufficiently. What is required to produce catastrophic cooling is that



**Fig. 7.26.** Density profiles for optically thin shocks. (a) Cool plasmas. Here  $\gamma = 5/3$ ,  $n = 4/3$ , and  $m = 2$ . Profiles are shown for  $R_r = 10^{18}$  (on the right) and  $R_r = 10^{19}$  (on the left). In this model, the density increases without limit as all the energy goes into radiation. (b) Hot plasmas. Here  $\gamma = 5/3$ ,  $n = -1$ , and  $m = 2$ , and  $R_r = 10^{21}$ , corresponding qualitatively to the behavior at temperatures above the minimum of the cooling function near 100 eV

$\partial\kappa/\partial T < 0$ . Then as cooling for some time period leads to a density increase and a temperature decrease, the rate of cooling increases so there is more cooling in the next time period. However, the behavior seen in Fig. 7.26a is not universal in all interstellar astrophysical shocks. As Fig. 6.5 showed, the astrophysical cooling function reverses slope at some temperature above a few hundred eV, corresponding to a shock velocity of approximately 300 km/s. Figure 7.26b shows the cooling that occurs for  $n = -1$  and  $R_r = 10^{21}$ . Some rapid cooling again takes place over a small distance, but following this there is gradual cooling over a much larger distance. Under these conditions, a shocked layer will cool slowly until  $\kappa$  reaches its minimum and  $\partial\kappa/\partial T < 0$ . Then rapid cooling to very low temperature will ensue. The rapid density increase may be described as a *density collapse* or *collapse of the shock*, because the thickness of the shocked layer decreases in inverse proportion to the increase of density.

#### 7.4.5 Radiative Shocks that are Thick Downstream and Thin Upstream

We turn now to a circumstance that is common in laboratory experiments with radiative shocks. The upstream medium, being limited in extent, may quickly all become heated and after that will be optically thin. Alternatively, for example, in a spherical experiment in gas, the upstream medium may quickly become optically thin out to a heat front (or an ionization front; see Sect. 7.4.9) where the radiation is absorbed. However, the optical depth of the heated upstream medium is likely to be small, as is the radiation flux from the precursor region back toward the shock. Correspondingly, for steady-state calculations we will take the view that the precursor region is a uniform plasma with some initial temperature (eventually this will approach

the final temperature of the optically thick downstream region,  $T_f$ , for reasons discussed above), but that essentially none of the radiation entering the precursor returns across the density jump.

Before considering the steady-state case, let us qualitatively analyze the evolution of such a shock from an initial, optically thin limit as a laboratory experiment begins. One has  $\partial\kappa/\partial T < 0$  under many conditions of interest, so the initial behavior will involve a density collapse like that discussed in the previous section. This could be limited by the maximum in  $\kappa$  at low temperature but is more likely to be limited by the transition at some temperature (and after some time) to an optically thick shocked layer. The resulting density and temperature of the shocked layer, making it optically thick, might be said to represent the initial attempt of the system to establish a steady state. However, the radiation from the cooling layer controls the ultimate steady state, as we discuss next. This radiation may heat the shocked matter to a final temperature above the initial value, establishing a steady state only when the shocked matter becomes optically thick at the temperature necessary for self-consistency.

Figure 7.27 illustrates the energy balance in a steady shock of this type. One can see how the cooling layer controls the final state. The net flux at the downstream boundary of the cooling layer must be zero, so the final temperature must increase until the thermal flux from the steady downstream layer equals the flux from the cooling layer. When this occurs, the net upstream radiation flux, lost from the system in our description, but perhaps in reality having the effect of extending the length of a precursor region, is  $2\sigma T_f^4$ . One self-consistency test for calculation of profiles as is described here is that the integrated radiation source, and the integrated change in material energy should both equal  $2\sigma T_f^4$ .

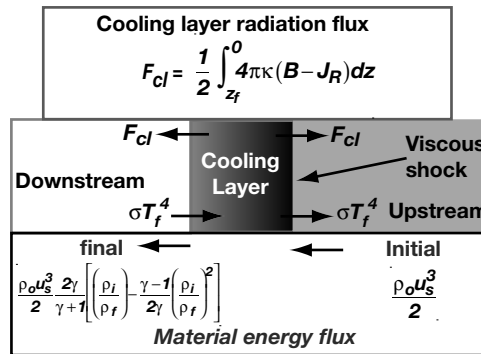


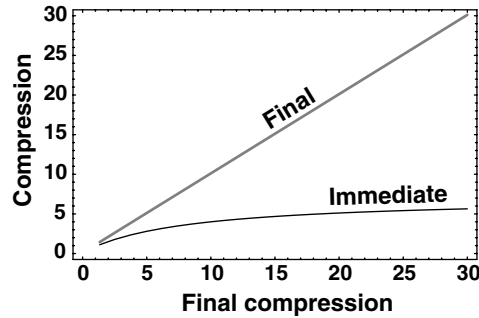
Fig. 7.27. Energy fluxes in thick-downstream, thin-upstream shocks. Note  $p_o = 0$

**Homework 7.17**

Assuming that the upstream radiation flux at the shock is  $2\sigma T_f^4$ , the intensity is isotropic, and the absorption and emission from the upstream medium contribute negligibly to  $J_R$ , find the steady-state temperature of the upstream medium.

We can develop solutions for the structure of such a cooling layer, as follows. Based on our discussion of optically thin precursors, the upstream temperature will be quite close to  $T_f$ . Here we will take it to be equal to  $T_f$ , so the normalized upstream pressure is  $p_{\text{on}} = \rho_o R T_f / (\rho_o u_s^2) = R T_f / u_s^2$ . From (7.80) this implies that  $p_{\text{on}} = R T_f / (u_s^2) = \rho_o / \rho_f$ . Knowing  $p_{\text{on}}$ , we can evaluate the flux balance equation (7.82) to find the conditions at the boundaries of the cooling layer. At the shock transition  $F_R = F_o = 2\sigma T_f^4$ , so given  $p_{\text{on}} = \rho_o / \rho_f$  one finds the initial inverse compression in terms of the final inverse compression, from (7.82),

$$\frac{\rho_o}{\rho_i} = \frac{\gamma - 1}{\gamma + 1} + \frac{2\gamma}{\gamma + 1} \frac{\rho_o}{\rho_f}. \quad (7.113)$$



**Fig. 7.28.** Compression in thick–thin shocks. The final compression (gray) and immediate postshock compression are shown as a function of the final compression, for  $\gamma = 4/3$

Figure 7.28 shows the implications of this equation, for  $\gamma = 4/3$ . One can see that a very large final compression will be required before the initial compression becomes as large as the nonradiative, strong-shock value of 7. The initial compression is smaller than this because of the finite pressure  $p_o$  in the precursor.

In the final state,  $F_R = 0$ , so using (7.82) with  $\rho_o / \rho = \rho_o / \rho_f = p_{\text{on}}$  one finds the net (normalized) radiation flux at the final density to be

$$\frac{-2F_o}{\rho_o u_s^3} = -1 + \left( \frac{\rho_o}{\rho_f} \right)^2, \quad (7.114)$$

which turns out to be independent of  $\gamma$  other than through the final density. Note that the limiting value of  $F_o$ , as the final density becomes very large, equals  $\rho_o u_s^3/2$  as it should (per Fig. 7.25).

To pin down the final state, we observe that the energy supplying the radiation flux comes from the shocked material, so the net radiation flux must equal the net energy flux lost from the material between the immediate, postshock state, and the final state. The behavior of the radiation flux in the shock is thus like that shown in Fig. 7.25, with three differences that are not dramatic on such a plot. The shape of the curve including the value of its maximum are altered by the finite value of  $p_{on}$ , the value of the postshock inverse compression is not the value for a nonradiative shock, and the final normalized flux is close to but larger than  $-1$ . By setting the net flux ( $F_R = F_o$ ) across the shock equal to  $\sigma T_f^4$  (see Fig. 7.27), we can solve (7.82) to find the final inverse compression. This turns out to depend on the radiation strength parameter we defined above as  $Q = 2u_s^5\sigma/(R^4\rho_o)$ , in terms of which

$$\frac{\rho_o}{\rho_f} = \sqrt{\frac{\sqrt{1+8Q}-1}{4Q}}. \quad (7.115)$$

This is independent of  $\gamma$ , although the detailed structure is not. For  $R = 10^{12}$  ergs/g/eV,  $u_s$  in km/s =  $u_{\text{kms}}$ , and  $\rho_o$  in g/cm<sup>3</sup>, one finds  $Q = 4 \times 10^{-11} u_{\text{kms}}^5/\rho_o$ . At large enough shock velocity and thus large  $Q$  the final inverse compression approaches zero, although by that point the system may be entering the radiation-dominated regime. This result may seem strange, given that the radiation gets stronger as velocity increases. But recall that a smaller final inverse compression corresponds to a decreasing fraction of the incident energy ending up as thermal energy, as more and more energy is radiated away. At small velocity (7.115) would take the inverse compression to 1. However, we require  $u_s^2 > c_{so}^2$ , where  $c_{so}$  is the upstream sound speed, to have a shock. Evaluating  $u_s^2/c_{so}^2$  we find

$$\frac{u_s^2}{c_{so}^2} = \frac{u_s^2\rho_o}{\gamma p_o} = \frac{u_s^2}{\gamma RT_f} = \frac{\rho_f/\rho_o}{\gamma}. \quad (7.116)$$

Thus, the final compression must exceed  $\gamma$ , or equivalently the inverse compression must be smaller than  $1/\gamma$ , in order to have a shock at all. Correspondingly, one can show that  $Q > \gamma^2(\gamma^2 - 1)/2$  in order to have a shock. The parameter  $Q$  depends primarily on  $u_s^5/\rho_o$ , which is the same as the ratio we found in our first, preliminary discussion of precursors. The temperature dependence of  $R$ , if included, would introduce additional complications in the solution. Here, to see the main qualitative behavior, we will assume  $R$  (and thus  $Z$ ) to be constant.

### Homework 7.18

Beginning with (7.80)–(7.82), derive the final inverse compression (7.115) under the assumptions of the present section.

Figure 7.29 shows (a) the dependence of the final compression on shock velocity for three densities and (b) the fraction of the energy radiated away under the same conditions. This figure uses  $Z = 17$  and  $A = 130$ , corresponding to the use of a high- $Z$  material to maximize the radiative effects. One sees that the radiation indeed carries away most of the energy as the system becomes more radiative. One sees that the compression can indeed become very high as shock velocity increases, but that this requires radiation of very nearly all of the incoming energy flux.

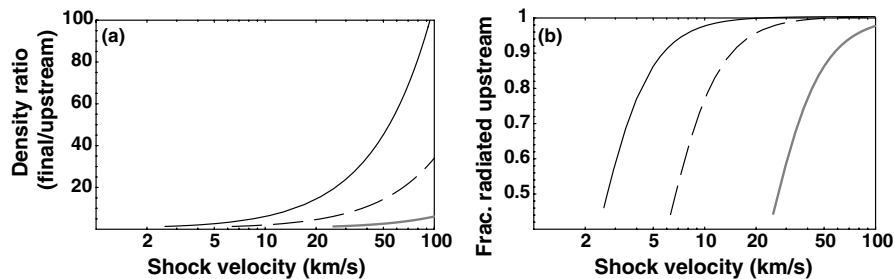
We are now ready to determine the profiles. To do so, we note that  $R_r$  evaluates to

$$R_r = Q \frac{\gamma + 1}{\gamma} \left[ \frac{2(\gamma - 1)}{(\gamma + 1)^2} - \frac{(1 - 6\gamma + \gamma^2)}{(\gamma + 1)^2} \frac{\rho_o}{\rho_f} - \frac{2\gamma(\gamma - 1)}{(\gamma + 1)^2} \left( \frac{\rho_o}{\rho_f} \right)^2 \right]^4. \quad (7.117)$$

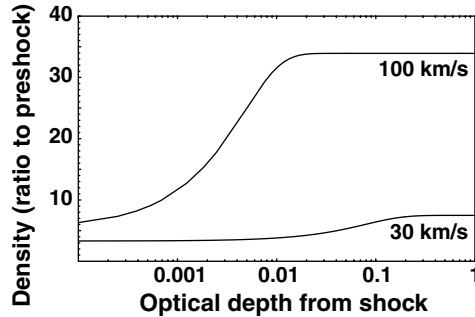
By integrating (7.83) as represented by (7.112), beginning at the shock transition, one can obtain the profiles shown in Fig. 7.30. The two cases shown correspond to  $R_r \sim 20$  and  $R_r \sim 1200$ , so one can see that here again the distance required for the cooling decreases as the shock velocity and hence  $R_r$  increase.

#### 7.4.6 Fluid Dynamics of Optically Thick Radiative Shocks

We now turn to a type of radiative shock that can exist only within an extensive system, in which both the upstream and the downstream mediums are optically thick but yet the shock is in steady state. One might be inclined in this case to consider the system to be in LTE, and to use a diffusion model to describe the dynamics. However, this is not strictly valid because the shock itself drives the plasma out of equilibrium with the radiation. We will take the point of view that the precursor region, far enough away from the shock, can perhaps be described by the modified Marshak-wave model developed in

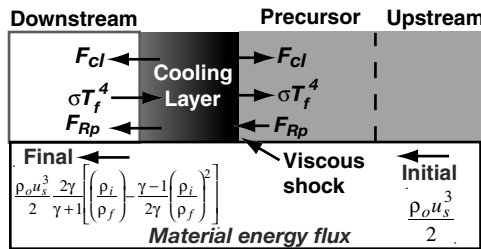


**Fig. 7.29.** Final state conditions for thick–thin shocks, for  $\gamma = 4/3$ ,  $A = 130$ , and  $Z = 17$ . The curves show results for  $1 \text{ g/cm}^3$  (gray),  $1 \text{ mg/cm}^3$ , (dashed), and  $10 \text{ } \mu\text{g/cm}^3$  (solid). (a) The ratio of final density to upstream density as a function of shock velocity. (b) The fraction of the incoming energy flux carried away upstream by radiation



**Fig. 7.30.** Density within the cooling layer, plotted as the ratio to the preshock density and against the optical depth from the shock transition. The curves show results for a shock velocity of 30 km/s (lower) and 100 km/s (upper). These are evaluated for  $\rho = 1 \text{ mg/cm}^3$ ,  $\gamma = 4/3$ ,  $n = 4/3$ ,  $m = 0$ ,  $A = 130$ , and  $Z = 17$

Sect. 7.4.3, and that the downstream region away from the shock is also in LTE. There are some general things we can say about this system. These are based on the energy flux balance, with reference to Fig. 7.31, and on the fluid dynamics, with reference to Fig. 7.32.



**Fig. 7.31.** Fluxes in thick-thick shocks

There are two places where the net radiation flux must be zero. These are at the head of the precursor, where all the net upstream radiation from the shock has been converted to heat, and at the boundary of the downstream region, where a new postshock steady state is established. In the radiation-dominated regime, radiation energy and pressure are convected through these boundaries, but there is still no net radiation flux through them. If the absolute value of the radiation flux generated in the cooling layer is  $F_{cl}$ , then at the downstream boundary of the cooling layer one has  $F_{cl} + F_{Rp} = \sigma T_f^4$ , where  $F_{Rp}$  is the radiation from the precursor plasma. (The cooling layer differs only in details from that discussed in the previous section. Recall that the cooling layer is optically very thin, so the fluxes from adjacent regions are fully transmitted.) The net radiation flux through the shock transition

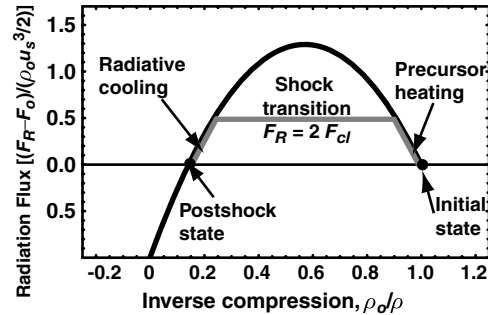


Fig. 7.32. Fluid dynamics of optically thick (thick-thick) radiative shocks

must balance the increase in convected energy flux in the precursor, which is negative and which we can designate as  $F_p$ . Thus  $F_{cl} + \sigma T_f^4 - F_{Rp} + F_p = 0$ , so that

$$2F_{cl} = 2\sigma T_f^4 - F_{Rp} = -F_p. \quad (7.118)$$

This also equals the net radiative flux across the shock.

We can take  $F_{Rp}$  to be approximately  $\sigma T_p^4$ , where  $T_p$  is the temperature of the first few optical depths of the precursor. As  $T_p$  approaches  $T_f$ , both  $F_{cl}$  and  $F_p$  become a smaller and smaller fraction of the radiation flux in either direction. However, these cannot become zero, and in fact we will shortly find the limiting value of  $F_p$  from the fluid dynamics. The implication is that  $T_p < T_f$  always. This once again casts some doubt on the traditional definition of a supercritical shock, which in fact exists only as a limiting case. One might imagine that the temperature could become continuous as the diffusion limit is approached, and that then the difference between  $T_f$  and  $T_p$  would represent the difference produced by the temperature gradient over a few radiation mean-free paths. However, this is not an accurate conclusion because the shock and cooling layer is always an out-of-equilibrium, nondiffusive structure.

It is next worthwhile to consider what the fluid dynamics of Sect. 7.4.3 may imply. The flux balance equation, if necessary including radiative pressure and energy terms, applies to all optically thick radiative shocks. No energy is “lost” from the system (except perhaps laterally in a laterally limited planar system, but we do not consider this here). This calculation naturally separates into flux-dominated and radiation-dominated cases. We take these up in turn.

#### 7.4.7 Optically Thick Shocks—Radiative-Flux Regime

Figure 7.32 shows the fluid-dynamics trajectory in the flux-dominated regime. The plasma is heated and compressed in the precursor, undergoes the shock transition at  $F_R = -F_p$  as just discussed, and then cools to a final state with



zero flux. For negligible initial upstream pressure, the final state thus has  $\rho_f/\rho_o = (\gamma + 1)/(\gamma - 1)$  and has a normalized temperature given by (7.80) as

$$RT_f/u_s^2 = 2(\gamma - 1)/(\gamma + 1)^2, \quad (7.119)$$

which is 0.12 for  $\gamma = 4/3$ .

The fluid dynamics also implies  $F_p$  as a function of  $T_p$ , as the difference between the net material energy flux reaching the shock from upstream (which includes recycled energy) and the incident material energy flux,  $\rho_o u_s^3/2$ . If one evaluates (7.82) in the precursor just before the density transition, taking  $p_o = 0$  and using (7.80) (which originates from the continuity equation, momentum equation, and equation of state), one finds that

$$F_p = -\frac{\rho_o u_s^3}{4(\gamma - 1)} \left[ (1 - \gamma) + 2(\gamma + 1) \frac{RT_p}{u_s^2} + (\gamma - 1) \sqrt{1 - 4 \frac{RT_p}{u_s^2}} \right]. \quad (7.120)$$

This goes to zero as  $T_p \rightarrow 0$ , as it should, but also goes to zero as  $T_p \rightarrow T_f$ . This cannot be a solution, because recycled energy is present for any  $T_p \neq 0$ .

Given  $F_p$  from (7.120)  $T_f$  from (7.119), and  $F_{Rp} = \sigma T_p^4$ , one can express (7.118) as an implicit equation for  $T_p$ :

$$\left( \frac{RT_p}{u_s^2} \right)^4 + \frac{1}{2Q} \frac{RT_p}{u_s^2} \frac{(\gamma + 1)}{(\gamma - 1)} + \frac{1}{4Q} \left( \sqrt{1 - 4 \frac{RT_p}{u_s^2}} - 1 \right) - 16 \frac{(\gamma - 1)^4}{(\gamma + 1)^8} = 0, \quad (7.121)$$

in which we again encounter the radiation strength parameter,  $Q = 2u_s^5\sigma/(R^4\rho_o)$ . Noting from (7.120) that the final term on the left-hand side in this equation is  $RT_f/u_s^2$ , one sees that  $T_p$  reaches  $T_f$  only in the limit that  $Q \rightarrow \infty$ , as expected from the discussion above.

To understand what is happening in this system, we can examine Fig. 7.33, which plots in (a) the solution of (7.121) for  $RT_p/u_s^2$  and in (b) the solution of (7.120) for  $2F_p/(\rho_o u_s^3)$ , which is the recycled flux as a fraction of the energy flux incident on the shock, both for  $\gamma = 4/3$ . We see that as  $Q$  increases above 1,000,  $T_p$  becomes a large fraction of  $T_f$ . We also see that, as this occurs, the

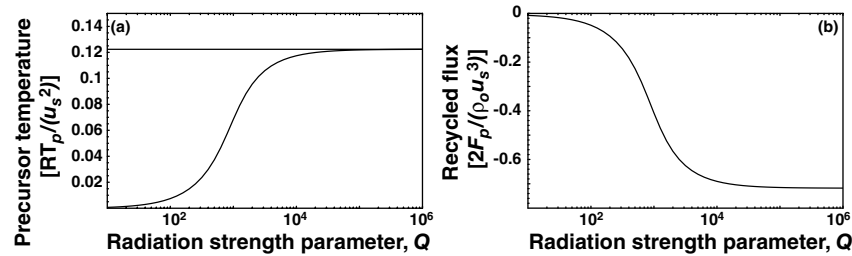


Fig. 7.33. Precursor properties in optically thick radiative shocks, for  $\gamma = 4/3$ . (a) Normalized temperature and (b) recycled flux

recycled energy flux asymptotes to about 70% of the incident energy flux. As the incident shock velocity (and thus  $Q$ ) increases further, the net recycled flux (which is the net radiative flux across the shock, and also twice the flux from the cooling layer) remains a fixed fraction of the incident flux while the radiative fluxes in each direction,  $\sigma T_p^4$  and  $\sigma T_f^4$ , increase much more rapidly.

The calculation of the structure of the shock in this regime is very similar that of Sects. 7.4.4 and 7.4.5. One can begin at the final state and integrate the energy flux toward the shock. One finds the inverse compression and hence the radiation flux and the temperature as a function of optical depth from the downstream end of the cooling layer. One reaches the shock when the upstream radiation flux reaches  $(\sigma T_f^4 - F_p/2)$  (recalling that  $F_p < 0$ ). The shape of the cooling layer will be similar to that found in the previous section.

The analysis of the precursor will have to proceed by in principle either by integrating upstream from the shock or by integrating downstream from the initial precursor state. At the shock, one knows the upstream radiation flux from the final state and from the cooling layer, and also both the downstream radiation flux and the material energy flux from the precursor. As an approximate solution, one can match this solution to the diffusive precursor profile found in Sect. 7.4.3. How realistic this will be is not so clear, as the flux just upstream of the shock *is* significantly out of equilibrium. To obtain a better solution, one will need to account for the actual radiation flux and the actual variation in density and other parameters. We will have to leave sorting out these details as an exercise for the motivated reader.

#### 7.4.8 Radiation-Dominated Optically Thick Shocks

For the radiative shocks in optically thick systems that are our subject here, one could hope to evaluate the structure in the radiation-dominated regime, using an approach similar to the one we just described for the radiative flux regime. However, this would be mathematically even more complicated, because the radiation pressure depends on the fourth power of  $T$ . We will leave the problem of the structure, and even more so of the structure in transition regimes, to the specialized literature. It is important to note, as Sect. 7.4.7 showed, that treatments of the internal structure near the shock that use only the diffusion approximation will be qualitatively wrong. Here we consider only the relation between initial and final states, where the radiative flux is zero.

In this case the continuity equation is unchanged. The momentum and energy equations (7.1 and 7.2) become, for steady-state planar shocks,

$$\rho u^2 + p + p_R = \rho_o u_o^2 + p_o + p_{Ro} \text{ and} \quad (7.122)$$

$$u \left( \rho \frac{u^2}{2} + \rho \epsilon + E_R + p + p_R \right) = u_o \left( \rho_o \frac{u_o^2}{2} + \rho_o \epsilon_o + E_{R_o} + p_o + p_{R_o} \right), \quad (7.123)$$

where as usual these equations are in the shock frame. We would like to develop useful relations from this, just as we have done previously. Because  $u_o = -u_s$  and  $u = -u_s(\rho_o/\rho)$ , we find

$$\frac{p + p_R}{\rho_o u_s^2} = \left( 1 - \frac{\rho_o}{\rho} \right) + \frac{p_o + p_{R_o}}{\rho_o u_s^2} \quad \text{and} \quad (7.124)$$

$$\frac{1}{2} \left( 1 - \left( \frac{\rho_o}{\rho} \right)^2 \right) - \left( \frac{\gamma p / (\gamma - 1) + 4 p_R}{\rho_o u_s^2} \right) \left( \frac{\rho_o}{\rho} \right) + \left( \frac{\gamma p_o / (\gamma - 1) + 4 p_{R_o}}{\rho_o u_s^2} \right) = 0. \quad (7.125)$$

Here we have two equations for three unknowns ( $\rho$ ,  $p$ , and  $p_R$ ). In the case that radiation completely dominates, so  $p$  can be neglected, these are readily solved for  $p_R$  and  $\rho$ . If  $p$  and  $p_R$  are both known functions of  $T$ , one can solve for  $T$  and  $\rho$ . Alternatively, if the medium is ionizing and has  $\gamma = 4/3$ , one can solve for the total pressure and  $\rho$ . We consider the third case here and leave the first two to homework.

If  $\gamma = 4/3$ , then one can substitute from (7.124) into (7.125) and express the total pressure as  $p_t$  to obtain

$$\frac{1}{2} \left( 1 - \frac{\rho_o}{\rho} \right) \left( 1 - 7 \frac{\rho_o}{\rho} + 8 \left( \frac{p_{t_o}}{\rho_o u_s^2} \right) \right) = 0, \quad (7.126)$$

in which  $p_{t_o}$  is the total pressure in the upstream state. The two solutions of this equation for the inverse compression ( $\rho_o/\rho$ ) give the total density change across the shock transition. These solutions are 1 (the upstream density) and  $(1/7)[1 + 8p_{t_o}/(\rho_o u_s^2)]$ . Thus, with negligible upstream pressure the density increases by a factor of 7 and this density increase gets smaller as the upstream pressure, normalized by the ram pressure, increases. The shock will vanish when the upstream pressure reaches 7/8 of the ram pressure. One can substitute for the inverse compression in (7.124) and solve for the final total pressure, finding  $p_t = (6\rho_o u_s^2 - p_{t_o})/7$ . At the most, this can be 6/7 of the ram pressure when the upstream pressure is negligible.

### Homework 7.19

Consider a truly radiation-dominated case, so  $p$  can be neglected in (7.124) and (7.125). Solve these equations for  $p_R$  and  $\rho$ . Find the dependence of the postshock  $T$  on the shock velocity, and compare it to the dependence of a non-radiative shock.

**Homework 7.20**

Express  $p$  and  $p_R$  as reasonable functions of  $T$  and solve (7.124) and (7.125) to find  $T$  and  $\rho$  in the postshock state. This may be a numerical solution, for which you should make reasonable choices about the parameters and show a few cases. Provide at least one graph based on these equations as part of the analysis.

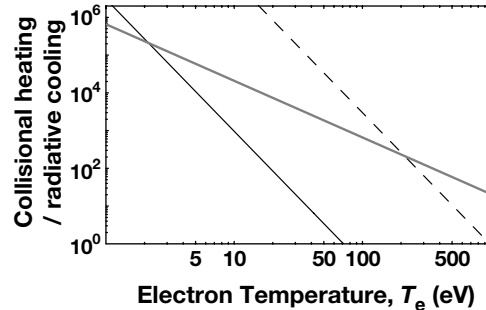
**7.4.9 Electron-Ion Coupling in Shocks**

To this point we have ignored the equilibration region behind the density jump in radiative shocks. The shock heating of the electrons is small [consider (7.77)] for electrons, so that it is the ions that are primarily heated. The ions then heat the electrons by Coulomb collisions, in a region we will designate the *equilibration zone*. At issue here is the slowing of the ions, which would be described in collision theory as the “test particles” in this case. The field particles, which interact with the ions and cause them to slow, are the electrons. The electron-heating coefficient is  $\nu_{ie} = 3.2 \times 10^{-9} n_i Z^3 \ln \Lambda / (A T_e^{3/2})$ , with  $T_e$  in eV and  $n_e$  in  $\text{cm}^{-3}$ . The evolution of the ion temperature is then

$$\frac{\partial T_{\text{ion}}}{\partial t} = -\nu_{ie} (T_{\text{ion}} - T_e). \quad (7.127)$$

The electron heating also may correspond to increased ionization of the ions. Typically the electron-heating coefficient and the temperature difference both decrease as the electrons heat. But curiously, if  $Z \propto \sqrt{T_e}$  as is approximately true in ionizing plasmas, then the electron-heating coefficient remains constant as the electron temperature increases, leading to more rapid equilibration. The exchange of energy between the electrons and the ions, including ionization and the heating of the new electrons, does not change the total postshock pressure. In contrast, radiation can affect the pressure, and also the rate of equilibration through the density. This makes it worthwhile to compare the radiative rates with the electron heating.

First we compare the heating or cooling rates. Then we consider more carefully the structure of the equilibration zone. The rate of energy emission, in power per unit volume, is  $2\kappa\sigma T_e^4$  at high density and  $n_e n_i \Lambda$  at low density (see Chap. 6). The rate of energy transfer per unit volume per unit fractional temperature difference ( $T_{\text{ion}}/T_e - 1$ ) is  $\rho c_V \nu_{ie}$ . Figure 7.34 shows the ratio of  $\rho c_V \nu_{ie}$  to the radiative cooling rate for both laboratory and astrophysical conditions. For electron temperatures that do not approach keV levels, electron heating clearly dominates, except at low density ( $0.01 \text{ g/cm}^3$ ), and using a density-independent scaling for  $\kappa$  (which may not apply in actuality). One concludes that radiative cooling of electrons would become important in shocks producing ion temperatures of many keV, and possibly under some conditions for somewhat lower temperatures.

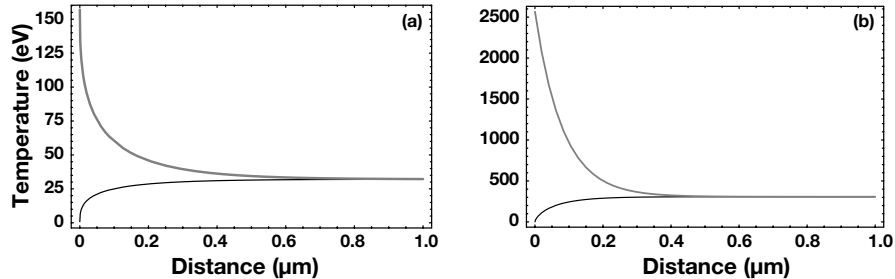


**Fig. 7.34.** Ratio of electron heating power to radiative emission vs.  $T_e$  for cases of interest. In the laboratory regime, with  $\rho = 0.01 \text{ g/cm}^3$  (black solid) and  $1 \text{ g/cm}^3$  (dashed), using the density-independent form of  $\kappa$  for Al from (6.45). In the astrophysical regime (gray), where the ratio depends only on  $T_e$ , using  $\Lambda = 10^{-22} \text{ erg cm}^{-3}$

However, Fig. 7.34 overstates the importance of radiation, because it is the net difference of absorption and emission that heats or cools the electrons. In a shock, the radiation from the final state will at first overwhelm the radiation emission by the electrons and will contribute to their heating. Later, when the electron temperature rises above the final plasma temperature, radiation will have a net cooling effect, in opposition to electron heating by the ions. However, in many cases these differences won't matter because the effect of the radiation on the equilibration zone will be negligible. Assuming that radiation plays no role, let us consider the structure of the equilibration zone produced by electron heating alone.

We can convert (7.127) into a spatial equation in the shock frame by dividing by the postshock fluid velocity,  $u = u_s(\gamma - 1)/(\gamma + 1)$ . There is a corresponding equation for the electrons in which  $T_e$  and  $T_i$  are exchanged and the right-hand side is negative. One can use (7.4.1) for the ions with  $Z = 0$  to get a characteristic initial value of  $T_i$ , and one can assume  $T_e \sim 0$  to start. Figure 7.35 shows the resulting spatial profiles of  $T_e$  and  $T_i$  for a shock velocity of 100 km/s. One sees that even for very low density, low- $Z$  gas (part a) the equilibration occurs within a few micrometers. For Xe gas at somewhat higher density, the equilibration occurs within a fraction of a micrometer.

Recall that in detail the shock transition itself is not instantaneous, but occurs over a distance of a few ion-ion mean-free paths. Both electron heating and radiation emission do occur simultaneously with the shock transition, so there may be regimes in which all three processes are simultaneously important. It will typically be adequate, though, to assume that the shock transition occurs instantaneously, that the equilibration zone is at most small, and that the radiation becomes important on a larger spatial scale. In simulations, the electron heating occurs on the scale of the shock transition for conditions



**Fig. 7.35.** Postshock electron-ion equilibration. Positive distance is downstream from the 100 km/s shock. (a) Be at  $\rho = 10^{-5}$  g/cm<sup>3</sup>. (b) Xe at  $\rho = .01$  g/cm<sup>3</sup>, assuming  $Z \propto \sqrt{T_e}$

that produce postshock temperatures of order 10 eV, and occurs on a larger (though still small) scale as the postshock ion temperature reaches hundreds of eV.

## 7.5 Ionization Fronts

The past two sections have dealt in part with radiation sources. Section 7.3 discussed the diffusive penetration of radiation into a medium. In the particular case of the Marshak wave, this was due to a temperature source at the edge of the medium. Section 7.4 discussed radiative shocks, which are shock waves that become strong enough to be a radiation source to the region ahead of the shock, making radiation essential to their dynamics. A third type of phenomenon that can occur when radiation penetrates a medium is the *ionization front*. An ionization front is a local region where energy delivered by radiation ionizes (or perhaps further ionizes) the medium it penetrates. To be an ionization front, as opposed to a diffusive radiation wave, this ionization must *not* lead to energetically significant emission of radiation.

This definition may sound like a somewhat special case, but in fact it is common. The opacity of many materials drops dramatically once they are ionized, which both reduces their emission and increases the transmission of the incoming radiation. This is particularly true of ionized hydrogen, which can no longer undergo bound-free transitions once it is ionized. When heavier atoms are ionized, their opacity also changes significantly. It decreases at photon energies near the ionization energy, and becomes larger at the higher photon energies required to ionize the next state. In consequence, a spectrum of photons that is able to ionize some specific state may be readily transmitted through the material after ionization but yet be unable to ionize the material further. It is this changing opacity that makes ionization fronts possible.

Examples of ionization fronts can be found in both the laboratory and astrophysics. They occur in the laboratory when a hot surface or a hot enclosed



**Fig. 7.36.** The Horsehead Nebula. Located in an intense UV environment, this is an example of a structure through which an ionization front has passed. Credit: N.A. Sharp/National Optical Astronomy Observatory/Association of Universities for Research in Astronomy/National Science Foundation

volume (a *hohlraum* as described in Chap. 8) irradiates a gas or a low-density foam material. They occur in astrophysics when bright stars emitting primarily in the UV irradiate molecular clouds containing mostly  $\text{H}_2$ . The Horsehead Nebula, shown in Fig. 7.36, is a structure that has developed, perhaps through hydrodynamic instabilities, in a molecular cloud through which an ionization front is passing.

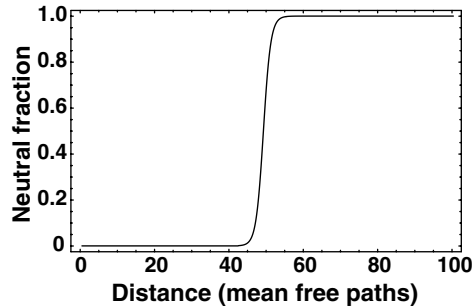
The speed of an ionization front depends on the photon flux that creates it. Suppose the flux of ionizing photons is  $F_\gamma$ , in number per unit area per unit time. This flux can be determined from the properties of the photon source including its geometry. Suppose in turn that the initial number density of atoms or ions to be ionized further is  $n_o [= \rho/(Am_p)]$  and that the local density of not-yet-ionized particles is  $n_i$ . The velocity of the front will from simple kinematics be the flux divided by the number density,  $v_f = F_o/n_o$  where the initial photon flux is  $F_o$ . It is straightforward to write equations for the evolution of the front, which are

$$\frac{dF_\gamma}{dx} = -F_\gamma n_i \sigma_o \quad \text{and} \quad \frac{dn_i}{dt} = -F_\gamma n_i \sigma_o. \quad (7.128)$$

These equations can be placed in dimensionless form by taking  $f = F_\gamma/F_o$ ,  $z = x/\lambda$ ,  $\eta = n_i/n_o$ , and  $y = v_f t/\lambda$ , where  $\lambda = 1/(n_o \sigma_o)$  is the mean-free path in the un-ionized medium. One can substitute for  $f$  to obtain the following integro-differential equation for  $\eta$ :

$$\frac{d\eta}{dy} = -\eta e^{-\int_0^z \eta(z') dz'}. \quad (7.129)$$

Figure 7.37 shows the resulting profile, from numerical integration of this equation, at a time of  $50 \lambda/v_f$ . The full scale in the figure is  $100 \lambda$ . One



**Fig. 7.37.** Structure of an ionization front. Neutral particle density  $n_i/n_o$  vs. distance in particle mean-free paths, in an ionization front that has evolved for a time of  $50\lambda/v_f$

sees that the ionization front has moved very nearly  $50\lambda$ , so it moves at the expected velocity. One also sees that the width of the front (from 10% to 90% ionization) is less than 5 mean-free paths.

Since the speed of an ionization front is determined by the photon flux, the front can be subsonic or supersonic relative to the sound speed in the upstream medium. It may also change from one of these to the other as the radiation source evolves in time. A supersonic front will encounter an unperturbed upstream medium. As subsonic front, in contrast, has the potential to produce very complicated behavior, for example, if it drives a shock wave that acts to reduce the opacity of the shocked material.

The heating and shock waves produced by a subsonic ionization front are of practical importance in the context of inertial fusion. The ionization process heats the ionized medium, as the electrons produced by ionization do not have zero energy. Such heating may be enhanced if there is some absorption, rather than zero as we assumed above, in the ionized material. In either case the pressure of the ionized material may be large enough to drive a shock wave into the upstream material. This might or might not significantly alter the opacity of the shocked material. It is possible for the ionization front and the shock wave to be coincident and supersonic, but the inertial fusion case is that the ionization front is subsonic in the postshock matter so the shock wave runs ahead of the ionization front. In the case of a low- $Z$  fuel pellet driven by soft x-rays for fusion, one ends up with a rarefaction wave at the surface of the pellet, an ionization front moving into shocked material at the sound speed of the heated pellet material, and a faster shock wave penetrating the pellet. This is discussed further in Sect. 8.2.3.

This concludes our discussion of the fundamental phenomena of radiation hydrodynamics. We have seen here that radiation hydrodynamic phenomena appear frequently in astrophysics and can readily be produced in the laboratory. We will see in the following that some radiation hydrodynamic effects are essential to the production of high-energy-density conditions and specifically to inertial confinement fusion.



## 9 Inertial Confinement Fusion

Chapter 8 represented a turning point in our discussion. The earlier chapters were focused on the physical fundamentals of high-energy-density physics. Chapter 8 showed how we could create such conditions, which in turn makes possible the application of high-energy-density systems to the pursuit of various goals. This and the next chapter are much more focused on these goals. The goal that has been and remains dominant in high-energy-density research is the development of *inertial confinement fusion*, or ICF. This is our topic in the present chapter.

Fusion is the joining of two nuclei. This leads to the production of various reaction products, which often carry significant kinetic energy. Whether or not nuclear fusion releases energy depends on the masses of the nuclei involved. If the total mass of the reaction products is less than the mass of the initial nuclei by an amount  $\Delta m$ , then the net energy released by the reaction is  $\Delta mc^2$ . It is by fusion that all the elements beyond the very lightest few were created. However, not all fusion events release energy. Figure 9.1 shows a plot of the nuclear binding energy versus atomic number. The *binding energy* is the energy one must invest to disassemble the nucleus into its component protons and neutrons. This is proportional to the mass difference between the mass of its constituent protons and neutrons and the mass of the nucleus. The most-stable nucleus is iron, with an atomic mass of 56. As a result, energy can be released by combining elements lighter than iron, or by

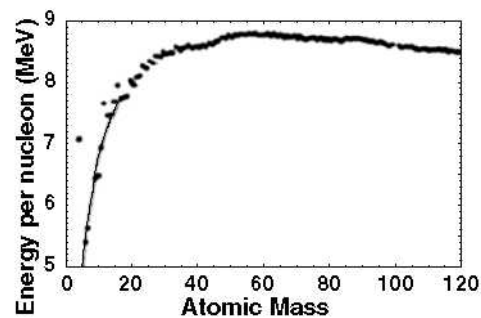


Fig. 9.1. Nuclear binding energy versus atomic number  $A$  up to  $A = 120$

dividing (by fission) elements heavier than iron. One can see that some light elements have relatively large binding energies – these are those with closed nuclear shells and correspond to elements that accumulate in stars.

Stars begin by assembling a very large mass of light elements. Through gravitational compression, their cores become dense and hot enough to initiate fusion burning. Through fusion, they begin to convert their light elements to heavier elements. Heavier elements require progressively higher temperatures to fuse, as the heavier nuclei have higher nuclear charge, so that it takes more energy to overcome the Coulomb repulsion. Low-mass stars like the sun accumulate He by burning H and eventually are able to ignite He, producing cores of C and O, but they cannot go further than this. High-mass stars (larger than about 8 solar masses) can create all the elements up to Fe, and accumulate significant amounts of Si in the process. The Fe proves to be the death of these stars, as it cannot burn, so it cannot sustain the pressure necessary to resist the gravitational contraction. The eventual collapse of the Fe core triggers some types of supernova explosions.

All this has much to do with elemental abundances in the universe. Elements up to iron can be created by stars during their lifetime, and the most abundant ones are those that represent stable endpoints during stellar evolution. The eventual stellar explosion creates an environment rich in neutrons and neutrinos, which rapidly process the material that exists, producing the elements heavier than Fe and altering the populations of the lighter elements. Arnett is a good first source for more details on this.

This context leads to natural questions. We can create conditions of high pressure and high temperature, if only briefly, using high-energy-density devices. Any concentrations of matter and energy we produce are confined inertially, not gravitationally. (That is to say, they blow apart in roughly one sonic transit time.) Even so, can we perhaps do this in a way that causes light elements to fuse, releasing energy? Can we perhaps release useful amounts of energy? Let us see. We will proceed from asking what conditions we have to end up with, moving to how we might get there and then to what might go wrong. Our approach here will be to use simple arguments to identify the important issues and resolve them. This will get us into the ballpark of actual ICF parameters. But real designs for producing ICF must consider every issue that can be identified, and not just the most important ones. Computer simulations are an important tool for including many of these details. A next level of detail may be found in Lindl and in Atzeni and Meyer ter Vehn.

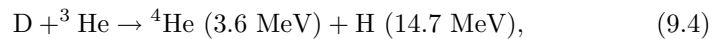
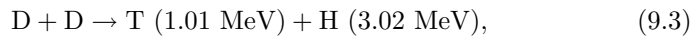
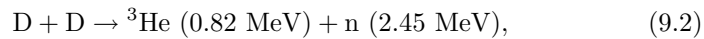
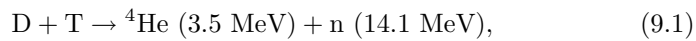
## 9.1 The Final State

To answer our question about the possibility of inertial fusion, we will proceed from the end toward the beginning. We start by asking what we might use for fuel. Then we will ask what physical conditions are required to make this fusion fuel burn and provide an energy yield. This will lead to the question

of how we can produce these conditions, and what the pitfalls might be in attempting to do so.

### 9.1.1 What Fuel, Under What Conditions?

A first question, determining much that follows, is what elements we might use for fuel. This would lead us to examine tables of nuclear reactions, from which we could find the following few that offer some potential for the easiest laboratory fusion systems:

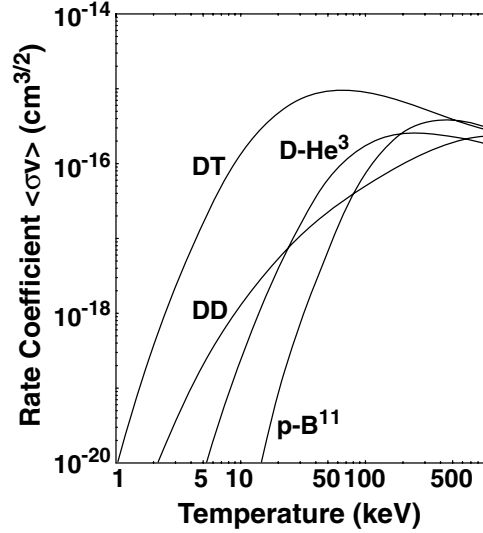


and



The first of these reactions (known as DT) is the focus of nearly all fusion research at this writing. Any plasma producing these reactions will produce the next three as well. We will see the advantage of this focus in a moment. The disadvantage, for applications such as the production of electricity, is that the energy emerges primarily as neutrons (designated  $n$  in the equations). One can only manipulate energy from neutrons by first converting the energy to heat, and heat cycles have limited efficiency. (The heat cycle efficiency is  $\sim 40\%$ , which applies after the conversion of neutron kinetic energy to heat.) This leads one to look toward *advanced fuels*, such as the reaction of  $p$  and  $\text{B}^{11}$ , that produce only charged-particle reaction products. In the longest run, these offer the potential to escape the need for heat cycles and to eliminate all the associated hardware from fusion power plants. In the shorter run, fusion systems that produce a lot of neutrons may prove more useful as breeders of fuel for power plants using nuclear fission. But we are not yet at the point of thinking about power production, so let us return to the question of whether one could do this at all.

Figure 9.2 shows the dependence on energy of the rate coefficients for these four fusion reactions, found by averaging the reaction cross section over Maxwellian distributions of interacting particles, just as we discussed in Chap. 2 for collisional processes. One sees that the rate coefficient for the DT reaction becomes large at temperatures far below those required for the other processes. A star may not care much about this. It can keep the material in place for a long time. But to attempt ICF we do care. We have to burn the fuel before it blows apart – getting the rate coefficient up near its maximum matters. One can see that DT is clearly the fuel of choice for initial attempts to achieve ICF.



**Fig. 9.2.** Rate coefficients for the DT, DD, D-He<sup>3</sup>, and p-B<sup>11</sup> reactions

Next suppose we have brought a clump of DT fuel into a final state, with conditions that encourage it to burn. How much of it burns? To answer this suppose that the density of deuterium nuclei in the clump is  $N_D$ , the density of tritium nuclei is  $N_T$ , and the density of pairs of reaction products is  $n$ . Also, ignore the DD reactions as we are seeking a simple estimate rather than a complete account. The rate equation describing the accumulation of reaction products is

$$\frac{dn}{dt} = N_D N_T \langle \sigma v \rangle_{DT}, \quad (9.6)$$

in which the rate coefficient for the reaction is  $\langle \sigma v \rangle_{DT}$ . Next suppose that  $N_D = N_T = 0.5N_o - n$ , and define the burn fraction  $\phi$ , given by  $2n/N_o$ . Then (9.6) becomes

$$\frac{d\phi}{dt} = \frac{N_o}{2} (1 - \phi^2) \langle \sigma v \rangle_{DT}, \quad (9.7)$$

Now if we make the approximation that  $\langle \sigma v \rangle_{DT}$  is constant as the fuel burns, and integrate over a time  $\tau$  during which the fuel burns, we find

$$\frac{\phi}{(1 - \phi)} = \frac{N_o \tau}{2} \langle \sigma v \rangle_{DT}. \quad (9.8)$$

Next we need to estimate how long the fuel burns. This time should be proportional to the fuel radius  $r$  and inversely proportional to the sound speed  $c_s$ , but is clearly less than the ratio  $r/c_s$ . We account for this by taking  $\tau = r/(3c_s)$ , where the factor of 3 would be only a guess without more-advanced knowledge. We take the temperature to be 30 keV in order to

evaluate the sound speed and rate coefficient. (This amounts to assuming that self-heating by the burning fuel will push the temperature to the value corresponding to the maximum of the rate coefficient.) We also convert  $N_o$  to obtain

$$\phi = \frac{\rho r}{\rho r + 6 \text{ g/cm}^2}. \quad (9.9)$$

Here we meet the quantity  $\rho r$  for the first time. This is the mass per unit area, defined as the *areal mass density*. We see that this quantity controls the burn fraction. The transmission of particles or photons through the fuel also depends on  $\rho r$ . Within the context of the approximations above, one can see that when  $\rho r$  is 3 g/cm<sup>2</sup> the burn fraction is 33%, while  $\rho r$  increases to 6 g/cm<sup>2</sup>, the burn fraction increases only to 50%. The returns are clearly diminishing, and the cost of  $\rho r$  is high, so let us assume that our final state before burning has  $\rho r = 3 \text{ g/cm}^2$ , producing a burn fraction of 33%.

### Homework 9.1

Plot the burn fraction versus  $\rho r$ . Discuss the impact of the assumptions made in deriving the burn fraction on this curve, and on the size of a system designed to produce a certain quantity of fusion energy.

The discussion just above applies most closely to uniform burning of an entire volume of fuel. We will see that actual fusion designs involve a *propagating burn*. A propagating burn is like a forest fire, in which the fuel begins burning at one location, after which the heating of nearby locations causes them to burn too.

We can pin down the final fuel conditions further by thinking about the total fusion energy released (the *yield*). The range of a 3.5 MeV  $\alpha$  particle (a <sup>4</sup>He nucleus) in DT is  $\rho r = 0.3 \text{ g/cm}^2$ , so the  $\alpha$  particles do not typically escape the fuel. Instead, they contribute to self-heating. This means that the energy release is the energy carried out of the fuel by the 14.1 MeV neutrons. One easily finds that the neutron yield,  $Y$ , is

$$Y = \phi \frac{m_f}{A m_p} \times 14.1 \text{ MeV} \approx 1.1 \times 10^{21} m_{\text{mg}} \text{ MeV} \approx 180 m_{\text{mg}} \text{ MJ}, \quad (9.10)$$

in which we have taken  $\phi = 0.33$  and the total mass of the fuel is  $m_f$ , also expressed in mg as  $m_{\text{mg}}$ . Since we are talking about the abrupt release of substantial energy, we need to place this in context. One kiloton of TNT is 4.2 GJ. The nuclear device exploded at Hiroshima in 1945 released about 10 kilotons of explosive energy. In a large laboratory device, we need to keep the yield small enough that it can be easily contained. For the present calculation, we will seek to produce  $\sim 0.1$  ton, or about 500 MJ, which will require 3 mg of fuel mass according to (9.10).

Thus, we have found that  $\rho r = 3 \text{ g/cm}^2$  and that the mass of the fuel is  $4\pi\rho r^3/3 = 3 \text{ mg}$ . We assume that the fuel volume is spherical, as this is

most efficient for both the assembly of the dense fuel and for its burning. Substituting for  $\rho r$ , we find  $4\pi r^2 = 0.003 \text{ cm}^2$ , from which we find  $r = 0.015 \text{ cm} = 150 \text{ }\mu\text{m}$ . This in turn implies that  $\rho = 190 \text{ g/cm}^3$ . Before we consider how to accomplish this, we would like to know that doing so would be worthwhile. This leads us to ask how much yield is needed.

### 9.1.2 Energy Gain: Is This Worth Doing?

If it took more energy to create this final state than fusion could release, then there would not be much point in pursuing fusion. The electrons are strongly Fermi degenerate in such highly compressed matter, unless it is quite hot. At  $190 \text{ g/cm}^3$ , and considering that the average  $A$  for DT is 2.5 while  $Z = 1$ , the electron density is  $n_e = 190Z/(Am_p) = 4.5 \times 10^{25} \text{ cm}^{-3}$ . In our discussion in Sect. 3.1 of Fermi degenerate systems, we saw that the Fermi energy is  $7.9(n_e/10^{23})^{2/3}$  with  $n_e$  in  $\text{cm}^{-3}$ . This is 464 eV for the compressed DT fuel. The corresponding pressure  $p$  is 13.5 Gbars. Perhaps more meaningful is the internal energy, which is  $3/2p$  (since  $\gamma = 5/3$ ). This is  $2.02 \times 10^{16} \text{ ergs/cm}^3 = 2.02 \times 10^9 \text{ J/cm}^3$ . In general, the specific internal energy of cold compressed DT fuel is  $10^7 \text{ J/g}$ . For comparison, the specific yield we expect, from (9.10), is  $1.8 \times 10^{11} \text{ J/g}$ . The implied yield of about 18,000 times the energy of compression certainly seems exciting, but unfortunately we have not considered the problem of igniting the fuel.

The simplest way to assure ignition would be to compress all the fuel so that its final state was hot enough to ignite. Suppose that the fuel at high temperatures behaves as an ideal gas with  $\gamma = 5/3$ . And suppose that we decide, from more detailed calculations, that a temperature of 5 keV will suffice for ignition. The specific energy of DT fuel at 5 keV temperature is  $6.4 \times 10^8 \text{ J/g}$ . We see that it would take 64 times as much energy to heat our fuel to ignition temperature as it would to compress it to the final density. This would reduce the ratio of neutron yield to internal energy to a bit less than 300. This still seems like a large number but in fact it is not, if one hopes to achieve a large *net gain* or to produce power by ICF. Here by net gain one refers to the ratio of neutron yield to the energy delivered to an ICF target. (This is the gain most commonly mentioned in books and articles on ICF.) To find the net gain, one must multiply the ratio of neutron yield to internal energy by the ratio of this internal energy to energy delivered to the target. This second ratio is of order 10%, depending on details. So a 1 MJ energy source might manage to create compressed fuel with of order 100 kJ of internal energy.

Continuing for a moment with the problem of power production, one must account for further inefficiencies. The energy delivered to the target is costly to produce; the efficiency of conversion of electricity to this energy may not reach 10%. This is why comparatively efficient energy sources such as heavy ion beams or KrF lasers are often discussed in the context of achieving significant net gain. The further conversion of neutron yield to electricity is

then less than 40% efficient, as mentioned above. Putting this all together, the ratio of neutron yield to compressed-fuel energy (i.e., the net gain) must exceed 250 to break even (i.e., to produce the amount of electricity used to create the fusion event). The net gain must be well above 2,500 for practical power production. Thus, ignition of the entire volume would have little value.

We seem to be in a quandary here, as we have to ignite the fuel but apparently cannot afford to. All the solutions to this quandary revolve around the range of the alpha particles in compressed DT fuel. We mentioned above that the alpha particles do not escape the fuel. In fact, their range is about  $0.3 \text{ g/cm}^2$ , or 10% of the  $\rho r$  of the entire compressed fuel. This corresponds to 0.1% of the fuel volume. If one could magically heat only such a volume to 5 keV, then it would begin to create fusion products, the alpha particles would heat the surrounding, cold fuel, and the fusion burn would propagate. The energy cost would still be  $6.4 \times 10^8 \text{ J/g}$  in the fuel that ignited, but averaged over the entire capsule the extra cost of ignition would be only  $6.4 \times 10^5 \text{ J/g}$ , which is much less than the cost of compression. Thus, some sort of *hot-spot ignition* is the key to obtaining an energy from fusion that is several thousand times the energy invested in the compressed and locally heated fuel. In consequence, the fuel will not burn throughout its entire volume all at once, but instead one will have a propagating burn.

Several approaches to producing such a hot spot have been proposed. The simplest and most thoroughly explored is to tailor an implosion so as to create the hot spot at the center of a fuel capsule. Other approaches are known as *fast ignition*. In all such approaches, some source of energy is released into the fuel after compression. The most studied option for doing this, usually called the *fast ignitor*, involves using a very-high-irradiance, short-pulse laser to create relativistic particles that penetrate the compressed matter and heat it.

### 9.1.3 Properties of Compressed DT Fuel

We have seen that the compressed DT fuel will have a pressure of at least 13 Gbars and a density of order 1,000 times the density of solid DT. Creating the necessary pressure costs money, and any increase in the required final pressure will increase the cost or decrease the performance of an inertial fusion system. For this reason we need to understand the relation of the pressure in DT fuel to the heating that may occur during compression. It is specifically helpful to understand the relation of pressure to entropy, because in practice compression by a factor of 1,000 must involve shock waves, and shock waves increase entropy (see Sect. 4.1.4). This leads us to explore further the fundamental properties of DT fuel. It is straightforward to consider the fuel as a collection of fundamental particles. We do this here, ignoring collective effects such as ionization and dissociation.

We will examine the properties of DT fuel with equal numbers of deuterons and tritons. Following Lindl, we consider the initial state of the fuel to be

solid DT at 11 °K and 0.25 g/cm<sup>3</sup>. The electrons and the tritons are fermions while the deuterons are bosons. Applying (3.13) to the initial state, we find that the Fermi energy for the electrons is 5.6 eV. This is more than three orders of magnitude above the initial temperature, with the implication that the electrons must be treated as fermions until conditions change greatly. In contrast, the initial Fermi energy of the tritons corresponds to a temperature of 7.4 °K. This implies that the tritons may be treated as a classical gas throughout the compression and heating, based on the discussion in Sect. 3.1.3.

As bosons, the behavior of the deuterons is not among the topics we discussed in Chap. 3. Upon referring to a statistical physics text such as Landau and Lifshitz, we find that the behavior of bosons varies across three temperature regimes. In the lowest temperature regime, particles accumulate in the lowest-energy state, which must be treated separately from the other states. The temperature,  $T_o$ , below which this occurs is

$$T_o = \frac{3.31}{g^{2/3}} \frac{h^2}{4\pi^2} \frac{n_D^{2/3}}{2m_p}, \quad (9.11)$$

in which  $h$  is the Planck constant,  $m_p$  is the proton mass,  $n_D$  is the number density of the deuterons, and  $g$  is their degeneracy (equal to 3, as they have spin 1). Evaluating  $T_o$  for the conditions given above, one finds that it is 3.7°K. The implication is that the deuterons in fusion fuel do not collect in the lowest energy quantum state but are instead distributed across many energy states. They are in the second temperature regime, in which the deuterons must be treated as bosons and an analysis involving integrals similar to those of Sect. 3.1.3 is valid. At some higher temperature, whose value depends on the accuracy one needs, the behavior of the deuterons becomes like that of a classical gas. Thus, to determine how to treat the deuterons for fusion fuel, we do need to evaluate their behavior as bosons.

The properties of bosons can be conveniently expressed in terms of integrals similar to those used for fermions. We define  $G_n(\phi) = \int_0^\infty x^n [\exp(x - \phi) - 1]^{-1} dx$ , in which  $\phi = \mu/(k_B T)$ . Then we have

$$n_D = \frac{12\sqrt{2}\pi}{h^3} (2m_p k_B T)^{3/2} G_{1/2}, \quad (9.12)$$

from which

$$\Theta = 1/(10^4 G_{1/2}), \quad (9.13)$$

where for convenience we again define  $\Theta = T/T_d$ , in which  $T_d$  is the degeneracy temperature of the electrons given by  $k_B T_d = \epsilon_F$ . Equation (9.13) defines the relation between the chemical potential (which is negative) and  $\Theta$ . The deuteron pressure,  $p_D$ , is given by

$$p_D = \frac{8\sqrt{2}\pi}{h^3} (2m_p)^{3/2} (k_B T)^{5/2} G_{3/2}, \quad (9.14)$$



while the specific entropy of the D in DT fuel,  $s_D$ , is given by

$$s_D = \frac{k_B}{5m_p} \left( \frac{5}{3} G_{3/2} - \phi \right). \quad (9.15)$$

Note that to obtain the specific entropy for the DT fuel we divide the entropy per particle by the average mass per D particle in the DT fuel ( $5m_p$ ). One can compare the results of these calculations with the classical pressure,

$$p_{Dcl} = 5\Theta/4, \quad (9.16)$$

and the classical entropy

$$\begin{aligned} s_{Dcl} &= \frac{k_B}{5m_p} \left[ \frac{5}{2} + \ln \left( \frac{5m_p}{\rho} \right) + \frac{3}{2} \ln(k_B T) + \frac{3}{2} \ln \left( \frac{2\pi 2m_p}{h^2} \right) \right] \\ &= 0.191 \times 10^8 \left[ 15.1 + \frac{3}{2} \ln(\Theta) \right]. \end{aligned} \quad (9.17)$$

One finds that the classical pressure and the boson pressure are identical to high accuracy for any  $\Theta$  above 0.001, while the classical entropy remains 5% to 20% below the boson entropy for all  $\Theta$  of interest here.

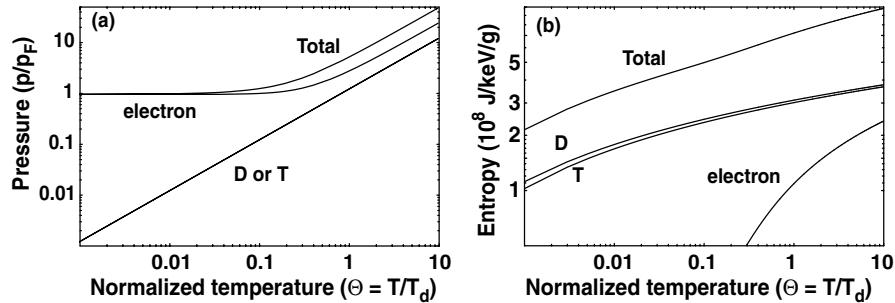
### Homework 9.2

Carry out the evaluation just described. For deuterium at a density of 0.1 g/cm<sup>3</sup>, plot the pressure as a function of temperature for deuterium treated as bosons and for deuterium treated as a classical gas. Discuss the comparison.

One can put this all together as follows. The total pressure of the DT fuel, normalized to the Fermi pressure of the electrons, is the sum of the electron pressure from Sect. 3.1.3 and the classical pressures of the deuterons and the tritons (each equal to  $5\Theta/4$ ). Figure 9.3a shows the resulting pressures. The total specific entropy is the sum of the specific entropy of the electrons, based on (3.26), the specific entropy of the deuterons, from (9.15), and the classical triton specific entropy, given by

$$\begin{aligned} s_{Tcl} &= \frac{k_B}{5m_p} \left[ \frac{5}{2} + \ln \left( \frac{5m_p}{\rho} \right) + \frac{3}{2} \ln(k_B T) + \frac{3}{2} \ln \left( \frac{2\pi 3m_p}{h^2} \right) \right] \\ &= 0.191 \times 10^8 \left[ 15.7 + \frac{3}{2} \ln(\Theta) \right]. \end{aligned} \quad (9.18)$$

Figure 9.3b shows the contributions to the entropy. The comparison of the two parts of Fig. 9.3 leads to the following conclusion. If one begins with cold, Fermi-degenerate DT fuel, one can heat this fuel until  $\Theta \sim 0.1$  before the pressure begins to increase. This is thanks to the degenerate electrons. Such heating corresponds to an increase in entropy. Equivalently, one could



**Fig. 9.3.** Origins of pressure and entropy in DT fuel. The pressure (a) is normalized to the Fermi pressure of the electrons,  $p_F$ . The specific entropy (b) is for DT fuel

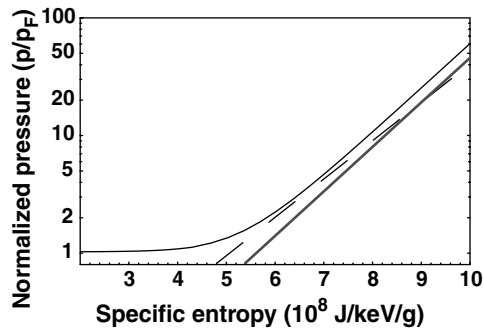
say that one can increase the entropy of the fuel, which is dominated by the ions, by some amount before the pressure begins to increase. This second point of view is useful if the entropy will be increased primarily by shocks.

By combining the results shown in the two parts of Fig. 9.3 one can obtain Fig. 9.4. The solid black curve in this figure shows the dependence of the normalized pressure on the entropy of the fuel. One sees that the pressure is constant up to some value of the entropy and increases exponentially with entropy at high entropy. It is a fortunate development for fusion that adding a certain amount of entropy imposes no cost.

The solid gray curve in Fig. 9.4 shows the classical result, whose derivation is left as a homework exercise. The dashed curve shows the equation given in Lindl, which is

$$p_{DT} = p_F \exp \left[ 0.75 \left( \frac{\Delta s}{10^8} - 4 \right) \right], \quad (9.19)$$

in which  $\Delta s$  is the difference in specific entropy from the initial state. We can see that this expression is a reasonable fit to the result of our calculation in



**Fig. 9.4.** The pressure increases only after enough entropy generation. The initial entropy of the fuel, in this model, is  $1.1 \times 10^8 \text{ J/keV/g}$

the important regime where the pressure is a few times the Fermi pressure. The entropy of the initial state in the present calculation is  $1.1 \times 10^8$  J/keV/g. Equation (9.19) describes the behavior of the pressure and the entropy, for densities above  $5 \text{ g/cm}^3$ , in QEOS or tabular EOS descriptions of DT that include atomic and molecular binding effects that we have ignored here. These effects do alter the equation of state at low densities. Because our calculation matches the results of more-sophisticated models, summarized in (9.19), one concludes that the behavior of highly compressed DT is dominated by the behavior of the individual particles (electrons, deuterons, and tritons). One also concludes that it is acceptable to increase the entropy during the creation of the final fuel state, but that ideally the increase ( $\Delta s$ ) should be kept no larger than  $4 \times 10^8$  J/keV/g. Above that level, the pressure required to obtain a desired state increases exponentially with increasing entropy.

### Homework 9.3

Derive the classical relation between entropy and pressure (normalized by the Fermi pressure of the electrons).

We have thus defined our task. Take 3 mg of DT fuel, compress it to a  $300 \mu\text{m}$  diameter sphere of density  $190 \text{ g/cm}^3$ , without adding too much entropy, and then start it burning. Table 9.1 summarizes the properties of this final state. This set of parameters poses two difficulties. First, the initial density of solid DT is  $0.25 \text{ g/cm}^3$ . Thus, to achieve fusion energy by ICF without blowing up the lab, we must compress DT fuel to  $\sim 1,000$  times liquid density. This evidently will take more than one simple shock. Second, the problem of igniting the fuel is nontrivial. We will take up these issues in turn.

**Table 9.1.** Parameters of a model high-gain ICF system without a hot spot

Parameter	Value
Fuel Mass	3 mg
Burn fraction	33%
Fuel $\rho r$	$3 \text{ g/cm}^2$
Final fuel density	$190 \text{ g/cc}$
Fuel radius	$150 \mu\text{m}$
Fermi energy	460 eV
Fuel pressure	13.5 Gbars
Maximum specific entropy	$4 \times 10^8 \text{ J/keV/g}$
Compressed fuel energy	30 kJ
Capsule kinetic energy	>60 kJ

## 9.2 Creating and Igniting the Final State

At this point we have some knowledge of the final fuel state, but without considering how we will create it, how we will ignite it, or how the final state might be modified so it can ignite. Here we discuss how one can create such a state, what is required to ignite it, and what is required for hot spot ignition or fast ignition.

### 9.2.1 Achieving a Highly Compressed State

We next face the problem of creating a state in which 3 mg of DT fuel has been compressed to a density of  $190 \text{ g/cm}^3$  with an energy of compression of 30 kJ and a final pressure of 13.5 Gbars, both of which require that the specific entropy remain below  $4 \times 10^8 \text{ J/keV/g}$ . One way to achieve this would be to take a sphere of solid DT and to compress it adiabatically to this final state. Unfortunately, there is no known source of compression that can reach 13.5 Gbars. So an alternative is needed. The idea that provides an alternative is that of a *capsule implosion*, in which a spherical shell of material is accelerated to a high velocity and is then decelerated to rest. We can develop this idea as follows. If we can accelerate our 3 mg of DT fuel to an *implosion velocity*  $v_{\text{imp}}$  of  $1.41 \times 10^7 \text{ cm/s}$  (140 km/s), the fuel will contain 30 kJ. If, without a significant loss of energy, we could decelerate this fuel to rest, using all of its energy for compression, then the fuel would reach a pressure of 13.5 Gbars. It is not quite this simple, however, since the deceleration cannot be accomplished without cost. But the basic notion of creating kinetic energy and then converting it to internal energy is the key here.

The geometry of choice for the final state, necessary to approach the burn fraction we calculated in Sect. 9.1, is a sphere. This allows one to use the matter at the core of the sphere to resist and eventually stop the implosion. This interior matter is necessarily a gas and is not Fermi-degenerate. The capsule may include this gas as part of its design. Even if no gas is included in the capsule, gas (actually plasma) will be produced when the shock wave that initiates the capsule implosion reaches the inner surface of the cold fuel, producing a rarefaction as discussed in Chap. 4. From the simplest point of view, the imploding cold, dense fuel expends kinetic energy doing  $pdV$  work on the gas, eventually slowing to a stop. At the same time, the gas does  $pdV$  work on the imploding fuel, compressing it further. In addition to this mechanical work, there are other losses, in particular by electron heat conduction or radiation from the gas into the cold fuel. Ignoring these losses and other considerations, but realizing that the work done by the gas and the fuel must be equal, would double the required energy. This would increase the required implosion velocity to 200 km/s.

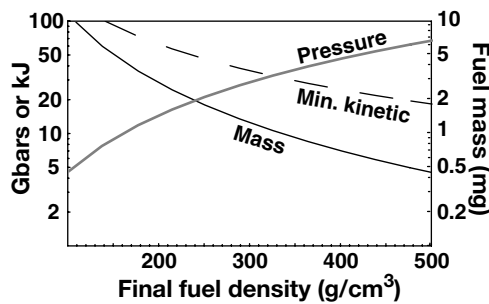
In more detail, the stopping of the capsule cannot involve only  $pdV$  work. If the capsule is initially empty, then the rarefaction of plasma from the

inner capsule surface produces a flow that stagnates at the center of the capsule. This creates a reflected shock that affects the incoming cold fuel. If the capsule contains gas within the shell, then the rarefaction from the inner capsule surface drives a shock through the gas until eventually the gas and then the rarefied capsule material stagnate. This again creates a reflected shock that affects the incoming cold fuel. However, in either case this shock must climb up the density gradient to the fuel, where it produces a weak shock that does not add much entropy. Shock reverberation ensues, accompanied by heating and compression of the inner gas. The shock heating, in addition to energy losses by electron heat conduction and radiation, may increase the implosion velocity and the energy of the imploding fuel that are required to achieve the desired final state. We will see that, for the case of ignition from a central hot spot, the velocity of the imploding capsule may exceed 300 km/s.

We can express these ideas in an equation. This gives us

$$E_{\text{kin, min}} = \frac{1}{2} m_f V_{\text{min}}^2 = 3p_f V_f, \tag{9.20}$$

in which  $E_{\text{kin, min}}$  is the minimum kinetic energy of the implosion,  $m_f$  is the fuel mass,  $V_{\text{min}}$  is the minimum implosion velocity,  $p_f$  is the final pressure of the fuel, and  $V_f$  is the final volume of the fuel. The rightmost term here is twice the final energy of the cold fuel, reflecting the  $pdV$  work done on the hot gas in slowing the fuel at the end of the implosion. Since  $m_f = \rho_f V_f$ , where  $\rho_f$  is the final fuel density, and  $p_f$  (Mbars) =  $2.13\rho_f^{5/3}$ , from Sect. 3.1 with  $\rho_f$  in  $\text{g/cm}^3$ , we can relate the kinetic energy, the final pressure, and the fuel mass for fixed  $\rho r$ . Figure 9.5 shows how these quantities vary with the final the fuel density, for  $\rho r = 3 \text{ g/cm}^2$ , for a spherical final fuel state. It is clearly an advantage to increase the final fuel density. However, larger implosion velocities are required for this and we will see that this is not trivial to achieve.



**Fig. 9.5.** Final fuel parameters versus fuel density. Variation of final electron pressure, minimum kinetic energy, and fuel mass with the final fuel density for a spherical fuel mass

**Homework 9.4**

Plot the minimum required implosion velocity, for  $\rho r = 3 \text{ g/cm}^2$ , versus final fuel density. Discuss the result.

At this point we have once again moved our task to an earlier one. The task now is to take 3 mg of cold fuel, in a spherical shell, and to accelerate this fuel to as much as 300 km/s without increasing the specific entropy above  $4 \times 10^8 \text{ J/keV/g}$ . When we do this in a high-energy-density system, the creation of the pressure will produce ablative acceleration. Thus, the inward motion will be a consequence of the exhaust of material outward, giving rise to the description of such an implosion as a spherical rocket. We need to provide some rocket fuel for the rocket, and the layer of material that is intended to be ablated is known as the *ablator*. In some circumstances, the ablator might also be composed of frozen DT, although for x-ray ablation it must contain Be or C (see Sect. 8.2.1). The ablator plus the fuel compose the entire mass of that is affected by the acceleration, so if the initial ablator mass is  $m_{\text{ab}}$  and all of this mass is ablated, then from (8.47) we will have a remaining mass fraction of

$$f_{\text{m}} = \frac{m_{\text{f}}}{m_{\text{f}} + m_{\text{ab}}} = e^{-v_{\text{imp}}/V_{\text{ex}}}, \quad (9.21)$$

in which  $V_{\text{ex}}$  is again the exhaust velocity. The exhaust velocity depends on the details of the ablation process, but is typically of order 1-4 times the desired implosion velocity. From the standpoint of the rocket efficiency (8.47), one benefits from smaller  $V_{\text{ex}}$  until one ablates more than 80% of the initial mass (see Fig. 8.11). If this were the only consideration, which it is not, then one would prefer to use x-ray ablation rather than laser ablation, because it produces a lower exhaust velocity. To be specific, suppose  $V_{\text{ex}} = 600 \text{ km/s}$ . Then the fuel is about 60% of the total mass, so the ideal rocket efficiency would be about 40% and the realistic rocket efficiency would be about 20% for x-ray irradiation and perhaps 10% for laser irradiation. Laser irradiation is less efficient because it involves an additional process: the transport of heat by electron heat conduction from below the critical density to the ablation density.

We now know that we will ablatively accelerate the fuel, using up about half the total mass. This must be accomplished before the capsule has moved too far, or it will begin to decelerate before we have injected the energy. We will suppose that we can accelerate it over a distance of  $r_{\text{s}}/2$ . Here  $r_{\text{s}}$  is the initial inner radius of the capsule. Assume this acceleration occurs at a constant pressure  $P_{\text{abl}}$  corresponding to a rate of mass removal  $\dot{m}$  of  $P_{\text{abl}}/V_{\text{ex}}$ , where  $P_{\text{abl}}$  is the ablation pressure. Also assume that the mass per unit area of the fuel plus the ablator is  $m_{\text{o}} = \rho_{\text{i}}\Delta r_{\text{i}}/f_{\text{m}}$ , where the initial compressed fuel density when the acceleration starts is  $\rho_{\text{i}}$  and the initial thickness of the compressed shell of fuel is  $\Delta r_{\text{i}}$ . (The thin-shell approximation for the geometry is valid for the compressed shell but not for the initial, uncompressed capsule.) Then we integrate the rocket equation to find

$$r_s/2 = V_{\text{ex}} \int_0^{t_a} \ln \left[ \frac{m_o}{m} \right] dt = -V_{\text{ex}} \int_0^{t_a} \ln \left[ 1 - \frac{\dot{m}t}{2\rho_i \Delta r_i} \right] dt, \quad (9.22)$$

where the integral proceeds until  $\dot{m}t_a = (1 - f_m)m_o$ . One can normalize and evaluate this integral to find the initial aspect ratio of the fuel capsule. This is

$$r_s/\Delta r_i = \frac{2\rho_i v_{\text{imp}}^2}{P_{\text{abl}}} \eta \left( \eta e^{1/\eta} - 1 - \eta \right) = \frac{2\rho_i v_{\text{imp}}^2}{P_{\text{abl}}} g(\eta), \quad (9.23)$$

where  $\eta = V_{\text{ex}}/v_{\text{imp}}$ . The ratio  $r_s/\Delta r_i$  represents an estimate of the *in-flight aspect ratio*. We will see that this quantity plays a role in determining the impact of the Rayleigh–Taylor instability. The implosion velocity is set by the fusion design and the exhaust velocity is determined by the approach to ablation, so that  $\eta$  might vary from about 1 to 5. Over this range,  $g(\eta)$  decreases gradually from 0.71 to 0.53. Equation (9.23) determines  $r_s$  because the fuel mass is known and  $m_f = \rho_i 4\pi r_s^2 \Delta r_i$ . One finds

$$r_s = \left( \frac{1}{2\pi} \frac{m_f v_{\text{imp}}^2}{P_{\text{abl}}} g(\eta) \right)^{1/3}. \quad (9.24)$$

As an example, we will evaluate this and other quantities using  $P_{\text{abl}} = 100$  Mbars,  $m_f = 3$  mg,  $v_{\text{imp}} = 300$  km/s, and  $g(\eta) = 0.6$ . From (9.24), this gives  $r_s = 1.37$  mm. Evaluating the acceleration time, which is the limit on the integral of (9.22), gives

$$t_a = \frac{r_s}{2v_{\text{imp}}} \left( \frac{\eta e^{1/\eta}}{g(\eta)} \right). \quad (9.25)$$

For the same parameters, this gives an acceleration time of 5 ns. Finally, one would like to know the initial aspect ratio,  $r_s/\Delta r_o$ , of the capsule, based on the initial shell thickness  $\Delta r_o$ . One can show that

$$\begin{aligned} \frac{r_s}{\Delta r_o} &= \left[ \left( 1 + \frac{3\rho_i \Delta r_i}{\rho_o r_s} \right)^{1/3} - 1 \right]^{-1} \\ &= \left[ \left( 1 + \frac{3}{2g(\eta)} \frac{P_{\text{abl}}}{\rho_o V_{\text{imp}}^2} \right)^{1/3} - 1 \right]^{-1}. \end{aligned} \quad (9.26)$$

Here the first equality is strictly geometric while the second uses (9.23). We find  $\Delta r_o = r_s/3.3$ , so  $\Delta r_o = 420$   $\mu\text{m}$ .

### Homework 9.5

Derive (9.26). Why do we need to express this result using a 1/3 power?

We will see when we discuss Rayleigh–Taylor instabilities that lower in-flight aspect ratios are strongly preferred, which pushes one toward high ablation pressures and shorter pulses. As a practical matter, the ablation pressure is limited to about 100 Mbars. Using direct laser irradiation, the limitation is due to laser–plasma instabilities, which become severe at irradiances much above  $10^{15}$  W/cm<sup>2</sup>. Using x-ray radiation, laser–plasma instabilities may contribute but in addition the other difficulties involved with hohlraums become severe at temperatures sufficient to produce higher ablation pressures.

### Homework 9.6

Suppose that one could apply a pressure  $p$  for a time  $t$ , using some energy source. With this source, we could accelerate some amount of mass per unit area,  $\rho_o \Delta r$ , to  $v_{\text{imp}} = 300$  km/s. Define a fusion capsule using the reflected pressure due to sunlight for 12 h as the pressure source. Approximate sunlight as light with a wavelength of 580 nm and an irradiance of 1 kW/m<sup>2</sup>. How long would such a capsule take to implode?

These results make our task seem straightforward. The recipe is apparently to apply 100 Mbars for a few ns to a capsule of the proper initial dimensions. Then one need only wait for the fuel to attain its final state. However, we still have the requirement to keep the specific entropy no larger than  $4 \times 10^8$  J/keV/g. Unfortunately, directly applying 100 Mbars to cold DT fuel would create too much entropy. The pressure of cryogenic, Fermi-degenerate DT fuel at 0.225 g/cm<sup>3</sup> is 300 kbars. A 100 Mbar shock at this density, for the appropriate value of  $\gamma$  (5/3), corresponds to a shock velocity of 260 km/s and heats the fuel to 326 eV. This is much hotter than the Fermi energy at this density, which is 5 eV. The increase in entropy produced by such a shock is  $9 \times 10^8$  J/keV/g. This is above the threshold value we need to stay below, so using a single 100 Mbar shock is not feasible.

We have already discussed the solution to the problem of applying a high pressure with minimum entropy increase in Sect. 4.1.4. To reach a 100 Mbar pressure without introducing too much entropy requires that we increase the pressure in stages, using a minimum of three shocks. Most current designs use four shocks. One could also attempt to achieve a shockless compression, using a precisely tailored pressure approach, but slight variations in pressure then have a tendency to launch unintended shocks, so the use of multiple, controlled shocks may be more reliable. This does introduce a new issue known as *shock timing*. If one is going to use a sequence of shocks to increase the ablation pressure to 100 Mbars, then one must time these shocks so that the later, stronger shocks do not overtake the earlier, weaker shocks too soon. Otherwise the resulting, stronger shock produced when two shocks coalesce would produce too much entropy. One desires to time the shocks so that they all coalesce just as they emerge from the inner surface of the capsule. Beyond that, one can try to tailor the coalescence to produce desirable properties in the matter ejected from the inner surface of the capsule.



### 9.2.2 Igniting the Fuel

Unfortunately assembling the high-density fuel is not sufficient to make it burn. One must also ignite some sort of hot spot. The challenge here is to keep a volume of fuel hot enough to sustain fusion despite the energy losses that are present. As a result, our problem boils down to a power balance equation. The net heating of the hot spot must be positive. We discuss this here. This section and the next two follow closely the treatment of Atzeni and Meyer ter Vehn. We use the subscript h for the properties of the hot spot. The equation for the internal energy of the hot spot,  $\epsilon_h$ , assumed to remain at constant density is

$$V_h \rho_h \frac{d\epsilon_h}{dt} = P_{\text{dep}} V_h - Q S_h - F_R S_h - p_h u_h S_h, \quad (9.27)$$

in which the hot-spot volume is  $V_h$ , the hot-spot surface area is  $S_h$ , and the hot spot density and pressure are  $\rho_h$  and  $p_h$ , respectively. This equation assumes that the kinetic energy of the hot spot material remains negligible. The rate at which fusion deposits energy in the hot spot is  $P_{\text{dep}}$ , while the radiation energy flux and thermal heat flux leaving the hot spot are  $F_R$  and  $Q$ , respectively. As the hot spot expands with radial velocity  $u_h$ , it does  $p dV$  work on the cold fuel, and this is represented by the final term. We now consider these terms in turn, assuming that in this dense matter  $T_e = T_i = T_h$ .

The fusion energy deposition is

$$\begin{aligned} P_{\text{dep}} &= \frac{dn}{dt} E_{\text{fus}} f_{\text{dep}} = N_d N_T \langle \sigma v \rangle_{\text{DT}} E_{\text{fus}} f_{\text{dep}} \\ &\approx C_\alpha \rho_h^2 T_h^2 \left( 1 - \frac{0.3}{4 \rho_h R_h} \right), \end{aligned} \quad (9.28)$$

where  $dn/dt$  is from (9.6). To obtain the final expression on the right for  $P_{\text{dep}}$ , we use here a standard fit to  $\langle \sigma v \rangle_{\text{DT}}$  for the range of 8–25 keV,

$$\langle \sigma v \rangle_{\text{DT}} \cong 1.1 \times 10^{-24} T_h^2 \text{ cm}^3/\text{s}, \quad (9.29)$$

with  $T_h$  in eV; one notes that in hot-spot ignition only the alpha particles provide significant heating, so  $E_{\text{fus}} = 3.5$  MeV; and one defines  $f_{\text{dep}}$  as the fraction of the alpha particle energy deposited in the hot spot. Equation (9.29) overestimates the rate coefficient for temperatures below 8 keV. For a first calculation, we could assume that we must make the hot spot large enough to absorb nearly all the alpha-particle energy, and take  $f_{\text{dep}} = 1$ . In more detail, energy deposition by charged particles in matter is complicated and is not represented by simple exponential functions. We will leave that as a detail, but do represent  $f_{\text{dep}}$  by the quantity in curved brackets, which accounts for the first departure from full absorption as  $\rho r$  decreases. Then with  $N_D = N_T$  we find  $C_\alpha = 3.8 \times 10^{11} \text{ ergs cm}^3 \text{ s}^{-1} \text{ g}^{-2} \text{ eV}^{-2}$ . Atzeni and Meyer ter Vehn provide more detail.

The heat flux  $Q$  is the Spitzer–Harm heat flux, which we derived in Chap. 8. One has

$$Q = -\kappa_{\text{th}} \nabla T_{\text{h}} \approx \kappa_{\text{th}} \frac{T_{\text{h}}}{R_{\text{h}}} \cong C_Q \frac{T_{\text{h}}^{7/2}}{R_{\text{h}}}. \quad (9.30)$$

Using (8.10) and (8.36), and including the factor of  $(1 + 3.3/Z)^{-1}$  due to electron–electron collisions, one can evaluate  $C_Q$ , finding, with  $T_{\text{h}}$  in eV and  $R_{\text{h}}$  in cm,  $C_Q = 5.9 \times 10^8 \text{ ergs eV}^{-7/2} \text{ cm}^{-1} \text{ s}^{-1}$ .

The radiation flux is the result of bremsstrahlung emission from the plasma in the hot spot. If the hot spot were optically thick, this would evidently be  $\sigma T_{\text{h}}^4$ . However, the hot spot is typically not optically thick. In the optically thin limit one has

$$F_{\text{R}} \cong \kappa_{\text{b}} R_{\text{h}} \sigma T_{\text{e}}^4 = C_{\text{b}} \rho_{\text{h}}^2 \sqrt{T_{\text{h}}} R_{\text{h}}, \quad (9.31)$$

in which  $\kappa_{\text{b}}$  is the spectrally averaged absorption coefficient [i.e., the Planck mean opacity of (6.39)].

Properly, one would determine the characteristic distance by integrating over the solid angle, accounting for the path length through the source volume. Here we assume that this gives a distance close to  $R_{\text{h}}$ . One finds  $C_{\text{b}} = 3.2 \times 10^{21} \text{ ergs cm}^3 \text{ g}^{-2} \text{ s}^{-1} \text{ eV}^{-1/2}$ .

### Homework 9.7

Derive the spectrally averaged absorption coefficient for bremsstrahlung in DT. Check your value against the value found in the NRL plasma formulary.

### Homework 9.8

Evaluate the appropriate integral of the radiative transfer equation over solid angle to obtain  $F_{\text{R}}$  from a spherical volume of DT. Find the value of the characteristic distance. Compare your result to the result in (9.31), which assumes that the integral over solid angle of the distance across the fuel gives  $\pi R_{\text{h}}$ . Extra credit: generalize this calculation to include arbitrary optical depth and discuss the results.

The approach to ignition determines whether and how much the hot spot expands. On the one hand, the implosion might be designed to produce a central hot spot. In this case the cold fuel and the hot spot are all at the same pressure at the moment of stagnation. This is described as an *isobaric* configuration. In this case,  $u_{\text{h}}$  is initially zero in (9.27). The hot spot will expand as it heats up, but because the rate of fusion increases strongly with temperature it is the initial heating that matters. So we can use (9.27) with  $u_{\text{h}} = 0$  to find an ignition threshold for an isobaric configuration. On the other hand, the implosion might be designed to assemble all the fuel, at

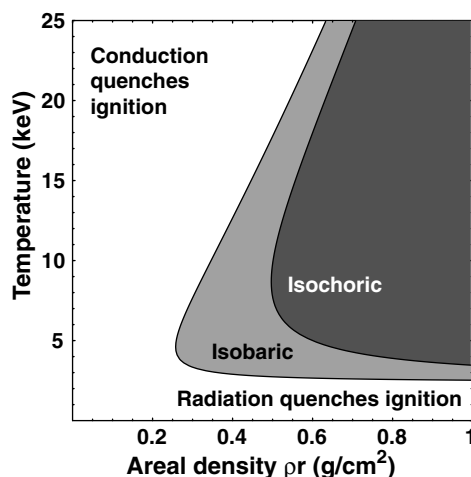
the end of which a hot spot will be created by some other means. In this case the cold fuel and the hot spot will initially have the same density. Such a configuration is described as *isochoric*. The hot spot will expand as it heats, and it is a reasonable estimate to take  $u_h$  to be the velocity of the fluid behind a strong shock entering the cold fuel. Then from (4.23) one has  $p_h = (\gamma - 1)\rho_s u_h^2/2$ , which determines  $u_h$ . Here  $\rho_s$  is the density of the shocked, cold fuel  $\rho_s = \rho_c(\gamma + 1)/(\gamma - 1)$  with  $\rho_c$  being the cold-fuel density. In addition,  $p_h = \rho_h T_h(Z + 1)k_B/(Am_p)$ . In this case with  $\rho_h = \rho_c$  one has

$$p_h u_h = C_h \rho_h T_h^{3/2}, \quad (9.32)$$

in which  $C_h = 5.8 \times 10^{17}$  ergs cm g<sup>-1</sup> eV<sup>-3/2</sup> for DT.

Returning now to (9.27), one can show that none of the four terms is always small throughout the regime of interest. As a result, the power of  $T_h$  of a given term determines where it will have its impact. One sees that the bremsstrahlung cooling term scales as the lowest power of  $T_h$ , so this will dominate at low temperatures. The  $pdV$  work term is next, scaling as  $T_h^{3/2}$ , so when present this term will increase the minimum temperature needed for ignition. Next is the fusion energy production term, scaling as  $T_h^2$ , so that eventually fusion energy production can overcome these first two losses. However, lurking at high temperatures is the heat-conduction term, scaling as  $T_h^{7/2}$ . At high enough temperatures, heat conduction will quench the ignition.

The ignition threshold occurs when the right-hand side of (9.27) equals zero. Figure 9.6 shows this condition for isobaric and isochoric configurations. The qualitative shape of the boundaries shown is correct. However, the lower boundaries of the ignition regimes are too low in this figure because



**Fig. 9.6.** Thresholds for ignition. Hot spots in the gray area will produce net heating and will ignite

(9.29) overestimates  $\langle\sigma v\rangle_{DT}$  at these temperatures, the actual lower boundaries straddle 5 keV. For the isobaric case, one indeed finds that  $\rho r \sim 0.3$  g/cm<sup>2</sup> and  $T_h \sim 5$  keV is the minimum ignition condition. For the isochoric case, relevant to fast ignition, one sees that one will need twice this  $\rho r$  and a bit higher temperature. This may change the optimum properties of the cold, dense, fuel, because doubling the  $\rho r$  of the hot spot at fixed density requires 8 times the energy invested in the hot spot. Depending on the cost of this energy, it might or might not make more sense to compress the cold fuel further.

It turns out that capsules that do not satisfy the threshold condition may ignite, if they are at temperatures where heat conduction quenches ignition initially. In this case the hot spot may cool, heating a surrounding region, and in effect creating a modified hot spot with smaller temperature and larger  $\rho r$ . This modified hot spot may then ignite. In addition, the fact that ignition occurs does not guarantee in principle that the resulting burn will propagate. If the heated region expanded too quickly relative to fusion energy production, then expansion cooling could quench the ignition. However, for the regime relevant to ICF the parameters work out favorably, and ignited capsules typically continue to burn. These last two effects are discussed further by Atzeni and Meyer ter Vehn. They can be summarized in as a condition for successful ignition and burn, given by

$$\rho_h R_h T_h > 6\sqrt{\rho_h/\rho_c} \text{ g cm}^{-2} \text{ KeV}, \quad (9.33)$$

in which  $\rho_c$  is again the density of the compressed, cold fuel. (A typical value of  $\rho_h/\rho_c$  for an isobaric case is 1/16). This condition is the analog for ICF of the well-known Lawson criterion for MFE, expressed in that case as a minimum value of density times confinement time.

### Homework 9.9

The Lawson criterion is generally written as  $n\tau > 10^{14}$  s/cm<sup>3</sup>, with density  $n$  and confinement time  $\tau$ . Find a way to relate this to (9.15) and comment on the comparison.

### 9.2.3 Igniting from a Central Hot Spot

It seems natural to try to get some benefit from the work done stopping the imploding capsule, by making the central gas become the hot spot that initiates the fusion burn. This places a constraint on the final  $\rho r$  of the hot spot, which must be 0.3 g/cm<sup>2</sup> to localize the alpha particles. It also places a constraint on the final temperature of the hot spot, which needs to be above 5 keV. If this were too low, the fuel in the hot spot would not ignite, but if it were too high the hot-spot fuel would begin burning too soon. The hot-spot

pressure must equal that in the fuel, and knowing pressure and temperature allows one to specify  $\rho_h$  from  $p_h = \rho_h T_h (Z + 1) k_B / (A m_p)$ . Then from  $\rho_h R_h$  one gets  $R_h$ .

If the pressure in the fuel were the minimum value of 13.5 Gbars derived above, this would imply  $\rho_h = 3.5 \text{ g/cm}^3$  and  $R_h = 850 \text{ }\mu\text{m}$ . Unfortunately this is too large. To work with this hot spot one would have to very greatly increase the fuel mass in order to obtain  $\rho r = 3 \text{ g/cm}^2$ . Thus, we cannot use a central hot spot with our initial definition of the compressed fuel. The only reasonable alternative is to increase the pressure in the fuel so as to decrease the radius of the hot spot. One can define the problem as follows. One can find  $R_h$  from  $p_h$  as just described. One also can find  $\Delta r_c$ , the cold-fuel shell thickness, from the cold, Fermi-degenerate fuel pressure,  $p_c = p_h = 2.1 \rho_c^{5/3}$ , knowing  $\rho_c \Delta r_c = 3 \text{ g/cm}^2$ . Then the fuel mass is found, from geometry, to be

$$m_f = \rho_c \frac{4\pi}{3} ((R_h + \Delta r_c)^3 - R_h^3), \quad (9.34)$$

in which all the variables on the right are now known functions of pressure. From this, for our desired fuel mass of 3 mg, one finds the parameters shown in Table 9.2

**Table 9.2.** Parameters of a model high-gain ICF system with a hot spot

Parameter	Value
Fuel Mass	3 mg
Burn fraction	33%
Fuel $\rho r$	3 g/cm <sup>2</sup>
Final fuel density	970 g/cc
Fuel outer radius	88 $\mu\text{m}$
Hot spot radius	57 $\mu\text{m}$
Hot spot temperature	5 keV
Fermi energy	1390 eV
Fuel pressure	200 Gbars
Max. cold-fuel specific entropy	$4 \times 10^8 \text{ J/keV/g}$
Compressed fuel energy	93 kJ
Capsule kinetic energy	>190 kJ
Implosion velocity	>350 km/s

The upshot of all this is that the cold, dense fuel must be accelerated to well above 300 km/s ( $3 \times 10^7 \text{ cm/s}$ ) in order to provide the energy necessary to create ignition from a central hot spot. The energy of the compressed fuel increases by a factor of 3 from  $\sim 30 \text{ kJ}$  to  $>90 \text{ kJ}$ . If the efficiencies do not change as the mass increases, one will need an energy source that is three times larger in order to create a fusion burn from a central hot spot than one

needs to compress cold fuel that one intends to ignite in some other way. It is because of this that fast-ignition schemes are potentially attractive.

### Homework 9.10

One choice in a central hot spot design is how much to increase the pressure above the minimum value of 13.5 Gbars. Increasing the pressure decreases the size of the hot spot but increases the energy required to create this pressure. Keeping the constraints on density and  $\rho r$  found above, consider the effects of scaling the pressure in the hot spot.

During the implosion, the hot spot obtains energy from  $p dV$  work and eventually from fusion burning, while losing energy through electron heat conduction and radiation. One can analyze the energy balance during the implosion using an energy balance equation similar to (9.27), except that  $\rho_h$  is no longer constant. Just as in the case of ignition, relatively cool hot spots lose more energy to radiation while relatively warm hot spots lose more energy to heat conduction. There is an optimum path in density and temperature to minimize the energy cost of assembling the hot spot. The further the hot spot is from that path during the implosion, the more energy will be required. This can affect the details of optimizing the shock timing and initial gas fill of the capsule.

The range of design options for ignition from a central hot spot is rather limited. One can, for example, vary the total fuel mass and then determine the implications for the final fuel pressure, final cold-fuel density, implosion velocity, and so on. Even so, when one considers the impact of the numerous possible approaches to creating that final implosion velocity, and the need to avoid the pitfalls discussed in Sect. 9.3, a challenging optimization problem remains. In addition, new inventions and discoveries with regard to the details offer the potential for improvements to the optimized design.

#### 9.2.4 Fast Ignition

In contrast to the case described in the previous paragraph, fast ignition in general is a much more complicated problem. One now has the potential to control the hot spot, so that in principle one can optimize the system across variations in heating energy, power, and radius (of the cylindrical zone that will be heated). Beyond this, there are numerous ways one might do this heating. One might use a beam of electrons, or ions, or photons. Or one might use a slug of dense matter to shock-heat the dense fuel. It has even been suggested (by Jim Hammer) that one might use a bubble of magnetic field. Here we first discuss the hot-spot properties required for ignition, and then discuss some of the options for producing these properties. The first published discussion of fast ignition was that of Tabak et al. Here we draw from work by Atzeni.

Since fast ignition begins from a nearly isochoric state, we draw from the analysis that produced Fig. 9.6, but using more-precise numbers from more detailed studies. The minimum value of  $\rho R_h$  required for ignition is  $0.5 \text{ g/cm}^2$ , at a temperature of 12 keV. If we consider this to be the optimum hot spot, we can estimate the amount of energy (deposited in the hot spot) that is required for ignition,  $E_{\text{ig}}$ , assuming that the hot spot is a cylinder of aspect ratio unity, as  $E_{\text{ig}} = \pi R_h^3 \rho (1 + Z) k_B T_h / (A m_p)$ , which is  $36(100 \text{ g/cm}^3 / \rho)^2 \text{ kJ}$  for this case. This seems quite hopeful, since achieving  $\rho$  above  $200 \text{ g/cm}^3$  seems plausible and since our simple scaling calculation indicated that the cost in fuel energy of producing a central hot spot was above 60 kJ. However, energy losses due to radiation, heat conduction, and hydrodynamic motion act to increase  $E_{\text{ig}}$  substantially.

In addition, the assumed parameters are not necessarily the optimum hot spot. The required temperature decreases as  $\rho r_h$  increases, reaching 4 keV at  $\rho r_h = 1.2 \text{ g/cm}^2$ . If, for example, one can readily and efficiently produce a beam whose radius is smaller than  $(0.5/\rho) \text{ cm}$  and which is whose absorption depth at the desired final density is roughly equal to its radius, then one would want to produce the high-temperature hot spot with the minimum  $\rho r$ . In contrast, if the beam one could efficiently produce had an absorption depth above  $1 \text{ cm}^2/\text{g}$ , then the optimum design would heat a larger-radius hot spot to a lower temperature. To make matters even more complex, the duration of the heating beam may be constrained or may be variable, and the beam may be limited in its maximum available power. Moreover we have yet to mention the question of how efficiently the heating energy can be produced and deposited, yet this is very likely the key technical issue that will determine the viability of fast ignition for fusion.

One approach to addressing this complexity is to separate the problem into components. Atzeni did so, asking what the conditions for ignition are for a set of rather general assumptions. He assumed that a beam of radius  $r_b$ , power  $W_b$ , and irradiance  $I_b$ , related by  $W_b = I_b \pi r_b^2$ , and with an absorption depth  $R_b$ , irradiates a constant-density fuel. He then used two-dimensional simulations to assess the parameters required for ignition. For  $0.15 \leq R_b \leq 1.2 \text{ g/cm}^2$ , ignition required

$$E_{\text{ig}} = 140 \left( \frac{\rho}{100 \text{ g/cm}^3} \right)^{-1.85} \text{ kJ}, \quad (9.35)$$

$$W_b = 2.6 \times 10^{15} \left( \frac{\rho}{100 \text{ g/cm}^3} \right)^{-1} \text{ W}, \quad (9.36)$$

and

$$I_b = 2.4 \times 10^{19} \left( \frac{\rho}{100 \text{ g/cm}^3} \right)^{0.95} \text{ W/cm}^2. \quad (9.37)$$

The second and third of these relations imply a beam radius of  $60 \mu\text{m} \times (100 \text{ g/cm}^3 / \rho)^{0.975}$ , which may or may not prove feasible in practice. The broad range of absorption depth over which these parameters apply reflects the rough balance between the fact that the overheated hot spots produced when the absorption depth is small ignite only after producing a larger volume at a lower temperature by heat conduction and that increased heating energy is required when the heating is dispersed because of large absorption depth. The net impact of this more-realistic analysis is to increase the optimum fuel density for fast ignition, perhaps to of order  $300 \text{ g/cm}^3$ , so that the deposited energy must be  $<20 \text{ kJ}$ . At the present level of the discussion this is a guess, since the important question in the end is how much energy must be expended to produce a given amount of deposited energy. The smaller the efficiency of energy deposition, the larger one will need to make the hot spot density. Then the question will be whether the beam radius can be as small as is required. One can see that fast ignition is a challenging problem involving the interplay of numerous constraints and of both physics and technology.

Now let us consider some possible methods for delivering the required energy to the dense fuel, at the required very large irradiance. A simple approach is to direct a sufficiently intense laser beam at the compressed fuel and hope that this leads to enough energy coupling. The most likely source of such coupling would be the generation of relativistic electrons by the intense laser beam. Indeed, beams of relativistic electrons are generated by intense, ultrafast lasers (see Chap. 11). Detailed questions follow. Can one generate enough electrons? Can one do so at useful energies? Can one do so close enough to the dense fuel that they couple efficiently? The dense fuel is surrounded by a formidable quantity of blown-off plasma. This leads one to consider various options. One might use a preliminary laser pulse to drill a hole in this plasma before one introduces the intense laser pulse that does the heating. Alternatively, one might implode a capsule that includes a high- $Z$  cone, whose purpose is to provide a region free of such blow-off plasma through which one can introduce the heating beam. Even so, electrons tend to have large absorption depths (though not as large as those of photons at relevant energies). Beams of electrons are also subject to disruption by filamentation or other instabilities. This leads one to consider using heavier particles.

Broadly speaking, the heavier particles might be protons, light ions, or heavy ions, and one might try to accelerate relatively few particles to higher energy or relatively more particles to lower energy. In the high-energy limit, beams of protons have also been observed in experiments with ultrafast lasers (see Chap. 11). If one can put enough energy into these beams, and if one can focus them, this might provide an alternative to the use of electrons. In the low-energy limit, one could try to drive a slug of solid material toward the target with enough energy to cause ignition by shock heating. If protons are still not absorbed readily enough, one could work on devising schemes to



deliver a sufficient irradiance of light ions or heavy ions to the target. At this writing, it is clear that there are many options but not yet so clear which may prevail.

Finally, we should not leave the topic of fast ignition without mentioning one of its major qualitative advantages. We address below the need for ICF implosions to be adequately spherical, both with regard to the symmetry of the ablation pressure and to the impact of the Rayleigh–Taylor instability. When using ignition from a central hot spot, small departures from a spherical implosion can permit the hot spot to be too cool or too convoluted to ignite. In contrast, for fast ignition one typically does not care what the shape of the gas within the imploded fuel capsule may be. Nor does one necessarily care if the implosion is asymmetric to some degree. So long as one can deliver the energy where it is needed to ignite some of the fuel, the fusion burn should be able to proceed.

### 9.3 Pitfalls and Problems

A pressure of 100 Mbars is fairly easy to produce. One can see from (8.42) that this requires a laser irradiance of  $1.6 \times 10^{15}$  W/cm<sup>2</sup> of 0.35  $\mu$ m light. The corresponding laser energy during the acceleration, for a capsule of 3.3 mm radius, irradiated for 2 ns, would be about 500 kJ. The laser energy required during the formation and propagation of the shocks would increase the total energy by some factor. Alternatively, one would need an x-ray temperature of 220 eV in a hohlraum to create a pressure of 100 Mbars. This again does not seem very difficult (see Chap. 8). Indeed, achieving such a pressure is easy enough that in the absence of limitations one would seek to use a larger pressure. However, there are three major problems that make ICF a challenge. We discuss these here.

#### 9.3.1 Rayleigh Taylor

An ICF implosion is Rayleigh–Taylor (RT) unstable during most of its development. Early on, during what is often called the acceleration phase, the low-density, hot ablated plasma is at a higher pressure than the cooler, higher-density layer being accelerated. This creates the condition of opposed density and pressure gradients that excites the RT instability (see Chap. 5). One could say that the low-density plasma is pushing on the higher-density plasma. The acceleration phase ends but before long the deceleration phase begins, when the low-density gas within the capsule pushes against and decelerates the denser, incoming fuel layer. Here again the density gradient and pressure gradient are opposed, so one has an RT unstable system. This necessitates understanding what limits RT may place on ICF implosions.

The number of e-foldings of RT growth,  $\gamma_{RT}t$ , is straightforward to estimate, assuming that the growth rate is the value for an abrupt, embedded

interface, which is  $\gamma_{\text{RT}} = \sqrt{A_n kg}$ . The density changes are large so we take  $A_n \sim 1$ . The most-damaging wavelength is related to the thickness of the capsule during the RT growth, which we will designate  $\Delta R$ . Wavelengths that are short compared to  $\Delta R$  will grow and saturate without creating large perturbations in the structure. Wavelengths that are long compared to the capsule thickness will distort the capsule and may decrease the compactness of the implosion, but they have less impact and a slower growth rate than wavelengths of order the capsule thickness, which can break up the capsule if they grow large enough. So we take  $k = 2\pi/\Delta R$ . If we approximate the acceleration and deceleration as constant, then we have  $R/2 = (1/2)g(t/2)^2$ , from which  $t\sqrt{g} = 2\sqrt{R}$ . Altogether, this implies  $\gamma_{\text{RT}}t = \sqrt{8\pi R/\Delta R}$ . The quantity  $R/\Delta R$  is known as *in-flight-aspect ratio* or IFAR. Equation (9.23) provides an initial estimate of this. The four or so shocks used to increase the pressure to 100 Mbars increase the density of the shell of order 100 times, from which our initial estimate of the IFAR would be  $\sim 200$ . However, this ignores the decompression that occurs after the shocks coalesce. Taking the IFAR to be 100, we would expect about 50 e-foldings of RT growth from this calculation. If the noise at such wavelengths corresponded to atomic displacements ( $\sim 1\text{\AA}$ ), and there were no nonlinear saturation and no spherical effects, the implied amplitude would be 1 km. The acceptable growth at such wavelengths is of order several, and certainly not more than 10, e-foldings. One concludes that ICF is not feasible in the presence of RT growth at such a rate.

Fortunately, the RT growth rate is actually smaller than  $\sqrt{A_n kg}$ , because the interfaces where the growth occurs are not sharp but rather have some scale length  $L$ . The growth is reduced much further during the acceleration phase because ablation carries away the material in which the modulations are growing. Suppose the ablation is carrying away material with some characteristic ablation velocity,  $V_A$  (this is not the exhaust velocity, but rather the velocity at which material flows away from the dense interface). Then recall that RT modes are surface waves with an exponential penetration depth of  $1/k$ . Given that the growth rate without ablation,  $\gamma_{\text{RT}}$ , sets a timescale of  $1/\gamma_{\text{RT}}$ , we would expect to see ablation quench the RT growth when  $1/k \sim V_A/\gamma_{\text{RT}}$  or  $kV_A \sim \gamma_{\text{RT}}$ . This is indeed what is seen, within small numerical factors, in simulations and in detailed analytic theory. A standard relation expressing the net growth is

$$\gamma_A = \sqrt{\frac{kg}{1+kL}} - \beta kV_A. \quad (9.38)$$

Here the coefficient  $\beta$  is  $\sim 1$  for x-ray ablation and  $\sim 3$  for laser ablation. The ablation velocity  $V_A$  is the velocity at which material leaves the RT-unstable region and equals the mass ablation rate divided by the density in the ablation layer,  $\dot{m}/\rho_2$ , where  $\rho_2$  is the density of the shocked ablator. The ablation velocity can be estimated from the discussion in Chap. 8. An

order-of-magnitude value of the product  $\beta V_A$  is  $5 \times 10^5$  cm/s, while the order-of-magnitude of  $g$  is  $3 \times 10^7$  cm/s per ns, which is  $3 \times 10^{16}$  cm<sup>2</sup>/s. For a steep interface, the maximum unstable wavenumber is  $k = g/(\beta V_A)^2 = 10^5$  cm<sup>-1</sup>, corresponding to a wavelength of about 1  $\mu\text{m}$ . Since the fuel shell is only a few  $\mu\text{m}$  thick during the acceleration phase (after compression by the sequence of shocks), the wavelengths whose thickness is of order the shell thickness are strongly stabilized by ablation.

It should be clear that a careful design must consider all possible RT modes in order to assure control of RT during the ablation phase. If the stabilizing effect of ablation alone is not enough, one can consider trying to increase the scalelength  $L$  at the ablation surface. Increasing  $L$  to a few  $\mu\text{m}$  can have an important effect. One way to try to increase  $L$  is to design the outer surfaces in the target so that they produce preheat that penetrates the ablator but is not energetic enough to reach and heat the fuel layer. This can be attempted in principle using either electrons or x-rays. Design work at this writing has focused on using x-rays. An alternative approach to increasing  $L$  is to launch the first shock into the fuel by using a brief impulsive load that rapidly evolves into a blast wave. The blast wave will decay as it moves into the capsule, so that the amount of entropy it produces will be larger in the outer layers of the ablator and smaller by the time it reaches the fuel. Either of these approaches, or perhaps another one, could be very helpful in the context of fusion by direct laser ablation.

During the deceleration phase there is little ablation, so only  $L$  can act to reduce the RT growth. Fortunately,  $L$  tends to be large enough to reduce the growth. This happens because the interior of the capsule has been heated by the coalescing shock waves and is much hotter than the cold fuel, in response to which the electrons transport heat into the inner layer of cold fuel, which in turn expands and lengthens the scale length at the inner fuel boundary. A typical estimate, from Lindl, is that at the inner fuel boundary  $L$  is roughly 15% of the final fuel radius. This has the consequence that the RT growth during deceleration is limited to about three e-foldings. The initial amplitudes that grow by this amount are determined both by the roughness of the inner surface of the fuel and by the perturbation of the inner surface due to the RT growth during ablation. This coupling of the outer surface to the inner surface is known as *feedthrough*.

The net result is that it appears that RT can be limited to a low enough level that fusion can succeed, but that doing so places difficult constraints on the initial roughness of the target surfaces and on the smoothness of the ablation pressure. As was mentioned in Chap. 8, the drive to invent ways to smooth the irradiation of surfaces irradiated by lasers came from ICF. Specifically, one needed to reduce the seeding of RT instabilities due to structure in the laser ablation. The precise constraints can be estimated now, but there is a potential to make them less severe through improvements in design of targets and in understanding of RT. The RT instability evidently

is present in any approach to fusion, whether initiated with lasers, with a z-pinch, or with sunlight.

### Homework 9.11

Evaluate the amount of RT growth for the sunlight-driven fusion system of the problem 9.6.

### 9.3.2 Symmetry

In any analysis of structure in spherical coordinates, it is natural to describe the structure in terms of spherical harmonics. These harmonics form a complete basis set that describes a system by breaking the structure into modes that correspond to having an integer number of wavelengths over  $360^\circ$  in azimuthal angle or  $90^\circ$  in polar angle. The RT modes that grow to the largest amplitude have mode numbers of several tens. (Without any stabilization, the most-unstable mode number would be much larger.) The low mode numbers, such as 2 or 4, correspond to variations in the structure that can be produced if the ablation pressure is not uniform over larger distances. For example, if two ends of a target (the poles) are driven more strongly than the middle of the target (the equator), then the imploded target will be flattened like a pancake. Such an imploded target is indeed described as pancake-like, or may be said to have pancaked. In contrast, if the equator is driven more strongly than the poles, the imploded target will be a long thin tube, for which the common metaphor is a sausage. In either case, the  $\rho r$  will be smaller than intended over a significant range of solid angle, and one will obtain less burning than one had hoped. One could say that ICF, in order to succeed, must not feed one breakfast.

A simple estimate of the required uniformity is straightforward to make. The fuel radius decreases by a factor of order 20 as the fuel implodes, from an initial radius  $R_i$  to a final radius  $R_f$ . We can ask what difference in velocity  $\Delta V$  would cause the fuel at one angle to be at twice the final radius when the fuel at another angle has reached its final radius. To do so, we estimate the implosion time for the fully imploded fuel as  $R_i/V$  and we ask when

$$\Delta V R_i / V = R_f. \quad (9.39)$$

Thus, to keep the asymmetry of the final state smaller than a factor of 2 requires the variation in implosion velocity to be less than  $R_f/R_i \sim 5\%$ . Since the implosion velocity is proportional to the ablation pressure we require the ablation pressure to be uniform to the same level. In reality, an asymmetry of a factor of 2 is far too large. A more realistic limit is that the ablation pressure must be uniform to 1% accuracy, at least in a time-averaged sense. This is a demanding constraint. With fusion by direct laser ablation, one can use many beams but even so only a few beams overlap

on any given point on the target. With fusion within a hohlraum, entrance holes for laser beams or for the flow of energy into the hohlraum from a z-pinch create a significant asymmetry in the radiation from x-ray heated walls. The asymmetry produced by the energy source must be compensated for by careful design. In all approaches, as the target begins to move (and as the plasma struck by the laser also moves), the irradiation symmetry may change. As in the case of RT instabilities, the requirement of symmetry places difficult constraints on ICF. These constraints are unlikely to be met without measuring the symmetry and fine-tuning the irradiation of the capsule to produce adequately spherical implosions.

A historical and cautionary note is worth making here. During the 1970's portions of the inertial fusion program in the USA attempted to create fusion within hohlraums without making any measurements of the implosion symmetry. This effort failed, as evidenced by neutron yields that were typically 100 times smaller than predicted. Only on the Nova laser, when implosion symmetry and of the other essential aspects of the fusion system were measured, was it possible to first achieve fuel densities above 100 times liquid density and to begin to see neutron yields not very far below those predicted by one-dimensional simulations. This required a close collaboration between experiment and simulation, measuring the details and using simulations to assess the implications of what was seen. Prior to that era, there was far too much reliance on simulation codes without verifying measurements, and far too much belief that the codes were the reality. A good scientist who does simulations understands that the simulation is an essential tool but cannot fully represent reality. In the view of the present author the fact that much of ICF was classified contributed to its failures in the 1970's. A major benefit of presenting results at open scientific meetings and of publishing in the refereed literature is that these activities force an improvement in the quality of the science being done.

### 9.3.3 Laser–Plasma Instabilities

We saw in Chap. 8 that laser–plasma instabilities can scatter laser light and can also produce populations of high-energy (suprathermal) electrons. (In z-pinch-driven fusion, these instabilities are not present but there is also some potential to produce populations of energetic electrons.) We saw that these instabilities, in particular stimulated Raman scattering and stimulated Brillouin scattering, can in some circumstances convert most of the laser light into scattered light or energetic electrons. We also saw that another instability, filamentation, can break the light into filaments, which can potentially change where the light goes or can trigger other instabilities. These processes have three types of adverse consequence. We discuss these first, and then the question of how to control the amount of such scattering.

First, stimulated scattering processes reduce the efficiency of the laser-fusion target. If half the laser light is scattered back toward the source,

then one will need to start with twice the energy to produce fusion. The energy sources (or *drivers*) are expensive enough that this would be a major problem.

Second, stimulated scattering and filamentation both may alter where the light is deposited, both in a directly driven capsule and in a hohlraum. This can affect the symmetry of the implosion, since ablation pressure or soft x-rays would then be produced in unintended places. If these processes were consistent in the amount and direction of the scattering that they produced, then one could tune out the resulting variations through a sequence of experiments that measured the symmetry. This would be fine, but these processes are observed to behave reproducibly in some regimes and irregularly in others. So one cannot count on being able to tune them out.

Third, stimulated Raman scattering and the decay instabilities produce energetic electrons. Some of the energy of these electrons reaches the fusion fuel, which heats it and adds to its entropy. We can recall that the entropy limit is then  $4 \times 10^8$  J/keV/g, but that most of this limit must be used in the process of compression. For purposes of estimates, let us suppose that the limit on the entropy from preheat is  $10^8$  J/keV/g. The amount of entropy produced depends on the temperature when the preheat occurs, since  $ds = dq/T$ . Thus, a serious design must address preheat in a time-dependent context. For purposes of a crude estimate, suppose that the preheat occurs when the final shock is being produced, so that the pressure is already  $\sim 30$  Mbars, the fuel density is  $\sim 5$  g/cc, and the Fermi energy is  $\sim 40$  eV, and also suppose that the fuel temperature is  $\sim 40$  eV. Under these assumptions, an entropy of  $10^8$  J/keV/g develops if the 3 mg of fuel absorbs 12 kJ of energy. If the final capsule energy is 140 kJ, the rocket efficiency is 10%, and the delivery of energy to the rocket fuel is 50% efficient, then the driver energy is about 3 MJ. Thus, our estimate is that if of order 0.3% of the driver energy is deposited in the fuel as preheat, the fuel entropy would increase above the desired value, the total compression would be less than desired, and the gain would decrease.

This seems like a rather difficult constraint, but it in fact is less severe than it seems. The only energy that does damage is the energy that heats the fuel. But to get to the fuel the electrons must first penetrate the ablator. Those that penetrate the ablator without much attenuation (at energies well above 30 keV) will also penetrate through the (lower- $Z$ ) fuel without depositing much energy. Those that cannot penetrate the ablator cannot heat the fuel. A suprathermal electron distribution with a 30 keV temperature has a mean range of about  $3$  mg/cm<sup>2</sup>, which is of order the initial areal density of the ablator. Thus, electrons below this energy will not tend to reach the fuel. In addition, because the electrons transport diffusively in the dense matter and can scatter out of the vicinity of the capsule, many of them may deposit their energy somewhere else, especially in fusion using hohlraums. Depending on details, one would expect that an ICF high-gain target could succeed even if

the fraction of the driver energy in suprathermal electrons were of order 1% or perhaps more. By the time this ratio reaches 10%, efficiency is becoming as much of a concern as preheat. In summary, the production of suprathermal electrons must be limited but need not be completely quenched.

It is clear that, if ICF is to succeed, then none of the laser-plasma instabilities can be allowed to grow to a large amplitude. There are at root two approaches to control these instabilities. The first approach is to use short-wavelength (UV) lasers for fusion. This helps in two ways. The growth rate for all the instabilities is proportional to the oscillation velocity of the electrons in the laser light wave, as shown in Table 8.1. The square of this oscillation velocity is proportional to the laser irradiance times the square of the laser wavelength (8.9). Thus, one reduces the growth rate by shortening the laser wavelength. In addition, shorter-wavelength laser light makes collisional effects more important; because critical density increases, all processes occur at higher densities. An added benefit of higher densities is that the laser energy is shared by more particles so the plasma is cooler. This further increases the importance of collisions. Furthermore, the strong collisionality of plasmas made with UV lasers leads to the absorption of the laser light at densities well below the critical density, so that processes at critical density become unimportant. The need for short-wavelength lasers was determined the hard way. Fusion programs using infrared lasers (at  $\sim 1 \mu\text{m}$  and  $\sim 10 \mu\text{m}$  wavelength) experienced severe problems with energetic electrons, leading in the long run away from such wavelengths as serious candidates for ICF.

The second approach to control laser-plasma instabilities is to actively suppress them or at least to reduce their threshold for growth. If high density and collisions are effective enough, this will not prove necessary. The myriad options for control are beyond our scope here but are also often not realistic possibilities for specific ICF facilities. Fundamentally, one can suppress instabilities involving ion waves, such as stimulated Brillouin scattering, by adding bandwidth to the laser beams. This is not practical for decay instabilities or stimulated Raman scattering. One can limit SRS by creating a smooth, steep density profile in which SRS finds it difficult to grow. Two-plasmon decay, which occurs only near the quarter-critical density, may or may not be a problem. If it is too much of a problem, this may limit the irradiance that can be used for ICF. At this writing, it is unclear whether active control of the laser-plasma instabilities will be needed for ICF.

## 8 Creating High-Energy-Density Conditions

Nature often creates high-energy-density conditions. At root, the cause is always gravity. At the center of the Earth, for example, the pressure is 3.6 Mbars, almost entirely due to gravity. Jupiter is similar, with a pressure of  $\sim 40$  Mbars at its core. In stars, the gravitational assembly of the stellar mass leads to heating by nuclear fusion, which produces much larger pressures – the pressure at the core of the Sun is roughly 0.2 terabars. Once fusion creates conditions that lead to supernova explosions, even larger pressures occur. For example, some supernovae produce neutron stars, and the magnetic field at the surface of a typical neutron star is near 1 teraGauss. The pressure of such a magnetic field is about 40 petabars.

Nature also creates conditions whose laboratory analog must be in the high-energy-density regime. One can consider, for example, shock waves that are fast enough to ionize matter and perhaps to cause radiative effects. A 1,000 km/s shock wave in the interstellar medium, where the density of particles is of order one per cubic centimeter, has a ram pressure of tens of femtobars. This pressure approaches nanobars when the density becomes large enough that radiative losses become important. In contrast, a laboratory system that is a good analog of this astrophysical shock wave might involve a shock wave at 10 km/s in a material that is  $\sim 1$  g/cc, for which the ram pressure is of order 1 Mbar.

In this chapter we take up the problem of producing high-energy-density conditions. It would have been convenient to take this up much sooner, but in fact many of the concepts we have already introduced are needed to understand how to do this. We will discuss the technology that makes this possible in the early 21st century. We will also discuss the conceptual, physical, and mathematical models that are necessary to understand how such conditions are produced. These topics subdivide naturally into five areas: direct laser irradiation, laser-driven hohlraums, Z-pinches and related systems, ultrafast lasers, and high-energy-density beams. The first three of these are covered here; the last two are covered in Chap. 11.



## 8.1 Direct Laser Irradiation

All *lasers*, from laser pointers to megaJoule systems, have certain features in common. They involve the preparation of a medium that can reach a lower-energy state by giving energy to a light wave. This often involves the excitation of a specific atom, such as neodymium (Nd), so that more electrons are present in the upper state than in the lower state of an atomic transition. All lasers also involve the initiation of a light wave within such a system, sometimes by thermal emission of radiation and sometimes from an external source. This light wave is then amplified coherently, as the medium gives energy to it. The resulting beam of light may be well collimated, but this depends in part on the geometry of the lasing system. In the present book we will not discuss these aspects of lasers in further detail, leaving this subject to other books. Instead, we will discuss the aspects of lasers that are specific to high-energy-density physics – high energies in this section and high powers in Chap. 11.

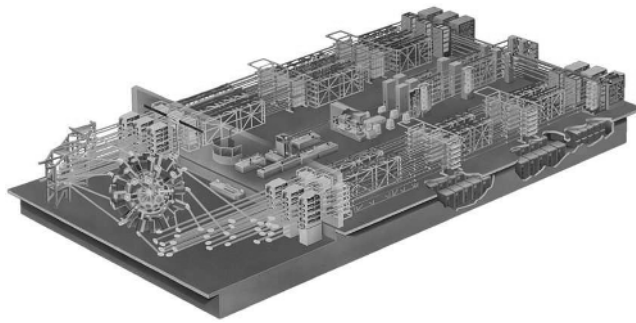
Before discussing the specifics of lasers, some discussion of the relevant units in common use is a good idea. The common units, also used here, are somewhat mixed. The energy of a laser pulse is typically given in Joules (J), or in related units such as kJ or MJ. Correspondingly, and considering that the timescale of the pulses is ns, the power is given in W, gigawatts (GW), terawatts (TW), or petawatts (PW). However, the practical unit of distance for real lasers is the cm, not the meter, so the power per unit area is typically given in  $\text{W}/\text{cm}^2$ , much to the horror of SI units purists. Perhaps more important is that the high-energy-density community has a habit of describing this power per unit area as an intensity, so one will see for example in the literature that the laser intensity in our experiment was  $10^{14} \text{ W}/\text{cm}^2$ . This horrifies the conceptual purists, such as the present author, as the general meaning of intensity involves power per unit solid angle (see Chap. 6). The correct term for power per unit area is irradiance, and this is the term we will use here.

### 8.1.1 Laser Technology

Now we turn to the specific issues involved in the lasers that produce high-energy-density conditions. A number of high-energy lasers have been constructed over the past few decades, motivated by the challenge of achieving inertial confinement fusion. The long-term goal is to create miniature fusion explosions with an energy gain of about 100. Such a laser system begins with a very high-quality laser beam, initially of low energy, which propagates through and extracts energy from Nd-doped laser glass. (We do not discuss here lasers based on gasses such as  $\text{CO}_2$  or KrF. No high-energy  $\text{CO}_2$  lasers remain in operation. The KrF lasers represent an important alternative to Nd-glass lasers for high-energy-density physics.) The first challenge for glass-based lasers is to prevent defects and diffraction from damaging the laser

components as the energy per unit area of the laser beam reaches high levels. The second is to extract a large fraction of the stored energy.

The first challenge was met in the 1970's. The key inventions here were *image relaying* and *spatial filters*. A laser system that uses image relaying first creates a high-quality, low-energy laser beam at a specific position (the object plane). It then designs the optics in the laser system so that the object plane is imaged onto or near the planes where the highest-irradiance laser light penetrates optical materials. These locations are often the lenses that inject the light into spatial filters. Spatial filters are also essential to reduce the structure in a laser beam. They do so by placing a pinhole at the focus of an input lens. This clips most of the energy in hot spots or other structures in the incident laser beam, as these small structures are focused to a much larger spot than the uniform beam is. The output lens then recollimates the remaining laser light into a smoother, more-uniform beam. These inventions led to several high-energy lasers capable of delivering  $>1$  kJ of laser energy to a target.



**Fig. 8.1.** A drawing of the Omega laser system. Credit: Laboratory for Laser Energetics

Figure 8.1 shows one example of such a laser system, the Omega facility (see Boehly et al., 1995), which can deliver 30 kJ to a target. The laser occupies approximately the area of one (American) football field. The capacitors in the basement accumulate energy for several minutes before delivering it to flashlamps in the laser amplifiers, preparing the Nd glass to amplify light. The initial laser beam, formed and amplified at the center of the laser bay, is split, amplified further, and eventually feeds the 60 amplifier chains that proceed down the sides of the laser bay toward the output end. Frequency conversion crystals then triple the frequency of these laser beams, decreasing their wavelength from  $1.05\ \mu\text{m}$  to  $0.35\ \mu\text{m}$ . Mirrors then direct the laser beams toward the center of the target chamber.

The second challenge cited above, of using the stored energy more efficiently, was met by large lasers constructed in the early 21st century. For

the laser beam to extract more of the stored energy it must pass through the Nd-doped glass several times, without destroying the quality or the focusability of the laser beam. To accomplish this it is necessary to clean up the laser beam between passes, using spatial filters, and/or to compensate for the phase differences across the amplifying optics, using *adaptive optics*. (An adaptive optic deforms an optical surface either continually or in small segments, allowing local adjustment of the distance that the light travels.)

### 8.1.2 Laser Focusing

It would seem that a simple lens would be sufficient to focus a high-energy beam to high irradiance for experiments, just as a simple magnifier can focus sunlight to start a fire. Unfortunately, a typical laser beam, especially when focused to a spot that is larger than the smallest (or best focus) spot that a lens can produce, creates a very irregular spot. This has a variety of adverse consequences, some of which we will touch on later. Inertial fusion, for example, requires the irradiation of a target by a very smooth laser beam. Small lasers often use a *Gaussian beam* to produce a high-quality laser spot. A Gaussian beam has a profile of irradiance that is approximately Gaussian as a function of radius (proportional to  $\exp(-r^2/a^2)$ , where  $r$  is radius and  $a$  is a distance). Such a beam can be image-relayed through an optical system to maintain high quality. This type of laser is comparatively inefficient, however, as most of the beam is at low intensity and does not extract much of the stored energy from the laser glass. High-energy beams must extract as much as possible of this stored energy, and thus must use much flatter irradiance profiles. Unfortunately, thorough studies proved that no practical optics could produce laser beams with flat irradiance profiles whose phase fronts were uniform enough that they would focus to smooth spots without some sort of extra processing. This has led to the invention of a number of techniques for so-called beam-smoothing. We discuss some of these here.

These techniques typically rely on the diffractive behavior of laser optics. When a light wave passes through a circular aperture, diffraction of the light by the aperture is well known to produce an Airy pattern (see Fig. 8.2). If such a light wave is focused, then diameter,  $d$ , of the first zero of the Airy pattern is given by

$$d = 2.44\lambda f, \quad (8.1)$$

in which the wavelength of the light is  $\lambda$  and the *f number*, the ratio of length to aperture, of the focusing system is  $f$ . The central maximum of the Airy pattern contains about 88% of the energy in the light wave. Thus, for example, for a 30 cm lens with a 3-m focal length,  $f = 10$  so for  $0.35 \mu\text{m}$  light  $d = 8.5 \mu\text{m}$ . This is also called the diffraction-limited spot size of the laser system. Typical high-energy laser systems, using only a focusing lens, produce best-focus spots that are larger than ten times the diffraction-limited spot size. The size of these best-focus spots and their structure is due to the

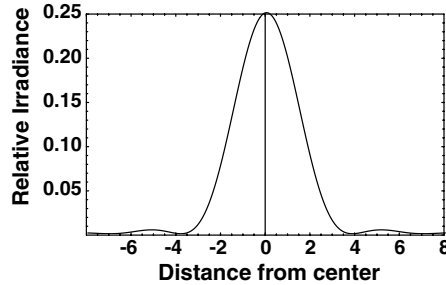


Fig. 8.2. An Airy pattern

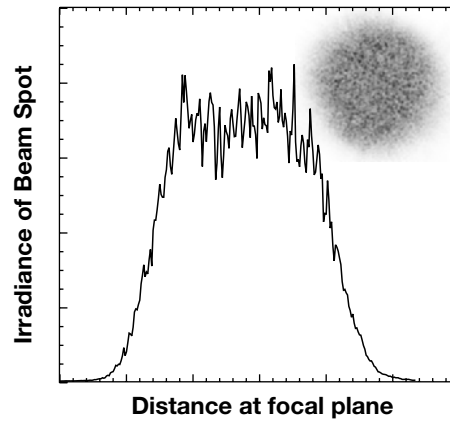
gradual variation of the phase of the light across the aperture of the lens, and to the interference of the beam from different portions of the aperture.

The simplest of the beam-smoothing systems, no longer in much use, is the *random phase plate* (RPP). A random phase plate passes the laser beam through an array of hundreds or thousands of adjacent optical elements, of randomly varying thickness, thus dividing the beam into small beamlets. The elements are typically hexagonal in shape. The elements are sized so that the diffraction-limited spot of each element is the size of the desired laser spot. Thus, to obtain a 1 mm spot with 0.35- $\mu\text{m}$  light at a distance of 3 m, (8.1) implies the aperture of an element must be 2.6 mm. This determines the overall size of the laser spot. (In actual experiments, one is most often concerned with the size of the spot that contains half the laser energy, or at the edges of which the irradiance is half its peak value. These numbers are somewhat smaller than the value from (8.1) and for real systems must be determined numerically or experimentally.) In addition, the beamlets from different elements interfere with one another, typically producing small, local maxima in the irradiance pattern, known as *speckles*. The minimum speckle diameter is produced by interfering beamlets that originate from opposing edges of the laser beam, and is given by (8.1) using the aperture of the entire laser beam to determine the  $f$  number. These speckles are actually very long and narrow structures. Their length,  $L_s$ , is

$$L_s = 7\lambda_o f^2, \quad (8.2)$$

so the ratio of length to width is roughly  $3f$ .

What has largely replaced the random phase plate in practice is the *distributed phase plate*, or DPP, in which the phase of each small element is controlled by design in order to determine the shape and structure of the resulting laser spot. In particular, this allows one to produce laser beams with flatter overall profiles of irradiance and with less energy in the wings of the laser spot. Figure 8.3 shows the typical irradiance pattern produced by a distributed phase plate. One sees a smooth overall profile modulated by many speckles. A related type of optic, often used in combination with a



**Fig. 8.3.** Pattern of irradiance from a distributed phase plate. The graph shows a line through the image, shown in the inset. Credit: Laboratory for Laser Energetics

DPP, is a *distributed polarization rotator* or DPR. A DPR uses birefringent optical elements to rotate the polarization of each beamlet by a controlled amount. Since only the components of two beamlets with parallel polarizations interfere with one another, this provides a further dimension of control when designing the shape of a laser spot. It also in principle allows one to tailor the polarization of the laser beamlets as they interact with the target.

Plasmas conduct heat easily, which will tend to smooth the effect of small-scale spikes like those seen in Fig. 8.3. This smoothing, combined with the tendency of shock waves to anneal as discussed in Chap. 5, implies that for some experiments the use of an RPP or DPP is sufficient to obtain high-quality results. However, for inertial fusion, at least in cases where the laser directly illuminates the fusion capsule, such smoothing techniques are not sufficient. The fixed location of each speckle creates lasting effects at a high spatial frequency. An improvement on this is to cause the speckles to move around, so that the profile of time-averaged irradiance across the overall laser spot is very smooth. There are two approaches to this.

The first approach is *induced spatial incoherence*, or ISI. One again breaks the laser beam into beamlets, but now one arranges that the difference in optical path length between the beamlets exceeds the distance over which the laser is coherent. This is only feasible using a comparatively broadband laser. The result is that the phase difference between beamlets varies in time, causing the speckles to move around on a timescale comparable to the coherence time of the laser. ISI was demonstrated on Nd-glass lasers, and is particularly well suited for implementation in KrF lasers, where it can be integrated into the laser design.

The second approach is *smoothing by spectral dispersion* (SSD), which has proven to be more practical for large glass lasers. In this approach one

produces a broadband laser pulse, disperses it in angle using a diffraction grating, and then collimates it to produce a laser beam whose frequency varies in the direction that was dispersed. One can also use two gratings (or complicated optics) to disperse the beam in two directions, producing 2D SSD. When such a beam is focused through a distributed phase plate, the phase difference between beamlets varies in time because the beamlets have different frequencies so their individual phases vary at different rates. This again causes the speckles to move around with a timescale determined by how large the differences in frequency are.

### 8.1.3 Propagation and Absorption of Electromagnetic Waves

Now that we have some idea how to irradiate a target with a high-energy laser beam that has been smoothed, we are ready to examine what happens when such a laser beam actually strikes a target. At the irradiances of interest, which are typically  $10^{12}$  to  $10^{16}$  W/cm<sup>2</sup>, the laser light immediately produces a plasma at the surface of the target. At the higher irradiances in this range, the electric field of the laser light is sufficient to directly ionize the atoms. At the lower irradiances, the process is more complicated but nonetheless a plasma is quickly produced. Figure 8.4 illustrates the three fundamental processes that occur when laser light penetrates a plasma. The laser light is refracted as it propagates through the density gradient in the plasma, it is reflected when it reaches a high enough density, and it is absorbed along its entire path of propagation. We will analyze these processes by examining the fundamental behavior of light in plasma.

To understand the behavior of laser light as it penetrates a plasma, we begin with Ampere's law, (2.18), in Gaussian cgs units, and use the standard vector and scalar potentials in the Coulomb gauge. This gives

$$\nabla \times \nabla \times \mathbf{A} = \frac{-1}{c^2} \frac{\partial^2 \mathbf{A}}{\partial t^2} + \frac{4\pi \mathbf{J}}{c} - \frac{1}{c} \frac{\partial \nabla \Phi}{\partial t}. \quad (8.3)$$

This equation has two parts, and it turns out that they separate completely. Any vector can be decomposed into a *transverse* (or rotational) part, the di-

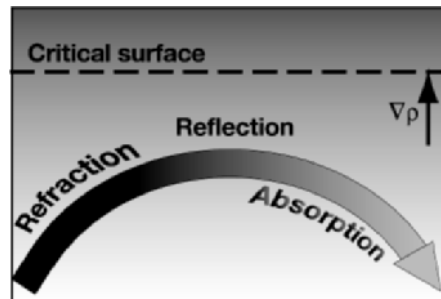


Fig. 8.4. The simple processes that occur when a light wave enters a plasma

vergence of which is zero, and a *longitudinal* (or compressive, or irrotational) part, the curl of which is zero. By definition,  $\mathbf{A}$  is purely transverse. Also, any gradient has zero curl, so  $\nabla\Phi$  is purely longitudinal. Taking the divergence of (8.3) yields a continuity equation for the charge in the plasma, in which the variation with time charge density (proportional to  $\nabla^2\Phi$  via Poisson's equation) is balanced by the divergence of a flux of charge (i.e., a current).

### Homework 8.1

Derive (8.3) from Maxwell's equations.

### Homework 8.2

Derive an equation for the conservation of charge from (8.3).

One can subtract this continuity equation from (8.3), and also expand its left-hand side, to obtain a fundamental equation for light wave propagation:

$$\frac{\partial^2 \mathbf{A}}{\partial t^2} - c^2 \nabla^2 \mathbf{A} = 4\pi c \mathbf{J}_t, \quad (8.4)$$

in which  $\mathbf{J}_t$  refers to the transverse part of  $\mathbf{J}$ . (One can construct  $\mathbf{J}_t$  from  $\mathbf{J}$  if needed. This is discussed for example in Jackson.)

For any specific plasma environment, the behavior of light waves is thus determined by  $\mathbf{J}_t$ . It is also true that the frequencies of the lasers of interest are so large that the ion motion is negligible on the timescale of the laser propagation. (Ion motion can have important consequences on longer timescales, some of which are discussed in Sect. 8.1.4) The net current carried by the electrons is  $-en_e \mathbf{u}_e$ ; we are seeking the transverse part of this current.

To find this transverse current, we work with the continuity and momentum equations for the electron fluid, which are (2.61) and (2.62). The momentum equation, written again here for electrons,

$$m_e n_e \frac{\partial \mathbf{u}_e}{\partial t} + m_e n_e \mathbf{u}_e \cdot \nabla \mathbf{u}_e = -n_e e \left( \mathbf{E} + \frac{\mathbf{u}_e}{c} \times \mathbf{B} \right) - \nabla p_e - \sum_{j=\text{ions}} m_e n_e (\mathbf{u}_e - \mathbf{u}_j) \nu_{ej}, \quad (8.5)$$

is key. In applying this equation here we can make several observations and simplifications. The velocity of any ions is negligible and can be neglected. In addition, the pressure gradient is an inherently longitudinal vector, so we can drop it as we are seeking the transverse velocity. We can also ignore  $\nabla\Phi$  after we again use the scalar and vector potentials. Finally, we can divide each term by  $n_e m_e$ . After these adjustments, we have an equation for the transverse electron velocity,

$$\frac{\partial \mathbf{u}_e}{\partial t} + \mathbf{u}_e \cdot \nabla \mathbf{u}_e = \frac{-e}{m_e} \left( -\frac{1}{c} \frac{\partial \mathbf{A}}{\partial t} + \frac{\mathbf{u}_e}{c} \times \nabla \times \mathbf{A} \right) - \mathbf{u}_e \nu_{ei}. \quad (8.6)$$

From standard vector identities,  $\mathbf{u}_e \cdot \nabla \mathbf{u}_e = -\mathbf{u}_e \times \nabla \times \mathbf{u}_e + \nabla u_e^2/2$ , but the gradient term here is also longitudinal and can also be dropped (This term will play a role in the coupling of laser light to longitudinal waves in the plasma). Substituting and rearranging, we have

$$\left(\frac{\partial}{\partial t} + \nu_{ei}\right) \mathbf{u}_e - \mathbf{u}_e \times \nabla \times \mathbf{u}_e = \left(\frac{\partial \mathbf{v}_{os}}{\partial t} - \mathbf{u}_e \times \nabla \times \mathbf{v}_{os}\right), \quad (8.7)$$

where we have defined  $\mathbf{v}_{os} = e\mathbf{A}/(m_e c)$ , which we will identify as the oscillating velocity of the electron within the light wave. One can see from (8.7) that in the absence of any collisional energy loss to the ions, one would have precisely  $\mathbf{u}_e = \mathbf{v}_{os}$ . The presence of  $\nu_{ei}$  introduces spatial damping of the electromagnetic wave and a phase variation between  $\mathbf{u}_e$  and  $\mathbf{v}_{os}$ . There are typically two simplifying aspects. The first, dealt with below, is that the spatial scale of the variation in  $\mathbf{A}$  is large compared to the wavelength of the light. The second is that  $\mathbf{u}_e$  is a small fraction of the speed of light. (This second is true for typical high-energy lasers, having pulses of order a ns, but not for sufficiently intense, high-power lasers, discussed in Chap. 11. To treat the relativistic regime one must use modified equations.) So one can divide (8.7) by  $c^2$  and note that the terms involving the curl are much smaller than the other terms because  $|\partial/\partial(ct)| \sim \omega/c \sim k \sim |\nabla|$ . This justifies the use of linearization and the dropping of the terms involving the curl. Then, assuming that  $\mathbf{A}$  and thus  $\mathbf{u}_e$  vary as  $e^{-i\omega_0 t}$ , we find

$$\mathbf{u}_e = \mathbf{v}_{os} \frac{1}{1 + i\nu_{ei}/\omega_0}. \quad (8.8)$$

Here the imaginary term produces the phase shift mentioned above, but this is small so long as  $\nu_{ei} \ll \omega_0$ . Also note that  $\mathbf{u}_e$  is purely transverse, because  $\mathbf{A}$  and  $\mathbf{E}$  for an electromagnetic wave are purely transverse.

It will be helpful for applications to connect  $\mathbf{v}_{os}$  and  $\nu_{ei}$  with practical units. The direction of  $\mathbf{v}_{os}$  is the direction of the electric field of the laser, typically described as the direction of polarization of the laser. The magnitude,  $v_{os}$ , can be related to the irradiance of the laser in vacuum,  $I_L$ , as follows. Because the energy density of the electromagnetic field is  $E^2/(8\pi)$ , and because this propagates with a group velocity in vacuum of  $c$ , one has  $I_L = cE^2/(8\pi)$ . The magnitude of this is  $I_L = \omega_0^2 A^2/(8\pi c) = v_{os}^2 \omega_0^2 m_e^2 c/(8\pi e^2)$ , from which

$$v_{os}/c = \sqrt{I_{14} \lambda_{\mu}^2/117}, \quad (8.9)$$

in which  $I_{14}$  is  $I_L$  in units of  $10^{14}$  W/cm<sup>2</sup> and  $\lambda_{\mu}$  is the wavelength of the light in  $\mu\text{m}$ . We discussed  $\nu_{ei}$  in Chap. 2, which in practical units is

$$\nu_{ei} = 3 \times 10^{-6} \ln A \frac{n_e Z}{T_e^{3/2}}, \quad (8.10)$$

with  $n_e$  in cm<sup>-3</sup>,  $T_e$  in eV, and where  $\ln A = \max(1, [24 - \ln(\sqrt{n_e}/T_e)])$ .



We then take  $\mathbf{J}_t = -en_e\mathbf{v}_{os}/(1 + i\nu_{ei}/\omega_o)$  in (8.4). This is completely accurate if  $n_e$  is constant. (However, if a variation in  $n_e$  is designated as  $n_{e1}$ , then the part of  $\mathbf{J}_t$  proportional to  $n_{e1}\mathbf{v}_{os}$  may have transverse and longitudinal components, depending on the direction of the gradient in  $n_e$ . This detail matters for wave coupling calculations, but we ignore it here.) Substituting into (8.4), simplifying, and rearranging, we obtain

$$\left( \frac{\partial^2}{\partial t^2} + \frac{\omega_{pe}^2}{(1 + i\nu_{ei}/\omega_o)} - c^2\nabla^2 \right) \mathbf{A} = 0, \quad (8.11)$$

in which  $\omega_{pe}$  is the plasma frequency defined in Chap. 2. This is the wave equation for a damped electromagnetic wave in a plasma.

It is worthwhile to examine and discuss the dispersion relation implied by (8.11), which is

$$\omega_o^2 - \frac{\omega_{pe}^2}{[1 + (\nu_{ei}/\omega_o)^2]} - c^2k^2 + i \frac{\nu_{ei}/\omega_o}{[1 + (\nu_{ei}/\omega_o)^2]} \omega_{pe}^2 = 0, \quad (8.12)$$

in which  $k$  is the wavenumber of the light wave. The imaginary term here introduces an imaginary component to the phase of the wave, which may be expressed as an imaginary part of either  $\omega_o$  or  $k$ . It causes damping as the wave propagates. We will examine this damping below. The real component of (8.12) describes the propagation of the wave. The reader may recall that a propagating light wave traveling through a stationary, unchanging medium experiences variations in  $k$  and not in  $\omega_o$ . Furthermore, when  $k$  is driven to zero the wave cannot propagate further and must be reflected and/or absorbed. Equation (8.12) implies that  $k$  is driven to zero approximately when  $\omega_o = \omega_{pe}$ . Physically, when  $\omega_o = \omega_{pe}$  the electrons resonantly oscillate at the frequency of the light wave, creating a reflecting surface like a mirror. This surface is known as the *critical surface* and the density there is the *critical density*,  $n_c$ . From  $\omega_o = \omega_{pe}$  one can show that

$$n_c(\text{cm}^{-3}) = 1.1 \times 10^{21} / \lambda_\mu^2. \quad (8.13)$$

Thus, for visible and UV lasers,  $n_c < 10^{23} \text{ cm}^{-3}$ . If one now looks again at Fig. 8.5, one can draw some implications for the absorption of light in plasmas. The critical density for visible and UV lasers is typically small enough that  $\nu_{ei} < \omega_{pe}$  there. Such laser beams propagate through the plasma and reflect, perhaps having been substantially absorbed in the process. In contrast, the critical density for x-rays is typically above any density present in the plasma. Thus, in the absence of absorption the x-rays would penetrate freely through the target. But absorption can be strong:  $\nu_{ei}$  becomes quite large at densities near or above solid density. The result is that soft x-rays, with energies below roughly 1 keV, are very strongly absorbed by collisions. Harder x-rays usually are not strongly absorbed by collisions, but the atomic absorption of these x-rays can be significant as discussed in Chap. 6.

Let us focus now on the *absorption of laser light* and assume that  $(\nu_{ei}/\omega_o)^2$  is small enough to be ignored. We are interested in the spatial rate of absorption, so we assume  $\omega_o$  to be real and take  $k = k_r + i\kappa_{EM}/2$ , with real and imaginary parts  $k_r$  and  $\kappa_{EM}/2$ , respectively. Here  $\kappa_{EM}$  is the spatial rate of absorption of the laser energy, proportional to  $E^2$ , and  $\kappa_{EM}/2$  is the spatial rate of change of  $E$  (or  $A$ ). We then solve (8.12) for  $k_r$  and  $\kappa_{EM}$ , ignoring the term involving  $\kappa_{EM}^2$  subject to verifying our assumption that the light is absorbed slowly as it propagates (i.e.,  $\kappa_{EM} \ll k$ ). We obtain

$$k_r = (\omega_o/c)\sqrt{1 - n_e/n_c} \quad (8.14)$$

and

$$\kappa_{EM} = \nu_{ei} \frac{\omega_{pe}^2}{\omega_o^2} \frac{1}{c\sqrt{1 - n_e/n_c}} = \frac{\nu_{ei}}{v_g} \frac{n_e}{n_c}, \quad (8.15)$$

in which  $v_g$  is the group velocity of the light wave in the plasma. Equation (8.15) is easy to understand – only that fraction of the energy in the light wave that participates in electron oscillations,  $n_e/n_c$ , can be affected by electron–ion collisions, and the spatial rate at which this effect occurs is the temporal collision rate divided by the rate at which energy propagates in space (i.e., the group velocity). Also note that our previous assumption that  $\nu_{ei} \ll \omega_o$  assures that  $\kappa_{EM} \ll k$ .

### Homework 8.3

Using the equation of motion for the electron fluid in the fields of an electromagnetic wave in a plasma of constant density, determine the time-averaged distribution of energy among the electric field, the magnetic field, and the kinetic energy of the electrons. Discuss how this varies with density.

The above discussion implies that the fraction of the incident laser light transmitted through a uniform plasma of length  $D$  is  $\exp[-\kappa_{EM}D]$ . Kruer shows how to determine the absorption in more complicated circumstances. Two of his results are worth quoting here. Let the electron-ion collision rate at the critical density be  $\nu_{ei}^*$ , let  $z$  be a spatial variable and let  $L$  be a scale length. Then, for a laser beam normally incident on a plasma with a linear density profile, so  $n_e = n_c z/L$ , the absorption  $f_A$  is

$$f_A = 1 - \exp\left(\frac{-8\nu_{ei}^*L}{3c}\right), \quad (8.16)$$

while for a laser beam incident at an angle  $\theta$  from the normal on a plasma with an exponential density profile, so  $n_e = n_o e^{-z/L}$ , the absorption is

$$f_A = 1 - \exp\left(\frac{-8\nu_{ei}^*L}{3c} \cos^3 \theta\right). \quad (8.17)$$

The third important process that occurs during penetration of a plasma by laser light is *refraction*. Refraction refers to the bending of rays of light

as they propagate. The concepts of light rays and refraction are valid in the geometric-optics limit, when variations in the medium occur on scales large compared to the wavelength of the light. In high-energy-density systems, refraction is a sensible concept when light, incident at some angle from normal incidence, penetrates a plasma that has been expanding from an initial solid surface for at 100 ps or more. In contrast, if the light penetrates to the critical density or if the plasma is only a few wavelengths in extent, one must analyze the light as a wave. Kruer provides a discussion of laser light reflection at the critical surface, where some energy can be absorbed by *resonance absorption*.

Continuing with the discussion of refraction, the variation of  $k_r$  in a plasma is given by (8.14). The variation in the wave vector  $\mathbf{k}$  is given by the equations of ray propagation (see vol. 2. of Landau and Lifshitz) as

$$\frac{d\mathbf{k}}{dt} = -\nabla\omega, \quad (8.18)$$

which is the analogy in geometric optics of the relation between the rate of change of momentum and the gradient of the Hamiltonian in mechanics. It is sensible to identify the components both parallel and perpendicular to the density gradient as  $k_{\parallel}$  and  $k_{\perp}$ , respectively. Thus, the component perpendicular to the density gradient does not change. Intuitively, this may be easiest to see by recalling the refractive behavior at a sharp interface, illustrated in Fig. 8.5. The phase velocity ( $\omega/k$ ) of the wave changes at the interface, implying that the distance between the phase fronts must change. However, the distance between the phase fronts along the interface is determined by the incident wave. As a result the component of the phase velocity (and  $\mathbf{k}$ ) that is perpendicular to the interface is what changes. One can view propagation up a gradient as the limit of propagation up a series of steps as the number of steps becomes large and the step size becomes small.

Also, because the boundary condition is that the fields in the wave must vary continuously across the boundary of the plasma, one has  $k_{\perp} =$

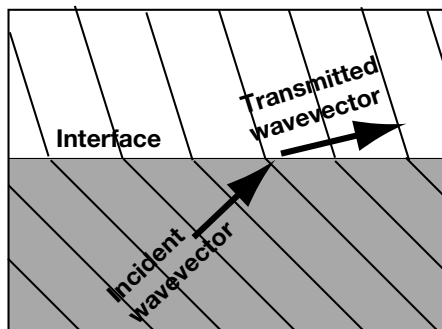


Fig. 8.5. Refraction at an interface, with solid lines showing phase fronts

$\sin(\theta)\omega_o/c$ , in which  $\theta$  is the angle of incidence, measured with respect to a normal vector. This implies

$$k_{\parallel} = (\omega_o/c)\sqrt{\cos^2\theta - n_e/n_c}. \quad (8.19)$$

This equation has an obvious interpretation: an obliquely incident light wave (in a planar plasma) reflects at a density such that  $n_e/n_c = \cos^2\theta$ . Thus, when the angle of incidence is  $45^\circ$ , the laser light is reflected at roughly  $n_c/2$ , and at  $60^\circ$  this decreases to  $n_c/4$ .

#### 8.1.4 Laser Scattering and Laser-Plasma Instabilities

Most of the applications in high-energy-density physics would be simpler and easier if laser beams did no more than propagate, refract, and absorb in plasmas. Even laser scattering from small fluctuations, discussed just below, would not disturb these applications. From this point of view, it is unfortunate that the underdense plasma produced by the laser is host to a variety of waves, and that these waves can couple unstably to the laser light wave. The unstable waves can become large and can have large effects, scattering large amounts of laser light, producing substantial populations of energetic electrons or ions, and even causing modulations in the laser absorption dynamics. From the point of view of such dynamics, *laser-plasma interactions* (LPI) is a tremendously exciting field. It even has a few astrophysical applications, relating for example to the dynamics in certain solar bursts and in the turbulence within stellar winds. One such application – the scattering of pulsar radiation by the plasma of a binary companion – might at times occur at a high energy density. But LPI is not our primary topic here. Our goal here is to cover as much of LPI as our reader should know to work intelligently in high-energy-density physics. Those interested in more details should start with the book on LPI by W.L. Kruer.

We now discuss the dynamics involved in *laser scattering* from density fluctuations in the plasma. We will consider only uniform plasmas. This is an oversimplification, as most scattering and instabilities actually occur in plasmas that are nonuniform. Here we ignore nonuniformity because it introduces complications without introducing many new ideas. Collisions may at times be important but we will ignore them too for the same reason. Thus, we take  $\mathbf{A} = \mathbf{A}_L + \mathbf{A}_s$ , in which s refers to a scattered light wave, and we take  $n_e = n_{e0} + \delta n_p$ . We will use L, s, and p as subscripts for the laser light wave, the scattered light wave, and the wave in the plasma, respectively. It is worth noting that the laser light interacts primarily with the electrons, because the ions move so much more slowly and thus carry far less current. However, the electron-density fluctuation  $\delta n_p$  may be produced by any wave in the plasma. On the one hand, it could be produced by an electron-plasma wave in which the ion density is effectively fixed. On the other hand, it could be produced by an acoustic wave in which the ion density fluctuates and

the electrons are forced by the ion-charge variations to move with the ions. We also assume that proper normalization would show that terms involving  $A_L$  alone are large in comparison with those involving only  $A_s$  or  $\delta n_p$ . We then substitute into (8.11) and linearize. We find that we can drop the terms involving only  $A_L$  because they cancel one another, so we obtain

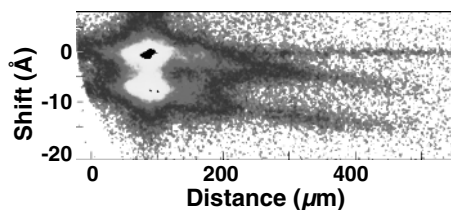
$$\left(\frac{\partial^2}{\partial t^2} + \omega_{pe}^2 - c^2 \nabla^2\right) \mathbf{A}_s = -\omega_{pe}^2 \frac{\delta n_p}{n_{eo}} \mathbf{A}_L. \quad (8.20)$$

With reference to (8.4), the interpretation of this equation is simple. The interaction of the laser light with the density fluctuation produces an additional transverse current proportional to the right-hand side of (8.20). Another aspect of (8.20) is that it is fundamentally an equation describing wave beating. The laser light wave and the electron density fluctuation beat together to drive scattered light waves. The reader should also recall, from elementary physics, that this will produce beat waves having both the sum and the difference of the frequencies and wavenumbers of the two driven waves. One may wonder where the energy in the scattered-light wave comes from. It is obtained from the laser light wave through a second-order term that slowly reduces  $A_L$ .

#### Homework 8.4

Derive (8.20).

Indeed, when scattering measurements are used to characterize the fluctuations in the plasma, a technique known as *Thomson scattering*, one sees both of the resulting scattered-light waves. Figure 8.6 shows an example of such data. This figure is a gated imaging spectrum, meaning: (a) the measurement was limited to a brief period (it was *gated*); (b) the instrumentation provides an image along a line through the object, horizontal in the figure; and (c)



**Fig. 8.6.** This spectral image from Thomson scattering in the collective regime, with wavelength varying in the vertical direction shows two peaks due to oppositely propagating acoustic waves. The feature near zero shows the wavelength of the probe laser. The colors on this image cycle from white to black twice as the intensity of the signal increases. Adapted from D.S. Montgomery et al., *Laser and Particle Beams* 17, 349 (1999)

the signal from this line was resolved spectrally in the orthogonal (vertical) direction. Thus, a vertical cut through the image shown in the figure provides a spectrum at that specific location. The plasma motion at any location shifts the entire frequency by the Doppler effect, providing a measurement of the local velocity. One can see that this average frequency shift increases with distance in the figure, which implies that the plasma flow velocity is increasing with distance. This is sensible because this image is from a plasma expanding from a laser-irradiated surface, as an isothermal rarefaction. The fact that the rarefaction is isothermal is confirmed by the constant spacing between the two peaks in the spectrum. The spacing is proportional to the frequency of the acoustic waves causing the thermal density fluctuation; thus it measures the sound speed in the plasma, which depends mainly on the electron temperature. The weakening of the signal from left to right in the image is due to decreasing plasma density in the rarefaction. Studies of the ionosphere employ similar Thomson-scattering techniques, also for the purpose of diagnosing density and temperature. Both the laser-plasma and the ionospheric applications take place in what is known as the collective regime of Thomson scattering, to which (8.20) applies. In contrast, the use of Thomson scattering in magnetic-fusion plasmas takes place in the single-particle regime, in which the scattering is effectively from individual particles. The different Doppler shifts of these particles produces a scattered-light spectrum from which temperature can be inferred. Sheffield treats the fundamental theory of Thomson scattering.

We now turn to cases in which one of the scattered-light waves can participate in an instability, producing much stronger scattering. We consider explicitly the case of *stimulated Raman scattering* (SRS). This process involves scattered-light waves, which we have just discussed, and electron-plasma waves, discussed in Chap. 2. To develop an initial understanding, SRS and the other instabilities are most profitably described in uniform plasmas irradiated by laser pulses that are not depleted by the action of the instability. In the simplest form of these instabilities, only the difference frequency participates, and we will selectively include here only those terms in our discussion here.

Before proceeding with this selective analysis, some words on notation are needed. All the complex quantities in any derivation must be related to real quantities by some convention, such as  $\mathbb{R}(\mathbf{A}_L) = (\mathbf{A}_L + \mathbf{A}_L^*)/2$ , in which  $*$  designates the complex conjugate. We will assume that the amplitude is real (an approximation that retains the essential physics) and will use a caret  $\hat{\phantom{a}}$  to designate the amplitude of each wave, so that for example  $\mathbf{A}_s = \hat{\mathbf{A}}_s e^{i(\mathbf{k}_s \cdot \mathbf{x} - \omega_s t)}$ . When doing theory with no nonlinear terms, we might typically write the time-and-space dependence of the scattered light as  $e^{i(\mathbf{k}_s \cdot \mathbf{x} - \omega_s t)}$ . When working accurately with equations that have nonlinear terms, one would have to write  $\mathbf{A}_s = (\hat{\mathbf{A}}_s e^{i(\mathbf{k}_s \cdot \mathbf{x} - \omega_s t)} + \hat{\mathbf{A}}_s^* e^{-i(\mathbf{k}_s \cdot \mathbf{x} - \omega_s t)})/2$ . The interaction of the two

real physical quantities in the nonlinear terms involves cross terms, and these are essential to accounting for the beat waves.

For a cursory analysis we will work with selected terms. Consider first the beat term in which the phase of the light waves varies as  $e^{i(\mathbf{k}_s \cdot \mathbf{x} - \omega_s t)}$  and that the phase of the plasma wave varies as  $e^{-i(\mathbf{k}_p \cdot \mathbf{x} - \omega_p t)}$ . One then has from (8.20)

$$\begin{aligned} (-\omega_s^2 + \omega_{pe}^2 + c^2 k_s^2) \hat{\mathbf{A}}_s &= \frac{-\omega_{pe}^2}{2} \frac{\delta \hat{n}_p}{n_{e0}} \hat{\mathbf{A}}_L \times \\ &\exp i [(\mathbf{k}_L - \mathbf{k}_s - \mathbf{k}_p) \cdot \mathbf{x} - (\omega_L - \omega_s - \omega_p) t], \end{aligned} \quad (8.21)$$

in which the exponential term is a phase-matching term. Its argument must be zero to obtain a nonzero averaged response. This imposes the beating condition that we expect – the frequency and wavevector of the driven wave must equal the difference of the values for the laser and for the electron–plasma wave.

To see the unstable behavior, we need to reconsider the derivation of the electron–plasma wave from (2.50) to (2.55), in the presence of light waves and informed by the discussion earlier in this chapter. We are looking for the ways in which light waves can affect plasma waves. The electron plasma wave is a purely longitudinal wave, so one can write  $\mathbf{u}_e = \mathbf{v}_{os} + \mathbf{v}_p$ , in which  $\mathbf{v}_{os}$  is purely transverse but may involve the sum of contributions from more than one light wave and  $\mathbf{v}_p$  is the purely longitudinal vector describing the motion of the electron fluid in the plasma wave. Substituting into (8.5) and again using scalar and vector potentials, one obtains an equation from which (8.6) can be subtracted. After linearizing in  $|\mathbf{v}_p|$  and ignoring collisions we find

$$\frac{\partial \mathbf{v}_p}{\partial t} - \frac{e}{m_e} \nabla \Phi + \frac{\nabla p_e}{n_e m_e} = -\nabla \frac{v_{os}^2}{2}, \quad (8.22)$$

in which we have also dropped a term proportional to  $\mathbf{v}_p \times \nabla \times \mathbf{v}_{os}$ , whose direction is orthogonal to  $\mathbf{v}_p$ .

### Homework 8.5

Derive (8.22). Calculate the energy density of the laser light wave and show how this is related to the source term on the right-hand side.

As in Chap. 2, one obtains a wave equation by taking the divergence of this equation, then using continuity and the equation of state to simplify all the terms on the left-hand side. The right-hand side of this equation represents the force known as the *ponderomotive force*. This term is the gradient of  $v_{os}^2$ , which is equivalent to the gradient in the energy density (or pressure) of the electromagnetic waves. The ponderomotive force can be important in other contexts, but here we focus on its role in instabilities. If the light waves present have vector potentials  $\mathbf{A}_L$  and  $\mathbf{A}_s$ , then  $v_{os}^2$  has three terms. However, only the cross term involves the beating of two waves. Keeping only the cross term, one obtains

$$\left(\frac{\partial^2}{\partial t^2} + \omega_{pe}^2 - 3\frac{k_B T_e}{m_e} \nabla^2\right) \frac{\delta n_p}{n_{eo}} = \frac{e^2}{m_e^2 c^2} \nabla^2 (\mathbf{A}_L \cdot \mathbf{A}_s). \quad (8.23)$$

This equation describes the driving of electron–plasma waves by the beating of light waves in the plasma. Once again the wave beating produces source terms that are upshifted or downshifted in frequency relative to the laser frequency, and once again only the downshifted source term is significant for the simple instability. To see the unstable coupling, consider the beat term in which the phases of the laser-light wave, the scattered-light wave, and the plasma wave vary as  $e^{-i(\mathbf{k}_L \cdot \mathbf{x} - \omega_L t)}$ ,  $e^{i(\mathbf{k}_s \cdot \mathbf{x} - \omega_s t)}$ , and  $e^{-i(\mathbf{k}_p \cdot \mathbf{x} - \omega_p t)}$ , respectively, and again use a caret  $\hat{\phantom{x}}$  to designate the amplitude of each wave (assumed to be real). One then finds

$$\begin{aligned} \left(-\omega_p^2 + \omega_{pe}^2 + 3\frac{k_B T_e}{m_e} k_p^2\right) \frac{\delta \hat{n}_p}{n_{eo}} &= \frac{-e^2 k_p^2}{2m_e^2 c^2} (\hat{\mathbf{A}}_L \cdot \hat{\mathbf{A}}_s) \\ &\times \exp i [(-\mathbf{k}_L + \mathbf{k}_s + \mathbf{k}_p) \cdot \mathbf{x} - (-\omega_L + \omega_s + \omega_p) t]. \end{aligned} \quad (8.24)$$

One sees here the same sort of phase-matching term we encountered in (8.21). If one now multiplies (8.21) by  $\hat{\mathbf{A}}_L$ , and substitutes for  $\delta \hat{n}_p$  from (8.24), one finds

$$(-\omega_s^2 + \omega_{pe}^2 + c^2 k^2) \left(-\omega_p^2 + \omega_{pe}^2 - 3\frac{k_B T_e}{m_e} k_p^2\right) = \omega_{pe}^2 \frac{k_p^2 v_{os}^2}{4}. \quad (8.25)$$

This coupled dispersion relation describes the growth of an instability. Physically, the laser light beats with density fluctuations to drive the scattered light and beats with the scattered light to drive density fluctuations. When phase-matching is satisfied – that is, when  $\mathbf{k}_L = \mathbf{k}_s + \mathbf{k}_p$  and  $\omega_L = \omega_s + \omega_p$  – the process is resonantly reinforcing. Note that the two sets of parentheses on the left-hand side each enclose the dispersion relation of one of the normal modes of oscillation of the plasma. In the absence of driving or coupling and for any given wavenumber, each set of parentheses would determine the frequency for each mode independently. The coupling represented by the right-hand side, in the presence of the phase matching that connects the mode frequencies and wavevectors, leads to instability growth.

Equation (8.25) implies an exponential growth rate for the instability. For the uniform-plasma case considered here it makes sense to look for temporal growth. Mathematically, the two driven waves grow as  $e^{\gamma t}$  while the amplitudes are small. One finds  $\gamma$  by identifying the real part of each frequency with the subscript r and the imaginary part of both of them as  $\gamma_o$ , with the sign corresponding to growth in time. For the components we have chosen to consider, this gives us  $\omega_s = \omega_{sr} - i\gamma_o$  and  $\omega_p = \omega_{pr} + i\gamma_o$ . We also assume here that the real part of each frequency is the normal-mode frequency, and thus cancels the other real terms in its part of the equation. If  $\gamma_o$  is much smaller than the wave frequencies, as is nearly always the case for SRS (and



is sometimes the case for other instabilities), then one finds the growth rate for SRS in a homogeneous plasma,

$$\gamma_o = \sqrt{\frac{\omega_{pe}^2}{\omega_{sr}\omega_{pr}} \frac{k_p v_{os}}{4}}. \quad (8.26)$$

The growth of SRS in other more complicated situations can be usefully expressed in ways involving this growth rate. SRS can occur in principle at densities up to  $n_c/4$ , where  $\omega_{pr}$  and  $\omega_{sr}$  are both  $\sim \omega_o/s$ . It more typically occurs near  $n_c/10$ , where  $\omega_{pr} \sim \omega_o/3$ ,  $\omega_{sr} \sim 2\omega_o/3$ , and  $k_p \sim 1.5\omega_o/c$ . The growth rate, for  $I_{14}\lambda_\mu^2 \sim 1$ , is  $\gamma_o \sim 0.002\omega_o$ . Thus, SRS indeed grows slowly on the scale of the wave cycles. But note that  $1/\omega_o \sim 1$  fs, so  $1/\gamma_o < 1$  ps. The instability, when present, thus grows extremely rapidly on the ns scale of the typical laser pulse.

We should confess that the derivation just provided involves cheating at several levels, in order to most simply make its physical point. Strictly speaking, one should express all the wave amplitudes as real quantities and follow through with all the wave-beating terms that arise. This is the only way one can obtain the factor of 2 that mysteriously appeared in (8.21) and (8.24). One also should not assume that the light wave and the electron plasma wave, which are normal modes of an undisturbed plasma, will be unchanged by the instability. Doing all this properly would involve several more pages, however, and in the end would produce the same result with many nuances. One could then proceed to consider other complications such as nonuniform plasmas and depletion of the laser pulse. A first level of improved analysis can be found in the book by Kruer. Doing better than that requires submersion in the archival literature.

Of greater importance than the details of the SRS theory is to understand that once energy is given to electron plasma waves (by any instability), it tends to be converted to energetic electrons. *Energetic electrons* are in most cases a terrible nuisance. They readily penetrate target materials, especially low- $Z$  materials, and so can preheat materials and alter the initial conditions for later evolution of the target. This is a crucial issue for laser fusion, as we discuss in Chap. 9. In addition, because laser irradiation produces large-scale magnetic fields that wrap around the target, the energetic electrons can very easily travel around almost any shielding to penetrate and heat surfaces that are distant from the laser spot. This can affect both the physics of an experiment and the signals seen by diagnostics.

Landau damping produces the energetic electrons. Landau damping typically dominates over collisional damping of electron plasma waves. (If collisions are strong, the waves will not be driven, and if collisions are weak, then the waves are Landau damped.) Some readers will recall that Landau damping operates by accelerating electrons in the wave, so that the electrons produced have an energy of order the phase velocity of the wave. This energy, for the typical SRS conditions given above, is then

**Table 8.1.** Laser-driven instabilities

Name	Driven wave 1	Driven wave 2	Where	Growth rate
Stimulated Raman scattering	Scattered light	Electron Plasma	$\leq n_c/4$	$\sqrt{\frac{\omega_{pe}^2}{\omega_1 \omega_2} \frac{k_2 v_{os}}{4}}$
Stimulated Brillouin scattering	Scattered light	Acoustic	$\leq n_c$	$\sqrt{\frac{\omega_{pi}^2}{\omega_1 k_2 c_s} \frac{k_2 v_{os}}{4}}$
Two-plasmon decay	Electron plasma	Electron Plasma	$\sim n_c/4$	$\frac{k_2 v_{os}}{4}$
Parametric decay	Electron plasma	Acoustic	$\sim n_c$	$\sqrt{\frac{\omega_{pi}^2}{\omega_1 k_2 c_s} \frac{k_2 v_{os}}{4}}$
Filamentation	Modulated light	Zero-frequency acoustic	$\leq n_c$	$\frac{v_{os}^2}{8(T_e/m_i)} \frac{\omega_{pe}^2}{\omega_o}$

$$\frac{1}{2} m_e v^2 \sim \frac{1}{2} m_e \left( \frac{\omega_{pr}^2}{k_p} \right)^2 \sim \frac{m_e c^2}{40} \sim 13 \text{ keV}. \quad (8.27)$$

This increases and becomes closer to 30 keV as the density approaches  $n_c/4$ . Two-plasmon decay, discussed below, can produce smaller wavenumbers, higher phase velocities, and higher-energy electrons. The penetration of such energetic electrons into materials is discussed at the end of Sect. 8.1.5.

Table 8.1 summarizes the instabilities driven by the laser beam in laser plasmas. In all cases, each of the two driven waves is coupled to the laser light wave so as to drive the other driven wave. The density where each instability occurs is given, as is the growth rate in a homogeneous plasma, with the two driven waves indicated by subscripts 1 and 2. In the column giving the growth rates,  $\omega_{pi}$  is the ion plasma frequency,  $\omega_{pi}^2 = 4\pi Z^2 e^2 n_i / m_i$ , with ion density  $n_i$  and mass  $m_i$ , and  $c_s$  is the sound speed, discussed in Chap. 2, but given approximately for a two-fluid plasma by  $c_s^2 = ZT_e/m_i + 3T_i/m_i$ . We now briefly discuss the instabilities shown in the table.

SRS is strongly reduced by collisional effects for short laser wavelengths and by the creation of smooth plasmas with SSD or other methods. *Stimulated Brillouin scattering* (SBS) is a direct analog of SRS in which the second-driven wave is an acoustic wave. It can be strongly reduced by the introduction of bandwidth in the laser and also saturates fairly easily under many conditions. However, both SRS and SBS have at times been observed to convert more than 50% of the laser energy, so they can be enormous.

In addition, there are two *decay instabilities* in which the laser light wave drives two waves within the plasma. These instabilities directly produce no scattered light. *Two-plasmon decay*, being localized at a single density surface in the plasma, saturates fairly easily. However, at this writing it appears to be the largest potential threat to laser fusion by direct laser irradiation. In most modern experiments, collisional absorption of the laser light prevents

the occurrence of the *parametric decay instability* near  $n_c$ . There is a variant of this instability involving a zero-frequency acoustic wave, sometimes described as the *oscillating two-stream instability*. The variant of SBS with a zero-frequency acoustic wave is *filamentation*, which in the nonlinear limit will break the laser beam into discrete intense beamlets. At present it appears that strong filamentation near the leading edge of the plasma may smooth the illumination of the denser regions, by producing beamlets that focus strongly and then spray their energy into a wide angular range. There are actually several types of filamentation. The growth rate shown in Table 8.1 is for ponderomotive filamentation, in which the ponderomotive force causes the plasma motion just as it does in SBS. In colder plasmas, thermal filamentation, involving differential heating, can be important. When  $v_{os}$  becomes relativistic, relativistic self-focusing can arise; this process involves variations in the effective electron mass.

One is led to wonder why all the laser energy is not consumed by SRS or some other instability (most of them grow very quickly on the scale of the laser pulse). The answer to this question is twofold. On the one hand, all the laser energy (more or less) is consumed by these instabilities if the laser wavelength is too long – roughly  $1\ \mu\text{m}$  or longer. This nearly led to the death of the laser fusion program, which initially attempted to use laser wavelengths of 1 to  $10\ \mu\text{m}$ . Such lasers produced some spectacular phenomena but not much progress toward fusion. Lasers with wavelengths in the visible and UV have two advantages –  $v_{os}$  is smaller for a given laser irradiance and collisional effects begin to play a role. On the other hand, so long as  $v_{os}$  is not too large, some of the instabilities saturate at low values and they all are strongly affected by plasma nonuniformity or laser bandwidth.

### 8.1.5 Electron Heat Transport

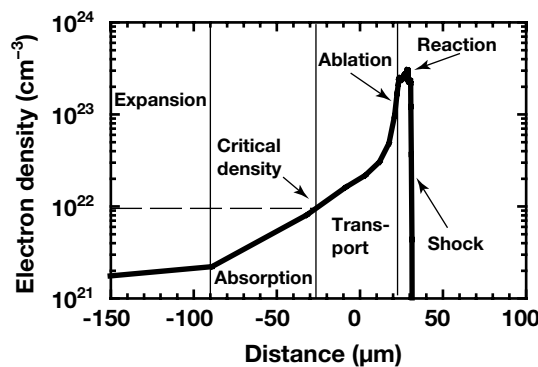
We have emphasized the mobility of electrons by comparison with ions, so that the electrons dominate for example the direct interactions of the laser with the plasma. This might lead one to expect that the thermal electrons would play a dominant role in transporting energy throughout all plasma systems. This, however, is not the case in the systems of interest here. Laboratory systems in the high-energy-density regime are typically so collisional that the electrons cannot manage to escape the ions and do not manage to affect the dynamics very strongly. We saw this quantitatively in Chap. 3. It is also remarkable that there are very few astrophysical systems in which the electrons carry significant heat. The electrons, because of their small mass, are very tightly bound to the magnetic field, and the magnetic field is typically tangled enough to keep them from accomplishing any large-scale heat transport. Systems involving instabilities in loops of magnetic field, which occur for example near the Sun, or involving magnetic reconnection, which occurs in many places, produce bursts of energetic electrons. These electrons in turn radiate, so that the electron radiation can be an important diagnostic

of the phenomena. However, the electrons do not dominate the overall dynamics of reconnecting systems. Likewise, the radiation from electrons has become an important indicator of cosmic-ray acceleration in supernova remnants, but the cosmic rays that actually reach the Earth are almost all ions. So electrons are important. However, with two crucial exceptions, they rarely carry much heat anywhere that matters.

One crucial exception is in the delivery of energy from a laser beam to dense material. This is an essential aspect of inertial fusion and any other high-energy-density experiment using lasers. A second exception is in the loss of energy from the burning region in inertial fusion. For these reasons, it is worthwhile to have some understanding of electron heat transport. Figure 8.7 shows the profile of a laser-irradiated plasma, taken from a computer simulation. The plasma expands but absorbs little laser light in the expansion zone while absorption takes place in the absorption zone, over some range of densities below the critical density. In the transport zone, electron heat transport carries the energy to higher densities above the critical density. Ablation occurs in this high-density material. In reaction to the ablation (or equivalently, in response to the ablation pressure), a shock wave propagates into the material, creating a region of high electron density.

To gain an understanding of electron heat transport, we will discuss here the classic derivation by Spitzer and Harm, after which we discuss the limit when transport becomes too strong for this derivation to be valid. Because this process involves the behavior of individual particles, we use the kinetic description of Sect. 2.4.3 and discuss the behavior of the distribution function  $f(v)$ , normalized for this purpose to that its integral over velocity space gives  $n_e$ .

Suppose we have a plasma with a gentle temperature gradient, a condition we would express mathematically as  $\lambda_{mfP} \ll T_e/|\nabla T_e|$ , where  $\lambda_{mfP}$  is the



**Fig. 8.7.** The electron density profile from a computer simulation, with various regions indicated. The simulation corresponds to a laser wavelength of  $0.35 \mu\text{m}$  and an irradiance of  $10^{15} \text{ W/cm}^2$

collisional mean free path of the electrons. The heat flux,  $Q$ , in a direction  $z$  within such a plasma is found by integrating the energy carried by each particle over the distribution:

$$Q = \int \left( \frac{1}{2} m_e v^2 \right) v_z f(\mathbf{v}) d^3v. \quad (8.28)$$

One can see that  $Q = 0$  if  $f(v)$  is Maxwellian or any symmetric function in  $v_z$ . Actual distributions are often nearly Maxwellian but are seldom fully symmetric, so that heat is usually carried by the electrons. The source of asymmetry can be found by thinking about what would happen if the plasma initially consisted of Maxwellian distributions with a slow spatial variation of the temperature. As temperature increases, the number of hot particles with energies above  $k_B T_e$  increases while the number of cold particles, with lower energies, decreases. As a result, the flow of particles through some point from a Maxwellian distribution in a warmer region will include a surplus of hot particles and a deficit of cold ones. From the opposite, cooler, direction, there will be a deficit of hot particles and a surplus of cold ones. Thus, if the distribution functions were initially Maxwellian but had a varying temperature, they would almost immediately develop non-Maxwellian structure, asymmetric in velocity, and thus able to carry heat.

To find an equation for the heat flow, we assume a plasma of constant density and slowly varying temperature. Note that the temperature gradient defines a unique direction within the plasma, and that the effects of interest involve motions in that direction, which we will call  $z$ . This motivates the definition of a polar angle  $\theta$  with respect to that direction and the expansion of the distribution function by means of Legendre polynomials. Keeping only the first term in this expansion, we have

$$f(v) = f_0(v) + f_1(v) \cos \theta, \quad (8.29)$$

in which  $f_0(v)$  is a Maxwellian and  $f_1(v)$  must be small to justify using only the first two terms in the expansion. In some experiments, such as those of Liu et al. in 1994, two or three coefficients of this expansion have been directly measured.

In addition, it is shown by Shkarofsky et al. that one can accurately treat the effects of collisions in this problem using a simple relaxation rate. Expressing this rate in terms of the electron-ion collision rate  $\nu_{ei}$  defined in Sect. 2.4 gives

$$\left( \frac{\delta f}{\delta t} \right)_C = \frac{-3}{4\pi} \left( \frac{2\pi k_B T_e}{m_e} \right)^{3/2} \frac{\nu_{ei}}{v^3} f_1(v) \cos \theta = \frac{-W}{v^3} f_1(v) \cos \theta, \quad (8.30)$$

which defines the coefficient  $W$  strictly for convenience in what follows. We can substitute (8.29) and (8.30) into (2.77) and keep only the terms proportional to  $\cos \theta$  to obtain

$$\frac{\partial f_1}{\partial t} + v \frac{\partial f_o}{\partial z} - \frac{eE}{m_e} \frac{\partial f_o}{\partial v} = \frac{-W}{v^3} f_1(v), \quad (8.31)$$

in which the electric field must be in the  $z$  direction from the symmetry of the problem. We ignore  $\mathbf{B}$  here, which is justified in the dense target material near the center of the laser spot, where collisions are large and the magnetic field produced by the laser is weak. In steady state, this implies

$$f_1(v) = \frac{-v^3}{W} \left( v \frac{\partial f_o}{\partial z} - \frac{eE}{m_e} \frac{\partial f_o}{\partial v} \right). \quad (8.32)$$

To find  $E$ , one can note that any net flow of charge would cause an electric potential to develop that would then shut off the flow of charge, so in steady state the net current in the  $z$  direction,  $J_z$ , must be zero. This gives

$$J_z = 0 = -e \int_0^\infty v_z f(\mathbf{v}) d\mathbf{v} = -2\pi e \int (v \cos \theta) f_1(v) \cos \theta v^2 \sin \theta d\theta dv, \quad (8.33)$$

from which

$$0 = \int_0^\infty v^3 f_1(v) dv = \int_0^\infty v^6 \left( v \frac{\partial f_o}{\partial z} - \frac{eE}{m_e} \frac{\partial f_o}{\partial v} \right) dv. \quad (8.34)$$

Integrating this, solving for  $E$ , and substituting for the derivatives of  $f_o$ , assumed to be Maxwellian, one finds  $eE = -4k_B \partial T_e / \partial z$ , so

$$f_1(v) = f_o(v) \frac{v^4}{2W k_B T_e} \left[ 8 - \frac{m_e v^2}{k_B T_e} \right] k_B \frac{\partial T_e}{\partial z}. \quad (8.35)$$

The heat flux per  $d^3v$  is proportional to  $v^3 f_1(v) \cos^2 \theta$  per (8.27) and (8.28). Also, the contribution at a large velocity reverses sign compared to that at a small velocity, as we expected from our qualitative analysis. Integrating (8.28) one finds the *Spitzer-Harm heat flux*,  $Q_{SH}$ , as

$$Q_{SH} = \frac{-128}{3\pi} \frac{n_e k_B T_e}{m_e \nu_{ei}} k_B \frac{\partial T_e}{\partial z} = -\kappa_{th} \frac{\partial k_B T_e}{\partial z}, \quad (8.36)$$

in which the heat transport coefficient is  $\kappa_{th}$ . Here  $k_B$  must convert  $T_e$  to the energy units used in  $Q_{SH}$ , and if  $T_e$  is expressed independently in some other units then the heat transport coefficient is  $\kappa_{th} k_B$ . Note that, through  $\nu_{ei}$ ,  $\kappa_{th}$  is proportional to  $T_e^{5/2}$ , so that the heat transported increases very rapidly with temperature. Remarkably, the heat transported is independent of density. The increase in the flux of particles with increasing density is precisely balanced by the increase in collision rate. This model is also known as a diffusion model or a description of diffusive heat transport, because the electrons carry the heat through a diffusive process. Indeed, when one uses (8.36) in a fluid equation for the electron energy, one obtains a diffusion equation (in simple limits). Note that  $k_B T_e / (m_e \nu_{ei})$  has units of a kinematic

diffusion coefficient (e.g.,  $\text{cm}^2/\text{s}$ ). Kruer points out that (8.36) overestimates the transport because the derivation ignores electron–electron collisions, and that this can be approximately corrected for by multiplying  $\kappa_{\text{th}}$  by  $g(Z) = (1 + 3.3/Z)^{-1}$ .

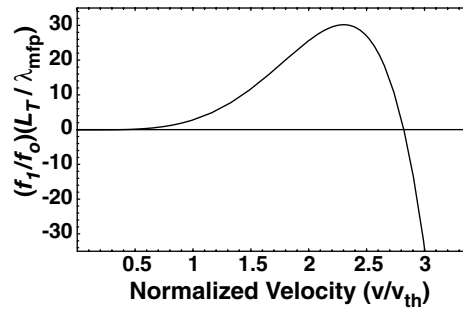
### Homework 8.6

Develop an energy equation for the electron fluid including a Spitzer–Harm heat flux, and show that it is a diffusion equation.

The Spitzer–Harm transport model, adjusted as just described, gives accurate results regarding the heat transport within limits we are about to define, despite the fact that it has some fundamental problems. These relate to the use of the expansion in (8.29), which is only valid if  $f_1/f_0 \ll 1$ . One can show from (8.35) that this ratio is

$$\frac{f_1}{f_0} = \frac{1}{\sqrt{2\pi}} \left( 8 \frac{v^4}{v_{\text{th}}^4} - \frac{v^6}{v_{\text{th}}^6} \right) \frac{\lambda_{\text{mfP}}}{L_T}, \quad (8.37)$$

in which the electron mean free path is  $\lambda_{\text{mfP}} = v_{\text{th}}/\nu_{\text{ei}}$ , the temperature scale length is  $L_T = T_e/|\nabla T_e|$ , and  $v_{\text{th}}^2 = k_B T_e/m_e$ . This particular definition of a thermal velocity  $v_{\text{th}}$  is common in laser–plasma interactions. (In various other areas of physics there is some numerical factor multiplying the right-hand side of this definition.) Figure 8.8 shows this ratio, normalized by  $\lambda_{\text{mfP}}/L_T$ . Two points are important with regard to this figure. First, the expansion of  $f(v)$  always breaks down at a large velocity. The only reason the Spitzer–Harm model ever gives accurate results is that negligible heat is carried by the high-velocity electrons. Mathematically,  $f_0$  decreases much more rapidly than  $|f_1/f_0|$  increases. Second, the heat is carried primarily by the electrons with energies of about three times  $k_B T_e$ , corresponding to the maximum in Fig. 8.8 for  $v/v_{\text{th}}$  between 2 and 2.5. In this range,  $f_1/f_0 \sim 30\lambda_{\text{mfP}}/L_T$ , so the Spitzer–Harm model will be accurate if



**Fig. 8.8.** The normalized perturbation to the distribution function from the Spitzer–Harm theory

$$L_T/\lambda_{mfp} \gg 30. \quad (8.38)$$

Unfortunately, condition 8.1.37 is only sometimes satisfied in laser-irradiated plasmas. As a rule of thumb, this condition is likely to be satisfied for UV lasers, may or may not be satisfied for visible lasers, and is not satisfied for infrared lasers.

The proper way to proceed when (8.38) is not satisfied is to find a better solution to the Boltzmann equation, which may also require finding a more sophisticated expression of the collision term. The standard, more-sophisticated version of the Boltzmann equation is known as the Fokker–Planck equation, but this equation must be solved numerically for all cases of interest. In addition, finding the heat flux from the Fokker–Planck equation is difficult enough that it cannot readily be included in the calculation of the overall dynamics of a laser target. This difficulty strongly motivates the search for simple models that can be of some use.

### Homework 8.7

Determine the range of electron velocities that contribute significantly to the heat flux, by plotting the first-order contribution to the argument of the heat-flux integral (8.28).

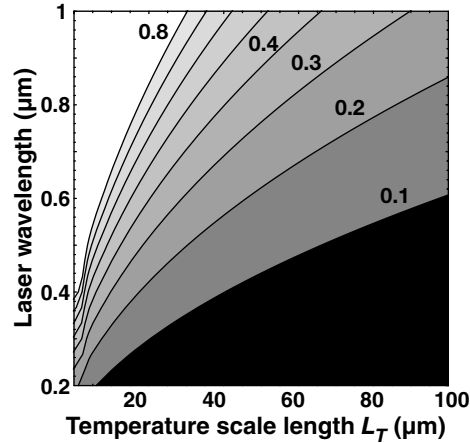
There is such a model, very crude but very widely used, known as *flux-limited transport*. The maximum possible flow of energy would occur if the thermal energy density in the plasma were transported as some characteristic thermal velocity, producing a *free-streaming heat flux*, equal to  $n_e k_B T v_{th}$ . In real physical systems, the maximum heat transported only approaches some fraction of this limit. This is described by introducing a *flux limiter*,  $f$ , so that

$$Q_{FS} = f \times n_e k_B T_e v_{th}. \quad (8.39)$$

Under typical conditions, matching the heat flux found in Fokker–Planck calculations requires taking  $f \sim 0.1$ . Figure 8.9 can be used to determine how likely this is. It plots contours of  $Q_{SH}/Q_{FS}$  as  $L_T$  and the laser wavelength  $\lambda_\mu$  (both in  $\mu\text{m}$ ) vary, for  $T_e = 0.5$  keV and  $Z = 3.5$ , appropriate to the region just beyond the critical surface in low- $Z$  plasmas. The ratio  $Q_{SH}/Q_{FS}$  scales with  $T_e^2$ . Here we see again the point made above – the Spitzer–Harm model may be accurate for visible and UV lasers, but produces too much heat flow for lasers of longer wavelength and in any condition with a very steep temperature gradient.

However, in some historical experiments, especially using infrared lasers and high intensities, the observations could only be explained using  $f$  as small as 0.01. This indicated that additional, noncollisional processes were impeding the heat transport in these cases. Possible explanations of these observations include the effects of laser-generated magnetic fields and of intense acoustic fluctuations produced by laser–plasma instabilities. A very common model





**Fig. 8.9.** Ratio of Spitzer–Harm to free-streaming heat flux for  $f = 1$ . Contours are shown of constant  $Q_{\text{SH}}/Q_{\text{FS}} = (128/(3\pi))(\lambda_{mfp}/L_T)$ , for  $T_e = 0.5$  keV and  $Z = 3.5$ . This ratio scales with  $T_e^2$

in simulations is to set  $f$  at some value and to take  $Q$  to be the smaller of the values given by (8.39) and (8.36). Some more sophisticated computer simulations employ a technique known as multigroup, flux-limited diffusion. In this technique, the electrons are divided into a number of groups, with each group being treated as either diffusive or flux-limited as appropriate.

The models just described are adequate for the calculation of the global influence of the laser on the material it irradiates. For example, they can be used to calculate accurately the production of pressure and shock waves by the laser. However, when it comes to the detailed structure of the target plasma near the irradiated surface, these models give results that are not correct. Unfortunately, although flux-limited transport can provide a quantitative estimate of the local flow of heat, it fails fundamentally to capture the dynamics of heat transport in strong temperature gradients. The reason is that such transport is at root *nonlocal*. The heat deposited at a given point is not determined only by the local conditions, but rather involves particles transported from a range of distances. As a result, the long-term evolution of the plasma differs from what would occur if heat transport were local. In particular, because the more energetic particles have smaller Coulomb cross sections and longer mean free paths, they tend to penetrate deeper into the target and to produce a warm foot ahead of the main heat front. As a result, experiments that hope to do sophisticated physics near the irradiated surface must also use very sophisticated calculations to interpret the results.

In the presence of laser–plasma instabilities, the reality becomes even more complex than this. As we discussed in Sect. 8.1.4, these instabilities produce electrons with energies of tens of keV. Such electrons penetrate far

deeper into materials than the electrons from the thermal population that transport heat inward. As a result, they can *preheat* the initially cold material in an experimental system, altering the initial conditions for the subsequent evolution. The distributions of electrons produced by instabilities are often observed to be exponential in energy, even though they do not arise from waves with a wide distribution of phase velocities. Thus, this is another of the many cases in which a plasma anomalously produces an exponential particle distribution. (The first was Langmuir's paradox, from the early 1900s, relating to the behavior of low density plasma in evacuated chambers.) The exponential energy distribution allows one to assign a temperature to the energetic electrons. These are most often described as *suprathermal electrons* or *hot electrons*, with a temperature designated by  $T_{\text{hot}}$ .

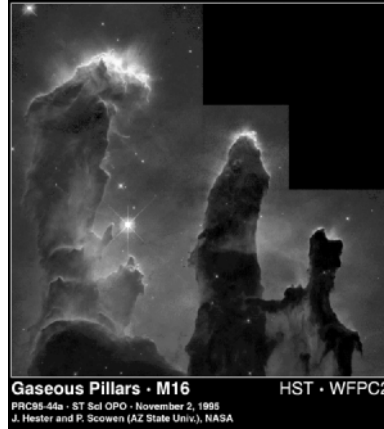
The penetration of such electrons into materials is complex, as they are strongly scattered by any nucleus they get close to. The result is that their penetration is diffusive, with a step size of an electron mean free path and a collision time of the mean free path divided by the velocity. However, the electron velocity decreases steadily as it loses energy in successive collisions and by drag on the electrons in the material. The net effect is that of diffusive penetration with a steadily decreasing diffusion coefficient. As discussed in Rosen et al., the resulting mean electron range,  $x_o$ , in g/cm<sup>2</sup>, is

$$x_o = 3 \times 10^{-6} \left( A/Z_{\text{nuc}}^{3/2} \right) T_{\text{hot}}^2, \quad (8.40)$$

where as before  $Z_{\text{nuc}}$  is the nuclear charge and  $T_{\text{hot}}$  is in keV. For  $A = 2Z_{\text{nuc}}$ ,  $Z_{\text{nuc}} \sim 4$ , and  $T_{\text{hot}} \sim 30$  keV, one finds  $x_o \sim 3 \times 10^{-3}$  g/cm<sup>2</sup>. For plastic at  $\sim 1$  g/cm<sup>3</sup> this is a 30  $\mu\text{m}$  mean penetration depth. This is enough to affect many experiments, and the penetration increases strongly as  $T_{\text{hot}}$  increases. For example, one might irradiate a 1 mm spot with 10 kJ of laser energy for 1 ns, producing an irradiance of  $1.3 \times 10^{15}$  W/cm<sup>2</sup>. If 1% of the laser energy were converted to hot electrons, at 30 keV temperature, that were deposited in a 30  $\mu\text{m}$  layer of plastic, the temperature of this material would increase to  $\sim 30$  eV and its pressure would be  $\sim 10$  Mbars. This is why preheat is very often a concern in laser experiments. (For different reasons, it is also a concern in Z-pinch experiments.)

### 8.1.6 Ablation Pressure

Although much of what we have discussed in the preceding two sections is confined primarily to laboratory environments, the ablation of matter by irradiation is found much more widely. Figure 8.10 shows an image of the Eagle Nebula, justly famous for its dramatic structures, referred to as Elephant Trunks. The Eagle Nebula exists within a star-forming region – a zone with many dense molecular clouds that can provide mass for very large new stars. These new stars are very massive and very bright, with much of their radiation in the deep UV. These photons have energies large enough to directly



**Fig. 8.10.** The Eagle Nebula. From Hubble Space Telescope, WFPC2, Credit: NASA, Jeff Hester and Paul Scowen Arizona State University

ionize the material they encounter, creating an ionization front that is also a region of comparatively high pressure. One hypothesis regarding the origin of the structures is that they might have been produced by Rayleigh–Taylor instabilities that developed when the hot, low-density, ablated plasma began pushing on the cooler, denser plasma behind it.

The intense lasers used in high-energy-density experiments also substantially affect the material they encounter. The photons in these lasers cannot individually ionize the material, but in combination they can and do ionize it. At most relevant irradiances the electric field of the laser can directly ionize the atoms. At lower irradiances the interaction with the target quickly produces plasma, although the mechanisms are more complex. (We leave it as an exercise for the student to find the threshold for direct ionization.) In this section we explore the ablation of matter and discuss its effects.

Even technically informed people often first imagine that the influence of light on a material is primarily due to reflection, as is the case for example in a solar sail. The magnitude of this effect is straightforward to estimate. The pressure produced by reflection,  $P_{ref}$ , is the rate of change of momentum by the reflection. The momentum of a single photon is  $\hbar k$  and the change of momentum upon its reflection is  $2\hbar k$ . The flux of photons,  $F$ , is the irradiance  $I_L$  divided by the energy per photon,  $F = I_L/(\hbar\omega)$ . The pressure is thus  $P_{ref} = 2\hbar kF = 2(k/\omega)I_L = 2I_L/c$ . If the photon is absorbed and not reflected, then the light pressure is half this value. If one chooses instead to take a microscopic view of reflection, one can do so by evaluating the ponderomotive pressure at the reflection surface. A first estimate of this, based on the relations given above, gives  $n_e m_e v_{os}^2/2 = I_L/c$ . One recovers  $2I_L/c$  by taking into account the doubling of the laser electric field during reflection and by taking the time-averaged value of  $v_{os}$ . In practical units,

$P_{ref} = 0.067I_{14}$  Mbars. However, reflection competes with ablation, which nearly always dominates the effect of the laser on the target material at solid density. Let us consider this.

Ablation is the process in which material is heated and then flows away from a surface. By Newton's third law, the surface experiences a reaction force equal to the rate at which momentum is carried away. Let us evaluate this. We will continue to use  $I_L$  for the laser irradiance, although our calculation really involves the absorbed irradiance. In effect, we are assuming the laser energy is completely absorbed. This is quite accurate for UV lasers and becomes less and less accurate at longer laser wavelength. We also assume that half the laser energy heats the plasma at low density, and that half the energy is transported through the critical surface. Using a flux-limited heat transport model, we then have

$$0.5I_L = fn_c k_B T_e \sqrt{k_B T_e / m_e}, \quad (8.41)$$

from which for  $f = 0.1$

$$T_e = 1.7 (I_{14} \lambda_\mu^2)^{2/3} \text{ keV}, \quad (8.42)$$

in which  $I_{14}$  is  $I_L$  in units of  $10^{14}$  W/cm<sup>2</sup> and  $\lambda_\mu$  is the wavelength of the light in  $\mu\text{m}$ . The inward ablation pressure is then equal to the outward flux of momentum through the critical surface. We will assume that this flow occurs at approximately  $\sqrt{2}$  times the sound speed, so

$$P_{abl} = 2M \sqrt{\frac{Zk_B T_e + 3k_B T_i}{M}} \times \frac{n_c}{Z} \sqrt{\frac{Zk_B T_e + 3k_B T_i}{M}}, \quad (8.43)$$

from which

$$P_{abl} = 2n_c k_B T_e \frac{Z+1}{Z} = 8.0 I_{14}^{2/3} \lambda_\mu^{-2/3} \text{ Mbars} \quad (8.44)$$

for  $Z = 3$  and for  $T_i = T_e/3$  (typical of laser coronae). A standard value (see Lindl) of the coefficient which is 8 here would be 8.6, based on computer simulations of the detailed behavior, using flux-limited, multigroup, diffusive electron-heat transport. Before proceeding, take a moment to note that the ablation pressure produced by these systems is an amazing number. Using late-20th century laser facilities, it is straightforward to irradiate a large ( $\text{mm}^2$ ) area with  $0.35 \mu\text{m}$  laser light at  $10^{15}$  W/cm<sup>2</sup>. This produces an ablation pressure of 75 Mbars, which is nearly twice the pressure at the core of Jupiter! The idea that one can produce pressures of many many millions of atmospheres in an Earth-bound laboratory is pretty exciting. Indeed, this is what has made the work described in this book as experimental astrophysics possible.

The exact value of the ablation pressure can be affected by several details. The flow through the critical surface might be at a velocity larger or smaller than  $\sqrt{2}$  times the sound speed (the discussion of rarefactions in Chap. 4 is

relevant here). The temperature may differ from that given by (8.42), as the heat-transport model used is only approximate. The temperature can also be affected by the lateral transport of heat, which is not considered here. Current computer simulations of laser absorption do not necessarily give a correct result for the ablation pressure, because they typically do not include the nonlocal effects of heat transport. For any specific type of experiment, it is often necessary to adjust the simulation parameters based on (direct or indirect) measurements of the ablation pressure, in order to obtain realistic results. Beyond these considerations, magnetic fields may play a role in some cases.

Returning to the comparison with reflection, the formulae we just obtained imply that the laser irradiance would have to reach about  $10^{21}$  W/cm<sup>2</sup> before reflection became dominant over ablation. This is not correct in detail because relativistic effects become important at  $10^{18}$  W/cm<sup>2</sup>, as we will discuss in Chap. 11. An accurate statement is that throughout the non-relativistic regime ablation is more important than reflection in the acceleration of material at or near solid density. This does not, however, imply that reflection and the ponderomotive force have no effects. The ponderomotive pressure grows to equal the plasma pressure at the critical density by the time the laser irradiance reaches  $10^{16}$  W/cm<sup>2</sup>. Above this intensity, the ponderomotive pressure steepens the density profile by pushing the critical surface inward. Even so, this does not prevent the outward flow of material that corresponds to the ablation pressure produced by heating and removal of solid matter at higher density.

Next we discuss the effects of ablation on the target itself. First, the ablation removes mass from the target. The calculations above would imply that the mass ablation rate is

$$\dot{m} = \frac{P_{\text{abl}}}{V_{\text{ex}}} = \frac{\sqrt{2}Mn_c}{Z} \sqrt{\frac{Zk_{\text{B}}T_e + 3k_{\text{B}}T_i}{M}} = 1.7 \times 10^5 I_{14}^{1/3} \lambda_{\mu}^{-4/3} \text{ g cm}^{-2} \text{ s}^{-1}, \quad (8.45)$$

for  $Z = 3$  and  $A = 6$ . The standard scaling from Lindl, which is based on simulations, has a coefficient of 2.6 rather than 1.7 here. The reality is that this coefficient depends on details that are beyond both this calculation and the standard simulations, so that it must be measured in any case where its precise value matters.

Second, the ablation pressure pushes on the target. The immediate effect of the ablation pressure on the target is to launch a shock into it, with consequences that were discussed in Chaps. 4 and 5. A short time after the shock wave has traversed the target, the entire target begins to accelerate. This is essential to laser fusion, for example, which needs initially to deposit as much kinetic energy as possible within moving material. To assess the acceleration of the target, which is like the acceleration of a rocket, we work with equations describing the conservation of momentum for such an object. We take the object to have an initial mass,  $m_0$ , and an instantaneous remaining mass

$m_r = m_o - m_a$ , where  $m_a$  is the total ablated mass. (Equivalently, all the masses may be replaced by mass per unit area.) One can derive the resulting behavior using the conservation of momentum, which must apply to the combined system of rocket and exhaust. Working in the lab frame, when an element of mass  $dm$  is ejected, the exhaust carries away (in the opposite direction) a momentum  $dm(V_{\text{ex}} - V)$ , in which the rocket velocity is  $V$  and the exhaust velocity is  $V_{\text{ex}}$ . The increase in momentum of the remaining rocket mass is  $(m_r - dm)(V + dV) - m_r V$ . Setting these equal and taking the limit that  $dm$  and  $dV$  are infinitesimal (so the product  $dmdV$  is negligible), one finds

$$dmV_{\text{ex}} = m_r dV = (m_o - m_a)dV, \quad (8.46)$$

where  $m_a$  is the variable whose increase is measured by  $dm$ . Integrating from  $m_a = 0$  to some value, and taking  $V = 0$  initially, one finds

$$V = V_{\text{ex}} \ln(m_o/m_r). \quad (8.47)$$

This shows that the velocity of the rocket increases rapidly at first and then more slowly, reaching  $V_{\text{ex}}$  when about 2/3 of the mass has been ablated. It would seem that one could reach an arbitrarily high velocity by ablating nearly all the mass. Unfortunately, the hydrodynamic instabilities discussed in Chap. 5 place a limit on the amount of mass that can be ablated without breaking up the target.

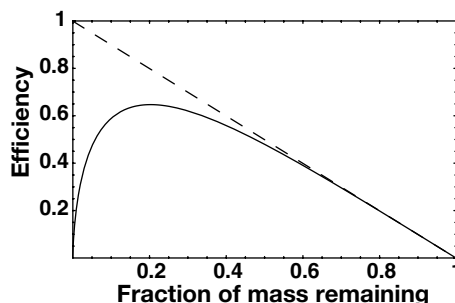
The *ablation efficiency* of such a rocket,  $\epsilon_R$ , is defined as the ratio of kinetic energy of the remaining mass to the total kinetic energy of the rocket plus the exhaust. This is the efficiency of an ideal system in which all the energy was kinetic and no energy went to heat. Thus  $\epsilon_R = m_r V^2 / (m_r V^2 + 2K_{\text{ex}})$ , where  $K_{\text{ex}}$  is the kinetic energy of the exhaust. To find this one must determine the total kinetic energy of the exhaust, from

$$\begin{aligned} K_{\text{ex}} &= \int_0^{m_a} \frac{1}{2} (V_{\text{ex}} - V)^2 dm_a \\ &= \int_0^{m_a} \frac{1}{2} V_{\text{ex}}^2 \left[ 1 + \ln \left( 1 - \frac{m_a}{m_o} \right) \right]^2 dm_a \\ &= \frac{1}{2} m_o V_{\text{ex}}^2 \int_{m_r/m_o}^1 \left[ 1 + \ln \left( \frac{m_r}{m_o} \right) \right]^2 d \left( \frac{m_r}{m_o} \right). \end{aligned} \quad (8.48)$$

This integral evaluates to  $1 - (m_r/m_o)[1 + \ln(m_r/m_o)]$ . Evaluating the efficiency, one finds

$$\epsilon_R = \frac{(m_r/m_o) \ln^2(m_r/m_o)}{1 - (m_r/m_o)}, \quad (8.49)$$

which is plotted in Fig. 8.11. When a small fraction of the mass has been ablated, one can expand the logarithm in (8.48), using the fact that  $m_a \ll m_o$ . Doing this, one finds to the lowest order  $\epsilon_R = m_a/m_o$ , so that the efficiency is equal to the fraction of the initial mass that has been ablated. *This is a*



**Fig. 8.11.** Ideal efficiency of rocket-based acceleration. The dashed line shows the simple estimate for small ablated mass

*very useful result.* One can see that it is reasonably accurate up to about 70% ablated mass fraction, which is far beyond its formal range of validity. The observed efficiency, in laser-plasma experiments and computer simulations, is roughly half this value. This reflects the fact, discussed above, that only about half of the laser irradiance actually reaches the densities above the absorption zone, where the ablation occurs.

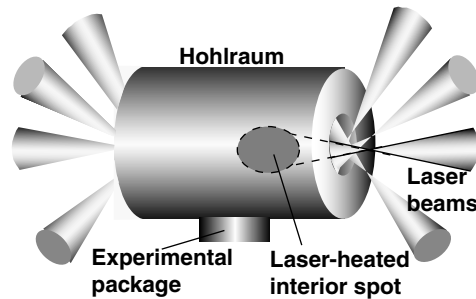
### Homework 8.8

Find the approximate expression for  $\epsilon_R$  to second order in the quantity  $m_a/m_o$ . Plot the corresponding rocket efficiency and the value of (8.49). Discuss the comparison.

## 8.2 Hohlräume

When Max Planck was exploring the fundamental nature of thermal radiation, in the late 19th century, he found it useful to conceive of a completely enclosed volume, within which the radiation field would reach equilibrium with the material in the walls of the volume. He designated such a volume a *hohlraum*. Placing a very small hole in such an enclosure disturbs it negligibly, and the image of an enclosed volume whose radiation emerges from a small hole has become a standard one in the study of blackbody radiation in courses on statistical and thermal physics. Evidently by heating such an enclosure one can increase the temperature of the radiation field. This is the notion behind laser-driven hohlraums, which allow one to produce radiation fields having very nearly the spectrum of blackbody radiation (described as Planckian) at temperatures of millions of degrees. In this section, we explore these devices and some of their fundamental effects.

Figure 8.12 shows an image of a typical hohlraum. They are often cylindrical, as this is easy to manufacture, but they can be made and have been



**Fig. 8.12.** Image of a typical hohlraum. The hohlraum is the cylindrical object, with a visible laser entrance hole to the right. The laser beams strike the interior walls of the hohlraum, as indicated in one case. An experimental package may be located within the hohlraum or on the wall as shown

made with a very wide variety of shapes. Their typical dimensions are a few mm or less. Hohltraums are composed of some high- $Z$  material. Gold is often used because good methods have been developed for producing gold hohltraums. Every so often, someone does an experiment in which the laser beams heat a (thin-walled) hohlraum by striking it on the outside. But it is more typical to do as shown in Fig. 8.12 – to provide one or more laser entrance holes, through which some number of laser beams enter the volume. In many cases beams enter the hohlraum from two directions, which makes it easier to create symmetric conditions inside the hohlraum. Some experiments, however, use only one laser entrance hole and a comparatively short cylinder. Such targets are known as halfraums, which is something of a pun as it confuses hohl with whole. The overall purpose of the laser beam and the hohlraum is to create a useful radiation environment. Z-pinchs, discussed in Sect. 8.3, can also be used to heat hohltraums. Here for the laser-heated case, we discuss the creation of the radiation, the establishment of a radiation environment, what can go wrong, and the application of the radiation to the ablation of matter. Sometimes the radiation is used to heat a sample rather than to ablate it, for example in order to measure the structure of the x-ray opacity.

### 8.2.1 X-Ray Conversion of Laser Light

The heating of a laser-irradiated hohlraum has two aspects – the conversion of the laser energy to x-rays and the transport of those x-rays. The x-rays are initially produced within the *laser spot* where the laser beams strike the gold walls. All of the elements of laser interaction with targets that were discussed in Sect. 8.1 are present in this case, but most of them are unimportant. The reason is that radiation production and radiative energy transport are the dominant physical processes. The high- $Z$  plasma becomes partially ionized, producing ions whose many atomic transitions interact strongly with the



soft x-rays. The first important process is that the electrons heated by the laser excite these transitions, which then radiate, with the net effect that the electrons give much of their energy to soft x-ray radiation. In addition, the transmission of these x-rays through the laser-absorption region is high. (In the language of Chap. 6, the optical depth is small.) As a result, a little less than half the absorbed laser energy is radiated outward and the same amount is radiated inward, into the dense target. The remainder of the energy goes into heating and motion of the gold plasma and to electron heat transport, which also carries energy deeper into the target. The approximate parameters of the underdense plasma that is produced by 1 ns irradiation of a gold target, based on simulations by Kent Estabrook (Drake et al. 1988), are

$$\begin{aligned} Z &= 42I_{14}^{0.067}\lambda_{\mu}^{-0.06} \\ T_e &= 2.3I_{14}^{0.045}\lambda_{\mu}^{0.85} \text{ keV} \\ L &= 114I_{14}^{0.067}\lambda_{\mu}^{-0.06} \mu\text{m} \quad \text{at } 0.1 n_c, \end{aligned} \quad (8.50)$$

in which  $L$  is the scale length of the density profile,  $(d(\ln n)/dx)^{-1}$ . One can show from these relations that the energy flux involved in this plasma expansion is, at  $I_{14} = 1$ , less than 2% of the laser irradiance.

Before continuing the story of the gold plasma, it is worthwhile to draw the contrast with the low- $Z$  plasmas discussed in Sect. 8.1. Laser-irradiated low- $Z$  plasmas reach lower temperatures than high- $Z$  plasmas do, because they transport heat away from the absorption region more quickly. (Recall that the coefficient of heat conduction is proportional to  $1/Z$ .) In addition, a much larger volume of material in the overdense region is heated in the low- $Z$  case, which also produces a lower temperature. The thermal radiation from a layer of temperature  $T$  in a target, as from any object, is the emissivity times  $\sigma T^4$ . Thus, the radiation from the low- $Z$  plasma is very much smaller.

Returning to the gold plasma, the dense material beyond the absorption zone is heated by the energy transported to it, which is dominantly radiation. However, the absorption length for this radiation is very small, so the penetration is diffusive and has the character of a Marshak wave, discussed in Chap. 7. In the limit that the radiation wave is a pure diffusion wave, the heat front would penetrate a distance proportional to  $\sqrt{t}$ . Because of the gradual expansion of the heated material, actual radiation waves penetrate somewhat more quickly. A standard estimate by M.D. Rosen, reported in Lindl, for the penetration depth,  $x_M$ , is

$$x_M = 0.53T_o^{1.86}t_{ns}^{0.75} \mu\text{m}, \quad (8.51)$$

in which  $T_o$  is the temperature of the material at 1 ns, measured in hundreds of eV, and  $t$  is as usual the time, measured in ns here. Note that  $T_o$  is the temperature at the outer boundary of the dense gold, and is not the much larger electron temperature in the absorption region. In this section on hohlraums, we will typically measure temperature in either eV or hundreds

of eV. Thus, the heated layer is of order 1  $\mu\text{m}$  thick – far smaller than the distance material will flow at the sound speed in 1 ns, which is hundreds of  $\mu\text{m}$ .

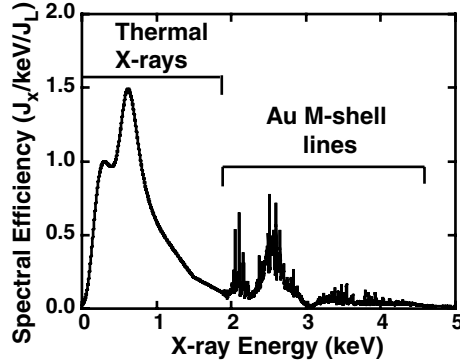
Let us consider the energy balance of a layer of a very opaque material on the surface of a hi- $Z$  object, heated by roughly half the laser irradiance. We can ignore the ions (except for their hydrodynamic momentum, when this matters) because there are many times more electrons. An equation for the balance of the energy flux is

$$\frac{d}{dt}(n_e k_B T_e x_M) = 0.5 I_L - \sigma T_e^4, \quad (8.52)$$

in which the left-hand side accounts for the energy contained in the Marshak wave while the right-hand side accounts for the energy fluxes through the surface. We assume that  $x_M$  varies according to (8.51). Because the diffusion wave penetrates so slowly, we can take the temperature and density to be nearly in steady state at any given time. Solving for the heat flow at 1 ns, one finds that nearly all the input energy is radiated outward, with only a few percent involved in the expansion of the Marshak wave. (Such a solution requires making some assumption about the level of ionization, which is temperature dependent, but the left-hand side of (8.52) is negligible no matter what one assumes.) As an initial estimate of the temperature, one can thus take  $\sigma T_e^4 \sim 0.5 I_L$  to find  $T_e \sim 148 I_L^{0.25}$  eV.

The total radiation from the laser spot will have a blackbody component with a temperature of this order and two harder (i.e., more energetic) spectral components. First, the emission from the hotter, underdense, absorption region will have a more-energetic, but nonequilibrium spectrum. Second, the gold material typically converts a few percent of the laser energy into radiation from M-band transitions (involving lower-state electrons whose principal quantum number is  $n = 3$ ), which have an energy near 2 keV. Figure 8.13 shows a typical spectrum of the x-rays emitted by a gold laser spot.

A rather dramatic aspect of the dynamics just discussed is the fraction of the incident laser energy that is reemitted as x-rays. Our rough estimate, assuming complete radiation dominance, was that half the energy was radiated in the absorption zone and all of the energy that reached high densities was radiated back through the surface. This would correspond to the conversion of 100% of the incident laser energy into x-rays. The observed values are typically in the range of 60 to 80% for irradiances of a few  $\times 10^{14}$  W/cm<sup>2</sup>, with 70% being a good standard estimate. The x-ray conversion efficiency increases with time, which we would expect because the power needed to sustain the Marshak wave decreases with time as it penetrates more slowly (as per (8.51)). The x-ray conversion efficiency is observed to increase at lower irradiances and to decrease as laser irradiance increases. The decrease presumably reflects the increasing importance of laser-plasma phenomena other than radiative energy transport at higher irradiance.



**Fig. 8.13.** A typical spectrum from an irradiated gold surface. Here  $J_x$  is the x-ray energy into  $2\pi$  steradians and  $J_L$  is the laser energy. The irradiance was  $\sim 5 \times 10^{14}$  W/cm<sup>2</sup>, at 0.35  $\mu$ m wavelength. The data below 2 keV is from a 10-channel x-ray diode detector system; above 2 keV it is from two crystal spectrometers. Credit: Robert L. Kauffman

Since the total radiation from the material is roughly 3/2 times the radiation from the Marshak wave, the effective temperature of the laser spot is near  $(3/2)^{1/4}$  times the value from the energy balance calculation, or

$$T_{\text{eff}} \sim 164 I_{14}^{0.25} \text{ eV}. \quad (8.53)$$

At this point we have a hohlraum within which there is an energy source, in the form of x-rays reemitted from the laser spots. This radiation from the laser spots illuminates the interior walls of the hohlraum. This is similar to the irradiation of the laser spots, although the x-rays penetrate to higher densities, where they contribute to heating, driving a Marshak wave, and reemission. But an arbitrary point on the wall of the hohlraum is not just irradiated by the laser spots. It is also irradiated by the other walls of the hohlraum that it can see, which further contributes to the local heating. Because the transit and reemission time is short compared to the timescale for the evolution of the system, we can express the total irradiance reaching the wall,  $I_w$ , as a series:

$$I_w = I_o (1 + \alpha f + \alpha^2 f^2 + \dots) = \frac{I_o}{(1 - \alpha f)} = \frac{\eta f A_L/A_w}{(1 - \alpha f)} I_L = \xi I_L, \quad (8.54)$$

in which  $I_o = \eta f I_L A_L/A_w$  is the average irradiance of the walls due to the laser spots. Here  $\eta$  is the x-ray conversion efficiency,  $A_L$  and  $A_w$  are the areas of the laser spot and the wall, respectively, and  $f$  is the fraction of the emission from the laser spots that reaches other walls. The fraction  $f$  is less than 1 because energy is lost through the laser-entrance hole and to capsules or other objects. The fraction of the radiation incident on a point on the wall that is reemitted by the wall is the albedo  $\alpha$ ; so the fraction of the irradiance

of the walls by the laser spots that is reemitted and is then absorbed again by other walls is  $\alpha f$ . There is in turn reemission of this radiation, and so on. One thus constructs the power series seen in (8.54). This series converges and the final result is that the wall irradiance is proportional to the laser irradiance. The constant of proportionality is  $\xi = \eta f (A_L/A_w)/(1 - \alpha f)$ . We assume here that this applies uniformly and in an averaged sense. In actual hohlraums, the wall temperature varies because the transport of energy is not uniform. We will return to this point in Chap. 9. The behavior of the heated wall is then described by (8.52), with  $0.5I_L$  replaced by  $I_w$  (once again, the energy going into the Marshak wave is negligible). Typical values of the parameters are  $\eta \sim 0.7$ ,  $f \sim 0.9$ , which is basically the ratio of entrance hole and package areas to the total area,  $\alpha \sim 1$ , because the reemission is much larger than the energy penetration into the walls, and  $A_L \sim 0.1A_w$ , because one wants  $A_L$  to be as large as feasible but must also inject the laser beam into the hohlraum and place the laser spot where the experiment demands. Taken together, one finds  $I_w \sim 0.6I_L$ . Thus,  $\xi$  can be near unity.

Take note of this result. Hohlräume are amazingly efficient devices. They can absorb the input energy and keep most of it bouncing around from wall to wall for many bounces. They can irradiate a capsule or an experimental package with an irradiance of soft x-rays that approaches the irradiance of the laser beams themselves. We will return to this point soon. In addition, it is now clear that we were not really justified in treating the laser spot in isolation, because the laser spot is also illuminated by the hohlraum walls and by other laser spots. Indeed, there is some evidence (see Lindl) that the effective conversion efficiency is higher within hohlraums than outside them. But the evidence is complicated and not entirely conclusive. We will leave its exploration to the interested reader.

Solving the approximate solution to the energy transport equation,  $I_w \approx \sigma T_w^4$ , where  $T_w$  is the wall temperature, we obtain an equation for the wall temperature (often called the hohlraum temperature or the radiation temperature,  $T_r$ ):

$$T_w = \left[ \frac{\eta f A_L/A_w}{\sigma (1 - \alpha f)} I_L \right]^{1/4} = 177 [\xi I_{14}]^{1/4} \text{ eV}. \quad (8.55)$$

The units of  $T_w$  are determined by those of  $\sigma$ , which must be consistent with those of  $I_L$ .

Next suppose we place a capsule or other experimental package, of area  $A_c$ , within or on the wall of the hohlraum. The irradiance experienced by such an object is  $I_w$ . The material is usually low- $Z$  as the goal is often to produce ablation pressure, and even targets irradiated for other purposes very often use a low- $Z$  outer layer to prevent motion of the interior materials (such a layer is called a *tamper*). The x-rays readily penetrate through the ionized, low- $Z$  material that has already been heated. (This material may be fully ionized but in any event has few atomic transitions to absorb the x-rays. It

also has a low enough collision rate to be weakly absorbing.) The x-rays are then absorbed in a short distance once they reach the cooler material that has not yet been heated. In most cases the object absorbs nearly all the energy incident upon it and thus contributes to  $f$ . The fraction of the laser power that is delivered to the capsule is

$$\frac{I_w A_c}{I_L A_L} = \frac{\eta f}{(1 - \alpha f)} \frac{A_c}{A_w}. \quad (8.56)$$

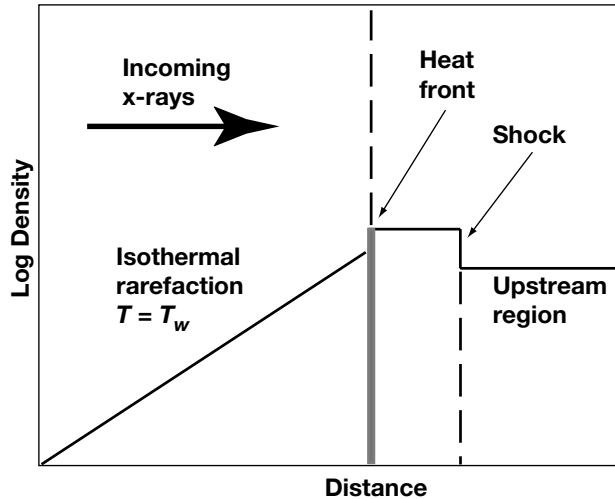
This can be a large fraction but cannot exceed 1 because  $\eta < 1$ ,  $\alpha < 1$ , and  $f < (1 - A_c/A_w)$ . One sees that the larger one makes the capsule, relative to the hohlraum, the more efficiently one will deliver energy to it. Unfortunately, this comes at a cost because larger capsules experience less uniform irradiation, as we will discuss further in Chap. 9.

### 8.2.2 X-Ray Production by Ion Beams

Another approach to heating hohlraums is to use beams of heavy ions. These can in principle be produced with a high efficiency and at a high repetition rate, making heavy ions a plausible source of energy for a power plant based on inertial confinement fusion. But the ion beams are not focusable to the degree that a laser is. An ion-beam-heated hohlraum will be irradiated from a single direction or from two opposing directions. The ion beams will deposit their energy within the outer wall of a hohlraum, or within a beam target placed within the hohlraum. This will produce a hot source that will irradiate those walls exposed to it, beginning the same process of absorption and reemission that occurs in a laser-heated hohlraum. For applications needing uniform x-ray heating of some object, the ion-beam-heated spot will typically be hidden from the object, which will be heated only by radiation from the hot walls.

### 8.2.3 X-Ray Ablation

To complete the basic story of the effect of hohlraums on capsules or packages, consider the ablation pressure produced by the soft x-ray irradiation of low- $Z$  materials in the hohlraum. Figure 8.14 shows the structure of this region, based on work by Hatchett. We will consider the response of some material, known as an *ablator*, to a Planckian spectrum of temperature  $T_w$ . This ignores the effects of M-band or other nonthermal radiation components, which may penetrate deeper into the material, causing heating in advance of the main thermal front, known as *preheat*. Once low- $Z$  material is ionized, its opacity to soft x-rays becomes very small. This is essential and is the reason ablaters rarely contain elements heavier than C. Based on the discussion of Sect. 7.5, one would expect an ionization front to form under such conditions. In the present context, it is important to note that the ionization front can also be



**Fig. 8.14.** The typical structure produced by x-ray ablation of low- $Z$  material

described as a *heat front*, as the x-rays deposit most of their irradiance in a very small volume. In typical cases of interest, the resulting high pressure drives a shock wave into the dense material ahead of the region where the energy is deposited. In order for x-ray irradiation to produce strong shocks, it is important that the shock heating does not render the material transparent to x-rays. For this reason, a pure (frozen) hydrogen layer would also be a poor ablator.

We will analyze this system by treating the heat front as a discontinuity like that of a shock, with the addition that there is a local energy input at the location of the heat front. We assume the system is a planar system. This will reveal the qualitative features of the behavior for spherical objects, such as capsules for inertial fusion, although some of the detailed spatial profiles will change. For such an analysis to be valid, the distance over which the x-rays are absorbed at the heat front must be small compared with the other distances in the problem. After an initial transient period during which the flow is established, this is a valid assumption for soft x-rays incident on low- $Z$  ablaters. In addition, in order to apply a significant ablation pressure, most of the incident x-rays must reach the heat front and must not be absorbed in the ablated plasma. This places an upper limit on the duration over which x-ray ablation is a practical way to apply pressure. However, the absorption in the ablated, ionized plasma is small enough that x-ray ablation is a useful option for high-energy-density applications.

We assume here for simplicity that the material blown off the ablator absorbs enough of the soft x-ray flux to maintain its temperature at  $T_w$ . (This assumption also turns out to be realistic.) As a result, the expansion of this material is an isothermal rarefaction, which we studied in Sect. 4.2.1.

In contrast to the situation with laser-ablated material, which is mostly at much lower densities, electron heat transport is not significant in sustaining the temperature of the expanding material in the present case. We can say that the x-rays propagate in the upstream direction, into the flow emerging from the heat front. The frame of reference of this rarefaction is more complicated than in the previous case, however. The rarefaction relative to a stationary upstream heat source produced fixed parameters at the original material surface and produced a rarefaction wave that penetrated the solid material at its sound speed. In the x-ray ablation case, the heated material flows out of the heat front at some speed while the heat front moves into the upstream material, which is also already moving. This motion can typically be ignored for first estimates, as the energy involved corresponds to a small fraction of the total energy unless the material has all been shocked and a large fraction of the mass has subsequently been ablated (Fig. 8.11).

It proves fruitful to analyze in more detail, the motion in the frame of reference of the heat front, which is analogous to the shock frame used in Chap. 4. First consider the isothermal rarefaction, with reference to the discussion of such systems in Sect. 4.2. Here take  $c_s$  to be the sound speed in the material emerging from the heat front, at initial density  $\rho_2$ . Equations (4.61) and (4.62) then apply, with  $\rho_0$  in those equations equal to  $\rho_2 e^{+1}$  here. One can show from those equations that the power per unit area required to sustain the rarefaction is  $3.4\rho_2 c_s^3$ , with  $1.4\rho_2 c_s^3$  being brought into the rarefaction by new incoming material and  $2\rho_2 c_s^3$  being the heating required to sustain the temperature. The (isothermal) sound speed within this rarefaction corresponds to the temperature  $T_w$ , being  $c_s = \sqrt{(Z+1)k_B T_w / (Am_p)}$ .

### Homework 8.9

By analyzing the isothermal rarefaction, derive the ratio of the energy required to sustain the rarefaction to the energy injected into the rarefaction at the heat front.

A first estimate of the ablation pressure (on the downstream side of the heat front) can be obtained as follows. Some fraction of the power delivered to the ablating object by the soft x-rays is deposited at the heat front, where it provides the flow of energy into the rarefaction at speed  $c_s$ . Balancing these gives the energy density at the heat front,  $\epsilon_{\text{hf}}$ , and the ablation pressure is  $(\gamma - 1)\epsilon_{\text{hf}}$ . The incident x-ray energy flux must also provide the required downstream heating and kinetic energy of the rarefaction and the energy for  $p dV$  work, so we assume that 50% of the incident x-ray energy contributes to the ablation pressure, which gives

$$P_{\text{abl}} = 0.50(\gamma - 1)\sigma T_w^4 \sqrt{\frac{Am_p}{(Z+1)k_B T_w}}. \quad (8.57)$$

Note that this is formally independent of density. However, based on the above discussion one will only find the conditions necessary to create the structure depicted in Fig. 8.14 with certain materials and over a certain density range. If we evaluate this for  $\gamma = 5/3$  and for Be, we find

$$P_{\text{abl}} = 4.4 \left( \frac{T_w}{100 \text{ eV}} \right)^{3.5} = 33 [\xi I_{14}]^{7/8} \text{ in Mbars.} \quad (8.58)$$

The standard estimate, based on simulations, in Lindl would replace 4.4 by 3 in the first part of this equation or would correspond to  $\xi = 0.68$  in the second part. In detail, the ablation pressure depends on materials and varies in time. It is also useful to have an equation for  $c_s$  subject to the same assumptions. This is

$$c_s = \sqrt{\frac{(Z+1)k_B T_w}{Am_p}} = 7.7 \times 10^6 \sqrt{\frac{T_w}{100 \text{ eV}}} = 1.03 \times 10^7 [\xi I_{14}]^{1/8}. \quad (8.59)$$

To analyze the behavior across the heat front in more detail, one needs to work with the generalized versions of (4.5) to (4.7), including the radiation flux on one side of the heat front in the energy balance equation:

$$\rho_1 u_1 = \rho_2 u_2, \quad (8.60)$$

$$\rho_1 u_1^2 + p_1 = \rho_2 u_2^2 + p_2, \quad (8.61)$$

and

$$\left[ \rho_1 u_1 \left( \epsilon_1 + \frac{u_1^2}{2} \right) + p_1 u_1 \right] = F_R + \left[ \rho_2 u_2 \left( \epsilon_2 + \frac{u_2^2}{2} \right) + p_2 u_2 \right], \quad (8.62)$$

in which subscript 1 applies to quantities just upstream of the heat front, subscript 2 applies to quantities just downstream of the heat front, and  $F_R$  is the radiative energy flux deposited at the heat front. This set of equations can be explored to look for heat fronts at which the density increases or drops and to look for heat fronts in which the velocities are subsonic or supersonic (see Hatchett). In all cases  $p_2 = \rho_2 c_s^2$ , and we define  $c_1^2 = p_1/\rho_1$  and  $M_1 = -u_1/c_s$ .

In general, there are two possible relations among the heat front, the shock wave, and the high-density end of the rarefaction. Figure 8.14 shows an expansion heat front, in which the rarefaction ends at the heat front and the shock wave pulls out ahead. In this case  $-u_2 = c_s$  and  $\rho_1 = \rho_o(\gamma+1)/(\gamma-1)$ . The shock pulls ahead of the heat front at less than 1/4 of the shock velocity in the lab frame (for  $\gamma = 5/3$ ), corresponding to small values of  $M_1$  and of  $c_1/c_s$ . For this case we can rearrange the first two of the above equations to show that

$$c_1^2/c_s^2 = 2M_1 - M_1^2. \quad (8.63)$$

This has a meaningful result only for  $0 \leq M_1 \leq 2$ , which is the range over which an expansion heat front can operate.



Alternatively, in the compression heat front case the heat front is supersonic, pulling away from the refraction and coalescing with the shock. Then we take  $c_1$  to be negligible,  $\rho_1 = \rho_o$ , and  $M_2 = -u_2/c_s$ . We then find from the same two equations, for  $M_2 > 1$ ,

$$M_1 = M_2 + 1/M_2. \quad (8.64)$$

In terms of these same variables, one can rewrite the energy flux equation as

$$\frac{F_R}{\rho_o c_s^3} = 2 \frac{M_1}{M_2} + M_1 \left[ \frac{M_2^2}{2} + \frac{\gamma}{\gamma - 1} \left( 1 - \frac{c_1^2}{c_s^2} \right) \right] - \frac{M_1^3}{2}. \quad (8.65)$$

This equation needs to be evaluated as just described in the two distinct cases. One also always has  $\rho_2 = \rho_1 M_1/M_2$  and  $p_1 = \rho_1 c_1^2$ . The expansion heat front develops at low fluxes, corresponding for typical low-Z solids to radiation temperatures of hundreds of eV. The transition to a compression heat front occurs when the left-hand side of this equation equals about 6.

### Homework 8.10

Evaluate the ablation pressure ( $p_1$ ) for the expansion heat front case, assuming the ablator is Be with a density  $\rho_o$  of 1.8 g/cm<sup>3</sup>, as a function of radiation temperature from 100 eV to 300 eV. Compare the result with the value given by (8.58).

We will return to x-ray ablation in Chap. 9, where we discuss its applications to inertial fusion. What is important at present is to see from (8.58) that it is also straightforward to obtain pressures of order 100 Mbars by using lasers to heat a hohlraum. Although the target is more complicated in this case, the resulting irradiation is inherently uniform, unlike that obtained even with smoothed laser beams. As a result, hohlraums are an option worth evaluating for any experiment that would benefit from highly uniform irradiation.

### 8.2.4 Problems with Hohlraums

In addition to their complexity, hohlraums have other limitations that affect their usefulness in certain experiments. We discuss three of these – plasma filling, plasma pressure pulses, and asymmetry – here. Plasma filling is jargon, as any hohlraum will fill with plasma at some density almost immediately. Beyond this, plasma filling refers to the establishment of conditions that permit the excitation of the laser–plasma instabilities discussed in Sect. 8.1.3. In particular, stimulated Raman scattering can become very strong in plasmas with a large volume near  $0.1n_c$  in density. So for this purpose we care about the development of underdense plasma flowing from the walls to the interior of the hohlraum. In detail, the underdense plasma dynamics are complicated.

One has the laser spots, in which the electron temperature is likely to be above 2 keV (8.50) and one has the plasma leaving the radiation-heated walls, with a temperature of order  $T_w$ . In addition, the electrons can carry heat between these regions, so that the temperature may be increased in some places and decreased in others. Moreover, magnetic fields on the one hand may limit this heat transport while collisions on the other hand may limit the effects of magnetic fields. Thus, to avoid difficulties caused by plasma filling may require some difficult work with sophisticated computer simulations. Here we discuss some aspects of plasma filling. A next level of detail is provided by Lindl.

We can gain a qualitative understanding of the main aspects of plasma filling from some simple calculations. Both the laser-heated spots and the x-ray heated walls contribute to the filling of the hohlraum. The x-ray heating takes time to develop, so that early in time the laser-heated spots dominate. Based on (8.50), the sound speed  $\sim \sqrt{ZT_e/M}$  is above  $3 \times 10^7$  cm/s, and the hohlraum radius is below 1 mm, so the timescale for filling is less than 3 ns. This is the regime in which hohlraums irradiated with infrared laser light operate. Once the walls of the hohlraum heat sufficiently, the sound speed of their plasma is about  $\sqrt{10}$  smaller. However, because the plasma expands from a much higher initial density, it needs to expand for a much shorter time to produce effects throughout the hohlraum. We can describe the expansion as an isothermal rarefaction (Sect. 4.2.1), so that the density is  $n = n_s e^{-x/(c_s t)}$ , where  $n_s$  is the electron density near the solid material, which we will take to be  $10^{24}$  cm $^{-3}$ , and we take  $c_s$  from (8.59). We ask how long it will take for the plasma density at the center of the hohlraum from one segment of wall to reach  $0.001n_c$ . This is about 14 e-foldings below solid density, and when this occurs there is a large volume in which the density is near  $0.1n_c$ , which is about eight e-foldings below solid density. For a hohlraum radius  $R_{\text{mm}}$  in mm we find

$$t = 0.61 \frac{R_{\text{mm}}}{\sqrt{T_w/100 \text{ eV}}} \text{ ns}, \quad (8.66)$$

with  $\rho$  in g/cm $^3$  and  $T_w$  in eV. For typical parameters, this gives times below 2 ns. This second model is more relevant to hohlraums irradiated with optical and UV lasers. In addition, there is some potential for plasma to fill the entrance hole through which the lasers penetrate the hohlraum. However, both the heating by these lasers and their combined ponderomotive force tend to resist this.

Typical experiments using hohlraums last at least a few ns. So it should be no surprise that plasma filling can be a significant factor. Both the energetic electrons and the laser scattering that laser-plasma instabilities can produce are of concern for experiments and especially for laser fusion. We discuss some of this further in Chap. 9. It is worth noting that these problems are very much larger in experiments with infrared lasers, because they have higher

oscillating velocities and smaller collisional effects at a given irradiance. In fact, the production of hot electrons as a result of plasma filling was both a severe and an unknown problem in early laser fusion experiments. The U.S. laser fusion effort was nearly canceled before the scientists involved discovered and demonstrated the effects of plasma filling. Some observers, including the present author, believe that this problem was not found sooner because the fact that the phenomena responsible for it, though known to exist, were not incorporated in the primary computer code used to model the experiments. It is very easy to mistake a limited computer model for reality. This is a cautionary note for students of this or any other science.

The second phenomenon worth discussing is the development of plasma pressure. A planar isothermal rarefaction, like that described above, has a velocity  $v = c_s \zeta$  with  $\zeta = x/(c_s t)$  and a mass density  $\rho = \rho_o e^{-\zeta}$ . The ram pressure,  $\rho v^2$ , of such a plasma is  $\rho_o c_s^2 \zeta^2 e^{-\zeta}$ , which has a maximum of  $\sim \rho_o c_s^2 / 2$  when  $\zeta \sim 2$ . Thus, for plasma from a gold wall with  $\rho_o \sim 19 \text{ g/cm}^3$  and  $c_s$  as given above, the maximum ram pressure is

$$P_{\text{ram}} \sim \rho_o c_s^2 / 2 = 1.3(T_w / 100 \text{ eV}) \text{ Gbars}, \quad (8.67)$$

in which  $T_w$  is in eV. This is already an enormous pressure, but if the hohlraum wall stayed hot long enough, it would be an underestimate, because the pressure at the center of the hohlraum would be increased by convergence effects. The nominal time required for the point with  $\zeta = 2$  to reach the center of the hohlraum would be  $t_{\text{maxp}} = R/(2c_s)$ , which is

$$t_{\text{maxp}} = 8.4 R_{\text{mm}} / \sqrt{T_w / 100 \text{ eV}} \text{ ns}. \quad (8.68)$$

This typically would be a few ns, but the heating pulse for hohlraums less than 1 mm in radius is typically of order 1 ns. So the pressure actually produced when the gold plasma converges is smaller than that indicated by (8.67). Nonetheless, it can be very large compared to the ablation pressure on a package irradiated by the hohlraum. What happens on this timescale in an actual hohlraum, as elucidated by Hurricane et al., is that a large pressure pulse propagates outward from the center of the hohlraum. The implication is that any experiment using x-ray ablation to drive a package has only a limited potential duration before the effects of the ablation are overwhelmed by the pressure pulse that will eventually follow. This places a real limitation on the design of experiments to examine the long-term evolution of hydrodynamic phenomena.

### Homework 8.11

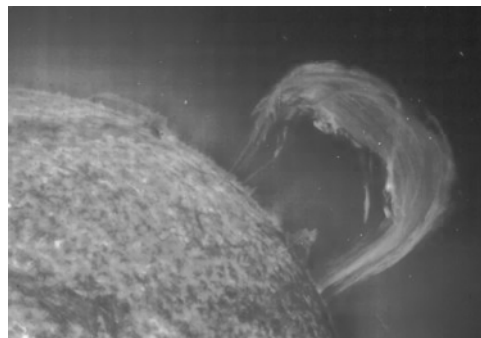
Assume that a hohlraum of 1 mm radius is heated for 1 ns at a temperature of 200 eV. Estimate the pressure produced at the center of the hohlraum when the plasma expanding from the gold walls reaches the axis. (Note: this is not

an application of (8.67). Instead, you will need to think about the rarefaction produced during the heating pulse.)

The final limitation of hohlraums, discussed in much more detail in Chap. 9, is that in practical experiments they do not produce isotropic irradiation. Using the equations of this section, one would be led in designing almost any experiment to maximize  $A_L/A_w$ . The result is that a given point on an experimental package sees larger x-ray emission (a hotter environment) from the laser spots. It also sees little x-ray emission (a colder environment) from the laser entrance holes. As one moves around the surface of a laser capsule, or even of a planar target, this can produce significant variations in the x-ray flux. This in turn can produce asymmetric pressures on the surface of the irradiated object. As a result, efficient designs require the use of viewfactor codes, which integrate a specified x-ray source distribution over solid angle at each point on a target, to assess quantitatively the radiation uniformity.

### 8.3 Z-Pinches and Related Methods

Figure 8.15 shows a loop of magnetic field near the surface of the sun, visible in soft x-rays because the plasma it contains is much hotter than the surrounding plasma. One contributor to such heating is the pinch force, which has the amazing effect of causing any channel of plasma that carries current to contract. As we will see, the pinch force is one consequence of the  $\mathbf{J} \times \mathbf{B}$  force. Since most astrophysical systems include magnetic fields, whose motions induce the flow of current, this force is present at some level in many circumstances. In the laboratory, modern *pulsed power* devices can deliver voltages of  $>1$  MV for  $>100$  ns. As we will see, this is enough to produce high-energy-density conditions. We will first discuss the traditional approach to this end, known as the Z-pinch. Later, we will discuss an alternative use



**Fig. 8.15.** A loop of hot plasma, held in place by a magnetic field, near the surface of the Sun. Credit: SOHO - EIT Consortium, ESA, NASA

of the same type of pulsed-power system – the magnetically launched flyer plate.

### 8.3.1 Z-Pinches for High-Energy-Density Physics

Andre-Marie Ampere showed that current-carrying wires exert forces on one another. We can revisit this briefly to set the stage for the more complicated discussions that follow. Imagine an infinitely long, straight wire carrying a steady current. If one integrates Ampere's law over the area of a surface centered on the wire and bounded by a circle, and applies Stoke's theorem, one finds, using SI units,

$$\mu_0 I_1 = \mu_0 \int_{A_1} \mathbf{J}_1 \cdot \hat{\mathbf{n}} dA = \oint_{\ell_1} \mathbf{B}_1 \cdot d\boldsymbol{\ell} = 2\pi r_1 B_1, \quad (8.69)$$

where  $\hat{\mathbf{n}}$  is a unit vector normal to the surface and subscript 1 represents the properties produced by the wire. One can show, from symmetry and the absence of magnetic monopoles, that  $\mathbf{B}_1$  is purely azimuthal. The direction of  $\mathbf{B}_1$  is given by the right-hand rule because the line integral is by convention always taken in the counterclockwise direction as viewed from the direction toward which  $\hat{\mathbf{n}}$  points. If there is a thin, parallel wire some distance  $R$  from the first wire, the force per unit length on this second wire,  $\mathbf{F}_2$ , due to the magnetic field from the first wire, is

$$\mathbf{F}_2 = \int_{A_2} (\mathbf{J}_2 \times \mathbf{B}_1) dA = \mu_0 \frac{I_1 I_2}{2\pi R} (-\hat{\mathbf{r}}_{12}), \quad (8.70)$$

where subscript 2 refers to the second wire and  $\hat{\mathbf{r}}_{12}$  is a unit vector from the first wire to the second wire. The minus sign means that the force is attractive (for  $\mathbf{J}_1 \parallel \mathbf{J}_2$ ), as one can verify from the right-hand rule. The general point is that parallel currents attract. This has the implication that any compressible medium carrying current will tend to contract.

This fact enables one to create a type of device known as a Z-pinch. A Z-pinch uses an axial current (in the  $z$  direction in a standard Cartesian coordinate system) to create a pinch force, with the aim of producing a high-temperature plasma. Some of the early approaches to magnetic fusion were based on this principle. One can find equilibria in which the inward pinch force, produced by current in a plasma, balances the outward pressure of the plasma. Unfortunately, these equilibria are not stable; if they were stable then we might indeed have had fusion power plants in the 1960's. Most modern Z-pinches are so-called *fast* Z-pinches, in which a rapidly rising current causes the implosive contraction of material. The imploding material is accelerated and then converts the kinetic energy of implosion to heat when the material stagnates on axis. Such implosions can occur with varying relative amounts of heating *versus* acceleration. As we shall see, the implosions of interest to high-energy-density physics are violent indeed. Such implosive pinches avoid the

slowly growing instabilities that plague equilibrium pinches. However, they create the transient growth of the Rayleigh–Taylor instability, discussed in Chap. 5, and this imposes some limitations on their operating range. Here we discuss the basic aspects of the implosion of a fast Z-pinch. More extensive discussions of Z-pinch physics can be found in the book, *Physics of High-Density Z-Pinch Plasmas*, by Liberman, De Groot, Toor, and Spielman, and in the article in *Reviews of Modern Physics* by Ryutov, Derzon, and Matzen.

We begin by considering the self-consistent behavior of a long cylindrical shell with a uniform current density in the  $z$  direction, given initially as a function of radius  $r$  by

$$\mathbf{J} = \mathbf{J}_o \text{ for } r_1 < r < r_2; \quad \mathbf{J} = 0 \text{ otherwise.} \quad (8.71)$$

Such a current produces no magnetic field in the  $z$  direction, and one can show from this and the absence of magnetic monopoles that there is no radial component of magnetic field. Applying Ampere’s law to the interior of the cylinder, there is also no azimuthal magnetic field inside the shell, so  $\mathbf{B} = 0$  there. Within the shell itself, Ampere’s law in SI units implies

$$\oint \mathbf{B} \cdot d\boldsymbol{\ell} = 2\pi r B = \mu_o \int \mathbf{J} \cdot \hat{\mathbf{n}} dA = \mu_o J_o \pi (r^2 - r_1^2) \approx \mu_o J_o \pi (2r\delta r), \quad (8.72)$$

where  $\delta r = r - r_1$ . The equation of motion relates the acceleration of the fluid to the inward force density  $\mathcal{F}$ , and is

$$\rho \frac{\partial \mathbf{u}}{\partial t} = \mathcal{F} = \mathbf{J} \times \mathbf{B} = -\hat{\mathbf{r}} \frac{\mu_o J_o^2}{2} \frac{(r^2 - r_1^2)}{r} \approx -\hat{\mathbf{r}} \mu_o J_o^2 \delta r, \quad (8.73)$$

where  $\hat{\mathbf{r}}$  is a unit vector in the radial direction. From this, the equation of motion for the radial acceleration of a fluid element is

$$\rho \ddot{r} = -\frac{\mu_o J_o^2}{2} \frac{(r^2 - r_1^2)}{r} \approx \mu_o J_o^2 \delta r, \quad (8.74)$$

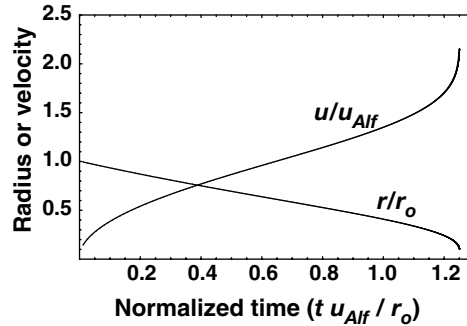
in which  $\rho$  is the mass density. One can integrate this over the cross-section of the shell to find the total inward force per unit length and thus the approximate equation of motion for the entire shell

$$\hat{m} \ddot{r} = -\frac{\mu_o I^2}{4\pi r}, \quad (8.75)$$

in which  $I$  is the total current and  $\hat{m}$  is the mass per unit length, and again this is in SI units. For constant current, one can integrate this equation to obtain

$$u_r^2 = \frac{\mu_o I^2}{4\pi \hat{m}} 2 \ln \left[ \frac{r_o}{r} \right] \quad (\text{SI}) = \frac{I^2}{c^2 \hat{m}} 2 \ln \left[ \frac{r_o}{r} \right] \quad (\text{cgs}) = u_{\text{Alf}}^2 2 \ln \left[ \frac{r_o}{r} \right]. \quad (8.76)$$

in which,  $u_{\text{Alf}}$  is the velocity of Alfvén waves at the initial outer edge of the pinch. (We won’t concern ourselves with Alfvén waves here, as they will



**Fig. 8.16.** Behavior of a constant-current Z-pinch implosion, showing radius as a fraction of  $r_o$ , and with inward velocity normalized to  $u_{AIf}$ , versus time, normalized to  $r_o/u_{AIf}$

not come up again. You can read about them in any plasma physics text.) One can in turn integrate (8.76) to find the time  $t_{imp}$  at which the implosion reaches a radius  $r$ , which is

$$t_{imp} = \frac{r_o}{u_{AIf}} \sqrt{\frac{\pi}{2}} \operatorname{erf} \left[ \sqrt{\ln(r_o/r)} \right], \quad (8.77)$$

where  $u_{AIf}$  is defined by (8.76) and erf is the error function. In practical units,

$$u_{AIf} = 3.3 \times 10^6 I_{MA} / \sqrt{\hat{m}(\text{mg/cm})} \text{ cm/s}, \quad (8.78)$$

where  $I_{MA}$  is the current in MA and the units of  $\hat{m}$  are shown. The behavior produced by (8.76) and (8.77) is shown in Fig. 8.16. One can see that the shell of current moves inward slowly at first, and that only late in the implosion does the acceleration greatly increase. The development of an actual pinch implosion, with a gradually increasing pinch current, is even more gradual. An essential phenomenon, included in the equation of motion only by the boundary condition that  $u_r = 0$  at  $r = 0$ , is that the pinch material must stagnate before it reaches the axis of the cylindrical shell, where the incoming matter will accumulate. In the simplest conception, the pinch material is accelerated inward, gaining kinetic energy, and is shocked and compressed as it stagnates when it symmetrically reaches the axis, converting the kinetic energy into thermal energy and later into radiation and an outward expanding plasma.

The convergence,  $r_o/r$ , enters into these equations. We can estimate a plausible radius at stagnation  $r_s$  in order to determine the maximum convergence. If, for example, one used a metallic shell whose density was  $10 \text{ g/cm}^3$ , with an initial mass of  $1 \text{ mg/cm}$ , and the imploded and shocked material on the axis had a density of  $40 \text{ g/cm}^3$ , all of which are plausible numbers, then the radius of the imploded pinch would be about  $30 \text{ }\mu\text{m}$ . If the initial

radius of the shell were 1 cm, then one would have  $r_o/r_s \sim 300$ . In actuality, instabilities limit the degree of implosion, which typically ends at  $r_o/r_s \sim 10$  to 20. Note that differences of a factor of 2 in  $r_o/r_s$  have a very small effect on the final implosion velocity because the convergence enters into the logarithm. Indeed, even increasing the convergence to  $r_o/r_s \sim 300$  would increase  $u_r$  by less than 50%. Pinch research in the 1950's was focused on creating a high-density matter where fusion would occur, for which high convergence is essential. Modern applications of Z-pinches to high-energy-density physics more often depend primarily on energy (and on its efficient conversion to radiation), and so are less sensitive to convergence.

Returning to the simple model of (8.77), and using  $r_o/r_s = 10$ , one can find the kinetic energy of the pinch material just at stagnation. Remarkably, this quantity depends only on the pinch current. It is

$$\text{K.E.} = 2.3\hat{m}u_{\text{Alf}}^2 = 2.3\frac{\mu_o I^2}{4\pi} \text{ (SI)} = 2.3\frac{I^2}{c^2} \text{ (cgs)} = 2.3I_{\text{MA}}^2 \text{ kJ/cm.} \quad (8.79)$$

With the currents above 20 MA that are now feasible, this energy can exceed 1 MJ/cm. The total energy of the pinch material may be higher than this, because it has been heated by Joule heating (i.e.,  $\mathbf{J} \cdot \mathbf{E}$ ) and by compression (i.e.,  $p dV$  work), but it also may lose energy to radiation before the end of the implosion. Assuming that the heated material stagnates symmetrically, all the remaining energy is momentarily converted to heat. The energy of stagnation initially develops in the ions, as they carry the kinetic energy, and is then transferred by collisions to the electrons. Once the temperatures have equalized, the heating produced by the kinetic energy of (8.79) gives a temperature in eV,  $T_{\text{eV}}$ , of

$$T_{\text{eV}} = \frac{\text{K.E.} \times Am_p}{k_B \hat{m}(Z+1)} = 20 \frac{I_{\text{MA}}^2}{\hat{m} \text{ (mg/cm)}} \left( \frac{A}{Z+1} \right) \text{ eV.} \quad (8.80)$$

In evaluating this equation, one may have to allow for the dependence of  $Z$  on  $T_e$  (Chap. 3). Such dense matter, at typical stagnation temperatures above 1 keV, is a very intense radiator. Note that one can adjust this temperature, to seek an optimum for some purpose, by adjusting the mass per unit length.

As was mentioned above, the current in an actual pinch is not constant. In fact, it often has a sinusoidal profile in time. This leads the implosion to develop more gradually than Fig. 8.16 shows. However, there are only limited analytic solutions for the motion of the pinch with more realistic, time-dependent current profiles. To make matters even more complex, the current is not fundamentally independent of the pinch and its dynamics. The pulsed-power machine provides a time-dependent voltage pulse to its load, which in this case is the pinch and the supporting structures for the pinch. The principal limitation on the current through the pinch is the inductance of the pinch itself. Thus, more accurate pinch modeling specifies the time-dependent voltage supplied to the pinch, determines the current by



calculating the instantaneous induction of the pinch and solving a circuit equation, and simultaneously solves an equation like (8.75) for the implosion of the pinch itself. One result of such circuit modeling has been that pinches have become shorter in recent years, and often now have a height that is only a fraction of their initial radius. By reducing the height, one can decrease the inductance and thus increase the current.

This has an application when one considers the duration of the pinch implosion. In actual Z-pinches the implosion time must be matched to the duration of the voltage pulse that can be produced by the pulsed-power system. Using (8.77) and (8.78), for initial radii of a few cm, masses within a factor of five times 1 mg/cm, and currents within a factor of 3 of 10 MA, one can see that the implosion time is within an order of magnitude of 100 ns. Of course, the reasoning actually must be done in reverse. Given the ability to deliver a voltage pulse of some duration, one must choose the mass and radius of the Z-pinch load to obtain an implosion of the same duration with the current that results. Let us explore this further.

The inductance  $L$  of a current-carrying cylinder of height  $H$  and radius  $r$ , with the return current carried at some larger radius  $r_{\text{ret}}$ , is easily found to be

$$L = \frac{\mu_0}{2\pi} H \ln \left( \frac{r_{\text{ret}}}{r} \right), \quad (8.81)$$

so with an available voltage  $V$  of duration  $\tau$  we solve  $V = LdI/dt \sim LI/\tau$  with  $\tau = t_{\text{imp}}$  from (8.77) to find

$$\begin{aligned} I_{MA} &= 10^{-4} \sqrt{\frac{V_{MV} \sqrt{\hat{m}}(\text{mg/cm})}{\sqrt{2}\mu_0^{3/2}}} \left( \frac{r_o}{H} \right) \frac{\text{Erf} \left[ \sqrt{\ln(r_o/r_s)} \right]}{\ln(r_{\text{ret}}/r)} \\ &= 1.47 (\hat{m}(\text{mg/cm}))^{1/4} \sqrt{V_{MV} \frac{r_o}{H}}, \end{aligned} \quad (8.82)$$

in which  $r_o$  and  $H$  must be in the same units and the second result is obtained using  $r_{\text{ret}}/r = 10$  and  $r_o/r_s = 20$  but depends weakly on the exact values of these ratios. We see that multi-MA currents are straightforward to achieve. Equation (8.82) makes it seem as though one can increase  $I_{MA}$  just by increasing the mass of the load. However, another look at (8.77) shows that to keep the implosion time  $\sim \tau$  one must keep the ratio  $I_{MA}/(r_o\sqrt{\hat{m}})$  constant. Under this constraint (8.82) implies that the ratio  $V_{MV}/(r_o\sqrt{\hat{m}})$  must also be constant. Taking these constraints together, for fixed  $V_{MV}$  one can increase  $I_{MA}$  by decreasing  $H$  while increasing  $r_o\sqrt{\hat{m}}$  by the same factor. Thus, one can only increase  $I_{MA}$  and thus the kinetic energy in the imploding material in proportion as one can increase  $V_{MV}$  or decrease  $H$ , while making whatever changes in  $r_o\sqrt{\hat{m}}$  are required to keep the implosion time  $\sim \tau$ . One can also see from these relations that if one desires to increase the temperature of the heated material, by decreasing  $\hat{m}$  according to (8.80), then to also keep the

$t_{\text{imp}}$  and  $\tau$  well matched requires that one increase  $r_0^2$  in proportion to the decrease in  $\hat{n}$ . The practical limits associated with this have to do with one's ability to produce a uniform, current carrying shell.

The applications of Z-pinches that are most relevant to high-energy-density physics at present involve the production of radiation. (Ryutov et al. discuss some other possible applications in their review paper.) The first such application is to use the pinch to produce the largest possible soft x-ray energy by blackbody radiation from hot, dense matter. For this purpose one implodes a high-Z material, typically tungsten (W). After the stagnation, the plasma both radiates and expands. We can evaluate the ratio of the blackbody radiation,  $\sigma T^4$ , to the power involved in plasma expansion  $(\rho/Am_p)k_B T c_s$  as follows:

$$\frac{\sigma T^4}{[Z\rho/(Am_p)]k_B T c_s} = 35 \frac{T_{\text{keV}}^{2.5}}{\rho} \left(\frac{A}{Z}\right)^{3/2}, \quad (8.83)$$

where  $T_{\text{keV}}$  is the temperature in keV. This implies that radiation will be strongly dominant above some temperature of order 1 keV. Such Z-pinch radiation sources are often produced within high-Z hohlraums, similar to those discussed in Sect. 8.2. These hohlraums can confine the pinch radiation and sustain for some time a high-temperature, thermal radiation environment. They have been used to irradiate packages either mounted on their walls, to study ablatively driven phenomena or radiation flow, or mounted within the hohlraum, to study radiation transport or photoionization effects. They have also been used to irradiate capsules for inertial-confinement-fusion research.

A second radiation-related application of Z-pinches is to use them to produce x-ray line radiation. For this purpose, one uses wires of a material whose  $K_\alpha$  x-rays have an energy of a few keV, such as titanium. The radiation balance is not as easy to estimate as it is in the case of blackbody radiation. The efficiency is large enough to produce useful yields for practical applications.

In order to maximize the power radiated by a Z-pinch during stagnation, one must maximize the stagnation power,  $P_s$ . Because the plasma expands during the implosion, the duration of the stagnation can be expected to scale with the implosion time, which by design one makes equal to the duration  $\tau$  of the voltage pulse. Thus, since the kinetic energy is proportional to  $I_{\text{MA}}^2$ ,

$$P_s \propto \frac{H I_{\text{MA}}^2}{\tau} \propto \frac{H I_{\text{MA}} V_{\text{MV}}}{L} \propto \frac{V_{\text{MV}}^{3/2} (\sqrt{\hat{n} r_0})^{1/2}}{H^{1/2}} \propto \frac{V_{\text{MV}}^2}{H}, \quad (8.84)$$

so for fixed pulsed-power parameters one can increase the stagnation power only by decreasing  $H$ . There are limits to this, as the implosion will be compromised near the ends of the pinch. Nonetheless, at around the turn of the century pinches less than 1 cm high were imploded with good results on the Z-device.

**Homework 8.12**

While one can vary the properties of the Z-pinch load from one experiment to the next, one can modify the pulsed-power device itself on a somewhat longer timescale. Such devices are typically characterized by the number of Volt-Seconds they can produce, and operate so that  $V\tau = \text{constant}$ . First, consider and then explain why Volt-Seconds is a reasonable way to characterize a pulsed-power device. Second, using the scaling relations developed in Sect. 8.3.1, discuss how to optimize the stagnation power for a device with  $V\tau = \text{constant}$ .

For many years the ability of Z-pinchs to actually produce x-ray radiation fell far below the expectations one would have from the scalings discussed above. This changed dramatically during the 1990's, with the development of Z-pinchs that use a load composed of hundreds of fine wires (typically  $\sim 10 \mu\text{m}$  dia.). All the previous approaches, which included wire arrays with fewer, thicker wires, solid cylindrical conducting shells (known as liners), and various schemes involving gas, performed far less well. It is tempting to infer that the use of many wires finally produced a structure resembling the uniform plasma shell of our simple modeling, while all the previous methods produced a less uniform plasma that did not stagnate as effectively. The success with arrays of fine wires lead to a large increase in the x-ray yield from such devices.

The observed implosion time of pinchs using arrays of many wires, determined from the timing of the x-rays produced at stagnation, is typically in excellent agreement with the time predicted by modeling of a uniform plasma shell. This has led some authors to conclude that the wires in such wire arrays do expand and merge so as to produce such a plasma shell. Further support for this conclusion has come from MHD modeling, which can reproduce the stagnation time and the size of the resulting plasma, although to do so one must assume very large initial perturbations to seed the Rayleigh–Taylor instability. However, the issues are not so simple and the evidence is rather complex. There are two ways that an array of wires can develop a comparatively uniform implosion. The simplest notion is that the wires explode into small plasmas and that if the wires are close enough then these plasmas will connect and the current will flow uniformly in azimuth. However, the evidence indicates that the wires, and especially those of materials such as Al and W that perform well, do not initially explode.

Instead the wires ablate because the current flowing on their surfaces heats them so strongly that plasma flows away from their surfaces. This can create a structure in which the plasma and magnetic field have merged but the wire cores remain. The likely behavior of the magnetic field is complex – the field is not frozen in. The magnetic diffusion time, from (2.72) and Fig. 2.2, for distances of fractions of a mm, with electron temperatures that are not so many eV, may be of order 1 ns and certainly is much smaller than the implosion time of order 100 ns. The field is initially strongest near the wire surfaces and will tend to diffuse outward into the developing plasma, where

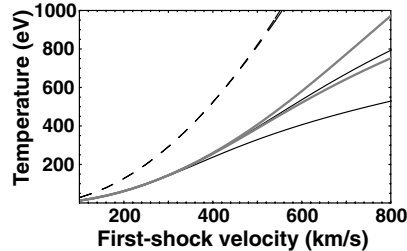
it can merge to form a more symmetric structure. The diffusion of the field also corresponds to a diffusion of the current, so the plasma experiences a  $\mathbf{J} \times \mathbf{B}$  force that accelerates it inward. In typical cases it appears that of order half the wire mass may be accelerated inward before the final phase of the implosion. Some magnetic field will be carried with such plasma, and more may diffuse into it.

It is unclear at this writing how much force is actually delivered to the wire cores, and whether the cores themselves actually move. On the one hand, if the wire cores eventually become small enough to explode into plasma, then they probably do move. On the other hand, there is some evidence that the wire ablation ceases once the wires develop gaps, which are likely the result of MHD instabilities in the wires themselves. At this time, there is no longer a source of plasma to sustain the current and magnetic field at the edge of the array, and the outer edge of the plasma will implode inward, sweeping up the interior mass in what is usually described as a snowplow implosion. There may be a fundamental underlying cause, but at the moment it seems amazing and fortuitous that modeling of this more complicated process produces implosion times that are nearly identical to those produced by a uniform plasma shell. Whatever these details turn out to be, the important consequence is that wire-array implosions can be efficient sources of x-rays for high-energy-density experiments.

### 8.3.2 Dynamic Hohlräume

One can also use an imploding geometry to produce an intense radiation environment known as a *dynamic hohlraum*. In a Z-pinch, one produces a dynamic hohlraum by imploding a cylindrical array of high- $Z$  wires onto a low-density cylindrical target that is a few mm in diameter. A dynamic hohlraum can also be produced with a laser by driving a cylindrical or spherical implosion in a high- $Z$  gas such as xenon. Here we discuss the fundamental behavior that makes a dynamic hohlraum possible. In a dynamic hohlraum, a radiative shock is driven by an imploding radiation case. We will make a preliminary analysis of the behavior and potential of the dynamic hohlraum. Our goal is not to find realistic results, but to see whether such an approach might merit further exploration. This is an example of the sort of back of the envelope scoping calculation that one does frequently, in order to assess which possibilities deserve a closer examination.

In order to heat the entire volume, the shock must be radiative and the upstream medium must not be too opaque. This leads one to drive a radiative shock through a low-density medium. However, the density must not be too low, as one must have enough optical depth to convert the post-shock thermal energy to radiation. The imploding container provides the necessary pressure, doing  $p dV$  work on the shocked matter. For simplicity, we assume that the shock velocity is kept constant as the shocked matter is compressed by driving a single shock through it. We further assume that all the thermal energy



**Fig. 8.17.** Temperatures from a simple dynamic hohlraum model. The curves show the temperature from the initial shock (solid), from the resulting stagnation (gray), and from the second shock and stagnation (dashed). When the two curves separate, the upper curve is for  $\rho = 0.01 \text{ g/cm}^3$  and the lower curve is for  $\rho = 0.001 \text{ g/cm}^3$

from the shock is contained within a fixed volume. Thus, we assume for now that the unshocked material is bounded by a fixed wall that is perfectly reflecting. (We saw in our discussion of hohlraums that this is a reasonable approximation for a high- $Z$  wall.) The combination of moderate opacity and confinement of radiation assures that the radiation and the shocked matter will equilibrate to the same temperature.

The shock, assumed to be strong and steady, converts 50% of the incident mechanical energy into thermal energy (in the laboratory frame, which is relevant here). Just as the shock reaches the axis of symmetry, the mechanical energy will still be contained in the moving matter and the thermal energy will be shared by the matter and the radiation. Allowing for the compression of the low- $Z$  matter by a factor of  $(\gamma + 1)/(\gamma - 1)$ , we find

$$\frac{4\sigma T^4}{c} + \frac{\rho_o RT}{(\gamma - 1)} = 0.5 \frac{\rho_o u_s^2 (\gamma + 1)}{2 (\gamma - 1)}. \quad (8.85)$$

Here  $\rho_o$  is the initial density,  $u_s$  is the shock speed,  $\rho$  is the postshock density (equal to  $\rho_o(\gamma + 1)/(\gamma - 1)$ ) and  $T$  is the temperature of the shocked matter and the radiation. If one were irradiating an object located inside the dynamic hohlraum, this is the largest temperature one could achieve before the mechanical impact also began to affect the object. Figure 8.17 shows this initial temperature as solid black curves, for two densities of a material with  $A \sim (1 + Z)$  and  $\gamma = 4/3$ . At low enough velocities, the radiation energy density is negligible and the temperature scales with  $u_s^2$  just as in Chap. 4. The curves deviate from this scaling as  $u_s$  increases, but over the range shown the system never becomes radiation dominated. One sees that shock velocities of a few hundred km/s can produce temperatures of a few hundred eV.

On the other hand, one might use the dynamic hohlraum to provide a directed radiation source out of the end of a cylinder or through a structure in a sphere. In the next phase of its evolution, the incoming material would stagnate at the center, converting its mechanical energy into thermal energy, decreasing the volume and increasing the density by another factor of

$(\gamma + 1)/(\gamma - 1)$ . During this period, we assume that the incoming, reflecting wall acts to sustain the pressure on the outside of the compressed material. At the end of this process, the energy balance would give

$$\frac{4\sigma T^4}{c} + \frac{\rho_o RT}{(\gamma - 1)} \left( \frac{\gamma + 1}{\gamma - 1} \right)^2 = \frac{\rho_o u_s^2}{2} \left( \frac{\gamma + 1}{\gamma - 1} \right)^2. \quad (8.86)$$

The resulting temperature is shown as the gray curves in Fig. 8.17. To obtain a large gain from this step, the incoming shock velocity must be quite large.

Following the stagnation phase, the imploding wall may launch an additional shock into the low-density material. Assuming that the incoming wall velocity and  $\gamma$  remain the same, the resulting ingoing shock and subsequent stagnation would double the energy and reduce the volume by another factor of the shock compression squared, giving

$$\frac{4\sigma T^4}{c} + \frac{\rho_o RT}{(\gamma - 1)} \left( \frac{\gamma + 1}{\gamma - 1} \right)^4 = \frac{\rho_o u_s^2}{2} \left( \frac{\gamma + 1}{\gamma - 1} \right)^4. \quad (8.87)$$

The resulting temperature is shown as a dashed curve in Fig. 8.17. Of course each of these calculations may be an overestimate if energy escapes from the system during the implosion to this point.

Figure 8.17 assumes  $\gamma = 4/3$ . When the energy content of the radiation is not important, the curves for different densities overlap. As usual, even when the energy content of the radiation is comparatively small, the radiative fluxes may dominate the energy transport. Implosion velocities of 300 km/s are not difficult to achieve. One can see that our estimated temperatures for this implosion velocity are 150 eV for the first shock and 300 eV for the second shock. This is definitely large enough to be of interest. At the present writing, dynamic hohlraums are being explored as drivers for inertial fusion. Looking to the future, the constraints on stability and on preheat for a dynamic hohlraum are much less demanding than those for inertial fusion. As a result, they may offer a path to the achievement of radiation temperatures above 1 keV either through shock reverberation or through very large implosion velocities.

### Homework 8.13

An alternative way to think about what could be done with an imploding radiative shock is to imagine that one can drive a converging shock in an optically thin system. Assuming that such a shock reaches steady state, plot the radiation flux and its characteristic temperature against shock velocity for densities of 0.01 and 0.1 g/cm<sup>3</sup>. Comment on the comparison with the above calculation.

### 8.3.3 Magnetically Driven Flyer Plates

As a final application of the pulsed-power technology that drives Z-pinches, we consider how such a source of current and voltage can be used to isentropically compress and/or accelerate samples. In the traditional Z-pinch uses the fact that nearby conductors carrying parallel currents attract, as we discussed in the beginning of Sect. 8.3. By running parallel currents through an array of low-mass conductors, one can make them implode. There is a return current in a Z-pinch, but it is placed at a large radius so that it has little effect on the implosion, as we discussed. However, it is also true that conductors carrying opposing currents repel one another. By placing the opposing currents in close proximity, one can create a large force that drives them apart. If one makes one conductor quite massive and gives the adjacent conductor a much-lower mass, then the low-mass conductor will be preferentially accelerated. This is the key to what is sometimes known as *magnetic drive*.

The reason for this name becomes more clear if one thinks about the magnetic fields that these currents generate. The two conductors generate a magnetic field between them, perpendicular to the currents and with a direction given by the right-hand rule. One way to think of the resulting drive is to consider that each conductor experiences a  $\mathbf{J} \times \mathbf{B}$  force, just as we did when we discussed the Z-pinch. A second way to think about magnetic drive is to note that the magnetic pressure drops to zero across a thin layer at the surface of the conductor where the current flows, so that one can say that the magnetic pressure is applied to the conductors. This magnetic field can be enormous.

#### Homework 8.14

Revisit the derivation at the beginning of Sect. 8.3. Consider two infinitely wide, plane parallel conductors carrying opposing currents. Find the force per unit area between them and express it in terms of the magnetic field magnitude. Discuss how the force per unit area compares to the energy density of the magnetic field.

Thus far this description shows how to apply a large force to accelerate an object. There is an additional aspect of this possibility that gives it more value. By adjusting the increase of the current with time, one can control the time dependence of the force. In particular, one can increase it slowly enough to avoid launching a shock into the driven material. Observing the response of the material to such an isentropic compression can provide substantial insight into the equation of state. Beyond this, by isentropic compression and acceleration one can launch a cold flyer plate at a higher velocity that can be produced by traditional flyer-plate launchers such as gas guns. At this writing, Al flyer plates have been launched isentropically at velocities above 30 km/s.

## 10 Experimental Astrophysics

In Chap. 1 we looked forward to potential connections between high-energy-density physics and astrophysics. Some of these connections arise because one can produce in the laboratory circumstances that actually exist in astrophysics and can measure the properties of these systems. We have seen examples of this in the areas of equations of state, in Chap. 3, and opacities, in Chap. 6. Other connections arise because high-energy-density experiments can produce hydrodynamic or radiation hydrodynamic behavior under conditions that are relevant to astrophysical systems. By this means, one can explore the dynamic processes in ways that allow precise reasoning from the experimental results to the astrophysical process. Even so, astrophysical phenomena involve spatial and temporal scales that are many orders of magnitude (sometimes 25) greater than the scales encountered in laboratory experiments, so one may wonder whether a valid comparison is possible. The issue of whether and how one can make this comparison is the issue of *scaling*. When an experiment and an astrophysical system can be directly compared, the experiment is said to be well scaled. The focus of the present chapter is on how one can establish scaling that relates laboratory processes to astrophysical ones. (This chapter draws heavily on work done by the author with Dmitri D. Ryutov and Bruce A. Remington, reported in a series of papers and reviews from 1999 through 2006.)

We begin with some historical remarks specific to this area. As applications of lasers were first developed, one of the great minds in plasma physics, John Dawson, suggested in 1964 that they might be useful in the context of astrophysics (see Dawson). With the lasers and instruments of that era, one could have blown up a speck of matter, taken a picture, and marveled at the exploding star. But such a picture, then or now, would have no relevance to exploding stars. The advances in lasers and experimental technique from that time through the 1980s set the stage for the use of lasers to study the astrophysical properties and processes. Effort in both areas began in the early 1990s. Work relating to equations of state and opacities was discussed in Chaps. 3 and 6. Among the early work in astrophysical processes were laser experiments at the Naval Research Laboratory relevant to blast waves in magnetospheres (see Ripin et al.) and in astrophysics (see Grun et al.). The first publication to specifically suggest the use of lasers to address



hydrodynamic processes in supernovae was by Takabe in 1993. There are several review articles describing the early years of such work (Drake 1999; Remington et al. 1999, 2000; and Takabe 2001, Remington, Drake, and Ryutov 2006).

Our task here is to discuss the fundamental principles that must guide laboratory experiments whose goal is to advance our understanding of astrophysical processes. Our specific focus is the areas that high-energy-density experiments can readily address – hydrodynamics and radiation hydrodynamics. There are three issues that arise in the context of any particular experiment. These might be labeled scaling, unique experimental contributions, and code validation. The issue of scaling is whether an experiment is possible that is completely well scaled to the astrophysical system, or is well-scaled with regard to certain key dimensionless parameters but not in all respects. The issue of unique experimental contributions is whether experiments can address questions that are important in the astrophysical context but that cannot be answered by theory or simulation. We discuss below two examples of this in the area of hydrodynamics. The issue of code validation is whether an experiment can be defined that provides a useful and meaningful test of a computational astrophysical simulation tool, whether or not the entire system may be well scaled or may make a unique contribution. In practice, there is no required relationship among these three issues. The third (code validation) is generally the easiest to identify and so is a likely starting point in any particular experimental effort. We turn now to the area of hydrodynamics, where we will first discuss the issue of scaling.

In this chapter, we discuss the general scaling issues for hydrodynamic experiments at some length, as this is an excellent example. We then discuss two types of such experiments. These explore hydrodynamic instabilities in supernovae and the destruction of clumps by shocks. Then we discuss scaling in radiation hydrodynamic systems, and radiative jets as a specific example. Before proceeding we should note that the laboratory study of processes in astrophysics is not limited to high-energy-density systems. Processes that do not require high Mach numbers or radiation can be studied at lower energy density. One example, at this writing, is the work of a number of groups who are advancing the study of magnetic reconnection, which is one source of energetic particle production in magnetized interplanetary and interstellar plasmas. Another example is the study of the dynamic behavior of magnetic flux tubes. There will doubtless be more examples in the future.

## 10.1 Scaling in Hydrodynamic Systems

We first consider the issue of scaling for processes that can be described by hydrodynamic equations. In the specific case of purely hydrodynamic systems, one can draw some general conclusions about scaling that are valid for all experiments. This section develops these conclusions and discusses how to

apply them to specific cases. The reader can find a more detailed discussion in Ryutov et al. (1999, 2001) and Ryutov, Drake, and Remington.

The reader may wonder why one cares about comparing systems whose behavior is hydrodynamic, so let us consider this first. It may seem as though we understand hydrodynamics, but it would be more accurate to say that we understand the equations that apply to hydrodynamics. We also know that these equations are nonlinear in ways that produce immediate complexity under many circumstances. This has two consequences. First, it means that there are questions that are too complicated to be addressed successfully in computer simulations. We will see one example of such a question below, in our discussion of supernovae. A second example involves the onset of turbulence, in the context of Sect. 5.8. The Reynolds number  $Re$  in astrophysical flows is typically far above the value of 10,000 at which the mixing transition can begin. In contrast, the computer simulations cannot achieve even  $Re \sim 10,000$  when modeling moderately complicated systems because of numerical viscosity. In addition, turbulence models are too uncertain to know which approach to an integrated description of turbulence might be best. Another aspect of the impact of turbulence is that it remains unclear whether fine structures in an evolving system can significantly affect the large-scale evolution of the system, a process described in the literature as “stochastic backscatter (Leith; Piomelli et al.). To address the presence and importance of these effects in the context of astrophysics, experiments that are both clever and well scaled are required.

Second, the complexity of realistic hydrodynamics implies that code validation is essential even for the processes that may in principle be simulated successfully. It is unclear how well we understand how to model numerically the very nonlinear evolution of hydrodynamic systems. Like a series solution to a differential equation, a computer simulation is only an approximation to the actual solution of the physical equations being solved. Unlike a series solution, it is very difficult to judge the error involved in the approximate computational solution. In addition, independent simulations often produce different results (Glimm, Grove, Li, et al.), especially on a scale below about 10% of the initial perturbation wavelength (Kane et al. 1997). Because of these difficulties, a code that works well for a certain class of problems may not work well for other problems. Thus, validation that is relevant to the dynamics of interest is important for each specific dynamical process.

Now we turn to the specific issues involved in hydrodynamic scaling. We are concerned with systems that obey the Euler equations, (2.1)–(2.3). Now consider the initial value problem for this set of equations. Let us present the initial spatial distributions of the density, pressure, and velocity in the following way:

$$\rho(t=0) = \rho^* f\left(\frac{\mathbf{r}}{L^*}\right), \quad p(t=0) = p^* g\left(\frac{\mathbf{r}}{L^*}\right),$$

and

$$\mathbf{u}(t=0) = u^* \mathbf{h} \left( \frac{\mathbf{r}}{L^*} \right), \quad (10.1)$$

respectively, where  $L^*$  is the characteristic spatial scale of the problem, and the other quantities marked by the asterisk denote the value of the corresponding parameter in some characteristic point; the dimensionless functions (vectorial functions)  $f, g$ , and  $\mathbf{h}$  have absolute magnitude of order unity. They determine the spatial shape of the initial distribution. We note that there are four, dimensional parameters determining initial conditions:  $L^*, \rho^*, p^*$ , and  $u^*$ . Let us then introduce dimensionless variables (which we denote by the tilde) in the following way:

$$\tilde{\mathbf{r}} = \frac{\mathbf{r}}{L^*}, \quad \tilde{t} = \frac{t}{L^*} \sqrt{\frac{p^*}{\rho^*}}, \quad \tilde{\rho} = \frac{\rho}{\rho^*}, \quad \tilde{p} = \frac{p}{p^*},$$

and

$$\tilde{\mathbf{u}} = \mathbf{u} \sqrt{\frac{\rho^*}{p^*}}. \quad (10.2)$$

When one expresses the set of (2.1) to (2.3) in terms of the dimensionless variables, one finds that the equations maintain their form, with all the quantities being replaced by their analogs bearing the tilde sign. The initial conditions presented in the dimensionless variables acquire the form

$$\tilde{\rho}(\tilde{t}=0) = f(\tilde{\mathbf{r}}), \quad \tilde{p}(\tilde{t}=0) = g(\tilde{\mathbf{r}}),$$

and

$$\tilde{\mathbf{u}}(\tilde{t}=0) = u^* \sqrt{\frac{\rho^*}{p^*}} \mathbf{h}(\tilde{\mathbf{r}}). \quad (10.3)$$

Now consider two different systems, say, an astrophysical system and a laboratory system. One sees that the dimensionless initial conditions for the two systems are identical if the dimensionless functions  $f, g$ , and  $\mathbf{h}$  are the same, and the single dimensionless parameter,  $u^* \sqrt{\rho^*/p^*}$ , remains unchanged. In other words, provided this parameter is invariant, and the initial states are geometrically similar (i.e., the functions  $f, g$ , and  $\mathbf{h}$  are the same), one would have the same dimensionless equations and the same dimensionless initial conditions for *any two hydrodynamical systems*. This implies that the systems will evolve identically in a scaled sense.

### Homework 10.1

Show that the Euler equations are in fact invariant under the transformations just described.

Normally, similarity arguments are used to ensure that some parameter (of interest in a particular problem) can be scaled between two systems. This can be, for example, the normalized spatial scale for radiative cooling in an astrophysical jet (Sect. 10.5), or the energy confinement time in a

plasma device (as in Connor and Taylor). We are demanding much more, that the whole dynamical evolution of two systems with properly scaled initial conditions be similar.

There is only one constraint on the four parameters determining evolution of the system. For the second system, one can choose the scale length,  $L^*$ , arbitrarily. One can then also choose arbitrarily two of the three parameters,  $p^*$ ,  $\rho^*$ , and  $u^*$ . Then by choosing the magnitude of the remaining parameter so as to maintain  $u^* \sqrt{\rho^*/p^*}$  constant, one can obtain a system which behaves similarly to the first one. This similarity was named in Ryutov et al. (1999) the Euler similarity, and the parameter  $u^* \sqrt{\rho^*/p^*}$  was named the Euler number,  $Eu$ . It is very important that the Euler similarity covers not only smooth solutions of the Euler equations, but also solutions containing shock waves or multiple shocks. The proof can be found in Ryutov et al. (2000).

There is a special case, often present in astrophysical objects (like SN explosions) and in the corresponding laboratory experiments, that is much simpler. Assume that there is a system with an arbitrarily distributed initial density, and with some initial pressure profile and initial velocities of the order of the sound velocities or less. Assume then that a planar (cylindrical, spherical) piston is moved into the system with a velocity *much greater* than the initial sound velocity. Considering as an example a spherical piston, we can describe its motion by the equation

$$r = L^* q_p(t/\tau^*), \quad (10.4)$$

where  $\tau^*$  is the characteristic time of the piston motion (the time within which it is displaced by the distance  $\sim L^*$ ); the dimensionless function  $q_p$  (with subscript p standing for the piston) and its argument  $t/\tau^*$  are of the order of unity. The initial density distribution will as before be  $\rho(t=0) = \rho^* f(\mathbf{r}/L^*)$ , with the function  $f$  being of the order of unity.

The strong shock propagating in front of the piston brings the plasma to a new state; the characteristic pressure in this new state is

$$p^* \sim \rho^* L^{*2} / \tau^{*2}, \quad (10.5)$$

and the characteristic velocity is

$$u^* \sim L^* / \tau^*. \quad (10.6)$$

This state is essentially independent of the pressure and the velocity in front of the very strong shock (see Chap. 4).

If one takes a second system, with the scale factors  $\tau^*$ ,  $L^*$ , and  $\rho^*$  arbitrarily changed but with the function  $f$  in (10.3) and the function  $q_p$  in (10.4) remaining the same (i.e., initial density distributions are geometrically similar, as are temporal dependences of the piston position), the two systems will evolve identically in the limit that the shock is strong enough.

For example, if in system 1 characterized by scaling factors  $\tau_1^*$ ,  $L_1^*$ , and  $\rho_1^*$ , the density is  $\rho_1(\mathbf{r}, t)$  then the function  $f$  is  $\rho_1(\mathbf{r}, t = 0)/\rho_1^*$ . The evolution of the density in system 2 will then be  $\rho_2 = (\rho_2^*/\rho_1^*) \times \rho_1(\mathbf{r}L_1^*/L_2^*, t\tau_1^*/\tau_2^*)$ . In this case (10.5) and (10.6) imply that  $u^* \sqrt{\rho^*/p^*}$  has the same value in both systems, so there is no need to impose this as a separate constraint. The implication is that all the characteristic parameters ( $\tau^*$ ,  $L^*$ , and  $\rho^*$ ) can be varied independently, and still the similarity exists. This very broad similarity can be extended to include the case where the piston surface deforms in an arbitrary fashion during the piston motion; to do that, one should just describe a piston by the general equation for a surface evolving with time:  $F(\mathbf{r}/L^*, t/\tau^*) = 0$ .

A necessary condition for the validity of the Euler equations is that the terms in a more general fluid description are negligible. We discussed the requirements for this in Chap. 2. The specific constraints are that the fluid description rather than the kinetic one must apply, that the Reynold's number must be large, that the Peclet number must be large, and that radiative energy transfer must be negligible. We discuss below the value of these numbers for one specific experiment and astrophysical system.

It is often the case that an experiment and a similar astrophysical system meet these constraints, but that the value of one of the scaling parameters is quite different. For example, high-energy-density experiments tend to have Reynold's numbers in the range of  $10^5$  to  $10^7$  while the Reynold's number of otherwise similar astrophysical systems is much larger. In such cases it is worthwhile to ask what the significance of these differences may be. This is not a topic we will explore here at any length. For the specific case of the Reynold's number, one would expect that the presence of an inertial range would matter (see Sect. 5.8), but that the ratio of the Taylor microscale  $\lambda_T$  to the Komogorov microscale  $\eta_k$  would not be important.

## 10.2 A Thorough Example: Interface Instabilities in Type II Supernovae

As an example of a well-scaled experiment, we will now discuss experiments to study interface instabilities in Type II *supernovae* in some detail. Overall, Supernovae (SNe) involve a very broad range of physical processes. (The term supernova is represented by SN, so the plural is SNe.) Their complete description requires the use of as disparate areas of science as particle physics and general relativity on the one side, and hydrodynamic stability and turbulence, on the other. Some aspects of SNe are amenable to study by experiments in the laboratory, while others are not.

A nice description of the SN phenomenology, as well as existing theories of their formation, can be found, e.g., in the book by Arnett. Other papers of general interest include Bethe, Woosley, and Woosley and Eastman. SNe are believed to explode by two fundamental mechanisms: collapse of the core (in

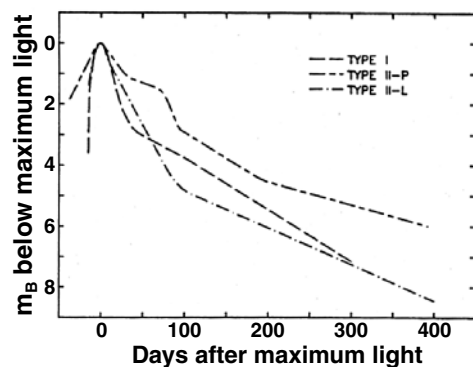
large stars) and thermonuclear explosion (in small stars below 8 solar masses at birth). The classification by types is based on spectra and is too involved to discuss here. The Type Ia SNe, which will not be our focus here, are predominantly thermonuclear. They occur when white dwarf stars, composed primarily of C and O, accumulate enough mass to overcome the pressure of their degenerate electrons. They then begin to gravitationally collapse, releasing enough energy to cause explosive fusion burning of their C and O. Our focus will be on the predominant core-collapse SNe of Type II.

### 10.2.1 The Astrophysical Context for Type II Supernovae

Any large-enough, pre-supernova star develops shells of material around an iron core. Iron accumulates in the core because Fe is the most stable nucleus (see Chap. 9). The inner shell is composed primarily of the elements Si and Ca, the second shell is primarily C and O, the third shell is mainly He, and the outer shell is mainly H. One such star became SN 1987A. Any such star is initially composed primarily of H. It develops the interior shells in succession as the gravitational pressure compresses and ignites the accumulating material that will form the next shell. This process stops with Fe, because the star cannot create further energy by converting the Fe to any other material. But once the star accumulates a “Chandrasekhar mass” (1.4 solar masses) of Fe, the gravitational pressure on the core overcomes the degeneracy pressure of the electrons and the core collapses. The collapsed core forms a neutron star that might later be detected as a pulsar.

The collapse of the Fe core is accompanied by the generation of a short but very intense burst of neutrinos (carrying away some 99% of the released energy). Some of the neutrino energy is coupled to the remaining matter that did not collapse, primarily near the core. The kinetic energy of the exploding matter in a typical SN event is  $\sim 10^{51}$  erg. This brief deposition of energy creates an almost classic point explosion case (except for the small hole in the middle). A blast wave (see Chap. 4) develops and propagates out through the star, blowing it apart. When the blast wave emerges from the star, this gives rise to the observed tremendous increase of luminosity. Only a few percent of the hydrodynamic energy is emitted as visible light. In more detail, all the elements of the description just given must be present, but may be insufficient to explain the actual generation of the blast wave. It may be that some nonsymmetric motions, even perhaps involving the generation of a jet during the process of core collapse, are involved in the generation of the blast wave.

Figure 10.1 shows typical light curves for three types of SNe. One should remember that the light as detected by optical telescopes comes not from the core, where the energy release has occurred, but rather from a photosphere, to which the energy is transported by a complex combination of hydrodynamic flows and radiative transport. Linking the energy release in the SN core to a visible light curve is a substantial challenge. A correct description of the

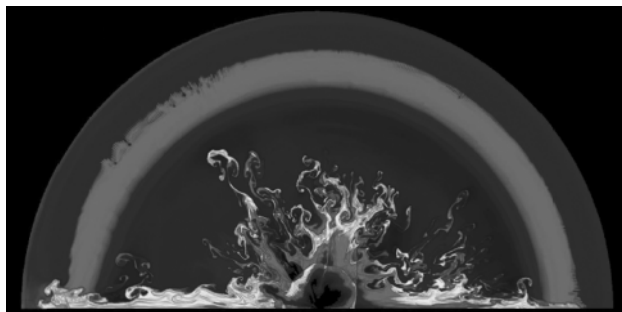


**Fig. 10.1.** Type I and II curves are normalized to their luminosity at the maximum (reproduced from Doggett and Branch 1985). Note the presence of two sub-classes of SN Type II, with one of them (P) having a plateau in luminosity, with the other (L) showing a regular decay

material opacities and of the transport of material and radiation is very important. In the present section we are concerned with the transport of material.

Among the broad array of problems related to SN explosions, we shall concentrate on the laboratory simulation of hydrodynamic phenomena in Type II SN explosions and specifically on the evolution of hydrodynamic instabilities at the interfaces in the star. We choose this topic because, on the one hand, multi-dimensional hydrodynamic effects are thought to be very important and, on the other hand, there already exist successful experiments of this type, related to the shock breakout through the He–H interface. In addition, the analysis of the scalability and other constraints can be nicely illustrated; this example can serve as a template for similar analyses of the other problems.

Hydrodynamic instabilities arise as follows during the stellar explosion. At each interface between shells there is a significant density decrease. The interfaces are not smooth but are structured by convection, rotation, and other dynamics. The blast wave is likely to be born with structures resulting from neutrino convection (one model of this leads to the structure seen in Fig. 10.2). In addition, the blast wave is perturbed by the structure at the interfaces as it propagates outward. Even though the shock at the head of the blast wave will tend to anneal as described in Sect. 5.7.1, only small-enough perturbations will anneal completely. As a result, each structured interface will experience in its turn the passage of a structured blast wave. That is, the blast wave communicates between interfaces, so that the structure at inner interfaces produces additional structuring of the outer interfaces. During the deceleration phase that follows the blast wave, the structures at the interface will grow first through the Richtmyer–Meshkov process (Sect. 5.7.3) and



**Fig. 10.2.** 2D simulation of SN 1987A, from Kifonidis, Plewa, Janka, and Muller, *Astrophys. J. Lett* **531**, L123 (2000)

then through the Rayleigh–Taylor instability (Sect. 5.1.2). All this had been understood for some time, but the ultimate nonlinear consequences were not clear before the advent of SN 1987A.

In observations of SN 1987A, emissions from the heavy elements, and other indications of their presence in the outer layers of the supernova, were observed only a few months after the explosion. The observed early appearance of heavy elements (like  $\text{Ni}^{56}$  and  $\text{Co}^{56}$ ) in the photosphere of SN 1987A (see Sutherland 1990, and references therein) is incompatible with a spherically symmetric expansion and seemed to indicate that the instabilities have important and observable effects. Even so, simulations of the explosion in two dimensions (see Arnett et al.; Fryxell et al.; Burrows et al.; Kifonides et al. 2000), which was all that was feasible computationally, did not produce rapid enough penetration to explain the observations. Partial simulation studies in three dimensions also indicated that the more-rapid penetration that would be found in three dimensions was not large enough to make up the difference.

In the context of our discussion above, in the introduction to this chapter, these circumstances created two roles for experiments. The first role is code validation. It was worthwhile to determine whether systems of this type behave in some way that existing simulations did not reproduce. A well-scaled experiment with two-dimensional structures would be sufficient for this purpose. We discuss such an experiment next. The second role is the direct observation of cases with realistic complexity. An important question, which cannot be answered by simulations in the current era, is to what extent in three dimensions the coupling between the interfaces in a diverging explosion actually manages to increase the outward penetration of the inner material. This is relevant not only to SN 1987A but also to other cases such as Cassiopeia A. Laser facilities to be completed during the early 21st century will be able to carry out this second role.



### 10.2.2 The Scaling Problem for Interface Instabilities in Supernovae

We now turn to the challenge of performing well-scaled experiments to simulate interface instabilities in SN 1987A, specifically at the He–H interface. This problem serves as an example for designing scaled laboratory experiments. It allows one to clearly identify the most important steps leading to the formulation of a scaled laboratory experiment. These steps, described in more detail in the following, are: (1) make sure that hydrodynamic description is adequate; (2) evaluate the role of transport processes (mass, momentum, and energy transport); (3) determine the equation of state; (4) characterize initial conditions; and (5) consider whether some simplification in geometry is possible (e.g., can some time segment of evolution of a planar system adequately describe the evolution of a spherically divergent system?).

As a representative set of plasma parameters in the He–H transition region (Table 10.1), we consider the set of parameters given in Müller et al. These are of course parameters from computer simulations, as observations are not possible. However, they should be fairly reliable because of the simplicity of the fundamental blast wave problem. With reference to the table, let  $h$  be the density gradient scale-length,  $u$  be the characteristic velocity,  $T$  be the plasma temperature (the electron and ion temperatures are equal), and  $\rho$  be the plasma density. The characteristic deceleration experienced by a given fluid element at the He–H interface in the SN following arrival of the blast wave can be estimated as  $v/\tau$ , with  $\tau \sim h/v$ , so  $v/\tau \sim v^2/h \sim 5 \times 10^5 \text{ cm/s}^2$ . The gravitational acceleration is much smaller and is also negligible in the laboratory experiment to be discussed. Table 10.1 also shows parameters for the laboratory experiment. These are again based on simulations, although in this case several measurements support the detailed numbers given. From the parameters in Table 10.1, we can derive the scaling parameters given in Table 10.2.

**Table 10.1.** Fundamental hydrodynamic parameters for a supernova experiment

Parameter	Supernova 1987A	Experiment
	(2000 s)	(21 ns)
Length scale (cm)	$9 \times 10^{10}$	0.0180
Velocity (km/s)	2000	35
Density ( $\text{g/cm}^3$ )	0.0075	0.4
Pressure ( $\text{dynes/cm}^2$ )	$3.5 \times 10^{13}$	$5.2 \times 10^{11}$
Temperature (eV)	900	7.4
$Z_i$	2.0	0.6
$A$	4.0	11.4
Density of nuclei ( $\text{cm}^{-3}$ )	$1.1 \times 10^{21}$	$2.1 \times 10^{22}$

**Table 10.2.** Derived scaling parameters for a supernova experiment

Derived Parameter	Supernova 1987A	Experiment
$u^* / \sqrt{p^* / \rho^*}$	2.9	3.1
Collisional mfp (cm)	$3.5 \times 10^{-3}$	$7.9 \times 10^{-8}$
Kinematic viscosity (cm <sup>2</sup> /s)	$7.0 \times 10^7$	0.334
Reynolds number	$2.6 \times 10^{11}$	$1.9 \times 10^5$
Thermal diffusivity (cm <sup>2</sup> /s)	$1.2 \times 10^6$	15
Peclet number	$1.5 \times 10^{13}$	$4.2 \times 10^3$
Radiation mfp (cm)	$6.8 \times 10^2$	$2.0 \times 10^{-4}$
Radiation Peclet number	$10^6$	$2.5 \times 10^9$

Now we proceed with the steps just described. First consider whether the Euler equations apply to these systems. The case of the experiment is straightforward – the plasma is quite collisional and radiation is not important. The SN is more complex. To be specific, we discuss properties of the He plasma. At  $T = 800$  eV it is fully ionized. Mean free paths with respect to electron–ion ( $\lambda_{ei}$ ) and ion–ion ( $\lambda_{ii}$ ) collisions are very short,  $\lambda_{ei} \sim 10^{-3}$  cm and  $\lambda_{ii} \sim 2 \times 10^{-2}$  cm. The electron–ion energy equilibration time is less than  $10^{-9}$  s. Therefore, the electron–ion component behaves as a strongly collisional gas with equal temperatures of electrons and ions. The particle pressure of a helium plasma for  $\rho = 4 \times 10^{-3}$  g/cm<sup>3</sup>,  $T = 800$  eV is  $p = 2.3 \times 10^{12}$  erg/cm<sup>3</sup> = 2.3 Mbar. The radiation pressure for  $T = 800$  eV is  $p_R = 2 \times 10^{13}$  erg/cm<sup>3</sup> = 20 Mbar. In other words, the radiation pressure dominates.

Despite the dominance of radiation, the Euler equations apply because the matter entrains the photons. The photon mean free path (mfp) with respect to Compton scattering (Thomson scattering at these low temperatures),  $\ell_C$ , is very short,  $\ell_C \sim 10^3$  cm (the mfp for inverse bremsstrahlung is much longer). In other words, the plasma containing the photon gas can be described as a single fluid, whose pressure is the sum of the photon and particle pressures, and which can be characterized by a single velocity of the mass flow  $u$ . The energy per unit volume in the case where the pressure is dominated by photons is  $E_R = 3p_R$ , thereby corresponding to the polytropic gas with  $\gamma = 4/3$ . Therefore, the Euler equations apply to both systems.

The second task is to evaluate the dimensionless parameters discussed in Sect. 2.3 and to consider their implications, with reference to the derived parameters in Table 10.2. First we discuss the SN. When evaluating viscous effects, all sources of viscosity must be added. An ordinary (particle) viscosity (from (2.39), for a helium plasma with  $T \sim 800$  eV,  $\rho \sim 8 \times 10^{-3}$  cm<sup>-3</sup>, and  $\ln A = 10$ , is  $\sim 2000$  cm<sup>2</sup>/s. The photon viscosity (from 2.40) is much larger than the ordinary viscosity,  $\nu_{\text{rad}} \sim 7 \times 10^7$  cm<sup>2</sup>/s. Accordingly, the Reynold's number, evaluated for  $h \sim 10^{11}$  cm,  $u \sim 2 \times 10^8$  cm/s, and  $\nu = \nu_{\text{rad}}$ , is  $2.6 \times 10^{11}$ . Hence, the viscous effects are very small.

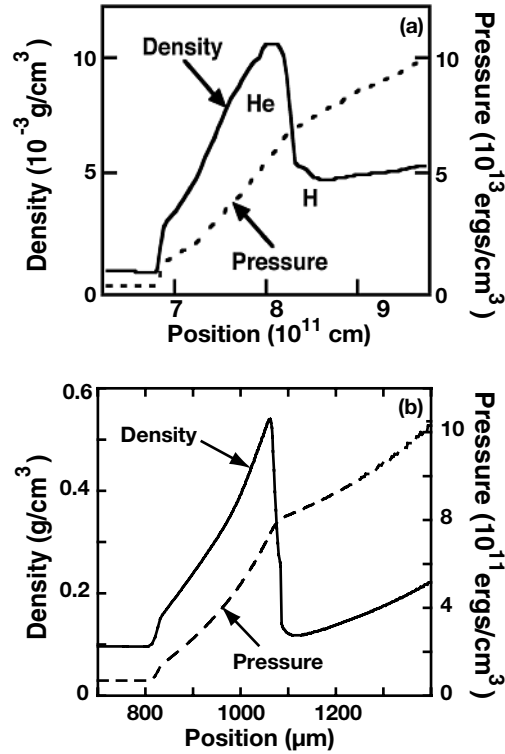
The Peclet number evaluated for thermal diffusivity (from 2.33) of the electrons (lines 5 and 6 in the table) is very large, but in fact heat conduction is dominated by the photons. The photon mean free path  $\ell$  is much less than the characteristic length-scale  $h$ . An estimate (see Zeldovich and Raizer) for the thermal diffusivity  $\chi$  of the photons is  $\chi \sim c\ell$ . For the case under consideration, where  $\ell$  is the mean free path for Compton scattering,  $\ell_c$ , one has  $\chi \sim 2 \times 10^{13}$  cm<sup>2</sup>/s. The corresponding radiation Peclet number is large,  $\sim 10^6$ , meaning that the system is essentially adiabatic.

Now consider the experiment. Assuming based on simulations that the average charge of the ions is  $\sim 0.6$ , and taking the Coulomb logarithm equal to 1, one finds (from 2.39) that the viscosity of CH<sub>2</sub> plasma is  $\sim 0.3$  cm<sup>2</sup>/s. Accordingly, the Reynold's number is  $Re \sim 2 \times 10^5$ . Although smaller than in the supernova, this is higher than a typical critical Reynold's number corresponding to the onset of the instability of the sheared flow ( $Re \sim 10^3$ ) and the mixing transition ( $Re \sim 10^4$ ). Therefore, it is clear that viscous effects will not determine the plasma behavior in the laboratory experiment.

Electron thermal diffusivity  $\chi$  in the laser experiment is (from 2.33)  $\sim 15$  cm<sup>2</sup>/s, so that the Peclet number corresponding to the particulate heat transfer is high,  $\sim 4,200$ . (Because the plasma in these experiments could be considered ideal only marginally, the aforementioned estimates of  $\nu$  and  $\chi$  should be considered as order-of-magnitude estimates.)

Turning to the role of radiative losses in the experiment, this subject is complicated by potential effects of optical depth. (The large value of the radiation Peclet number shown in the table implies that diffusive heat conduction by radiation is small but does not preclude large radiative losses from plasma that is not optically thick.) One can give an upper estimate for the effect of radiation as follows. The maximum possible energy loss from the surface of the plasma slab is that corresponding to the blackbody radiation at the plasma temperature,  $2\sigma T^4$ . On the other hand, plasma energy content per unit area of the plasma surface is  $(3/2)h(n_e + n_i)k_B T$ . Dividing the second by the first, one finds a lower-bound estimate for the characteristic plasma cooling time. Taking parameters of Table 10.2, we find that it is  $1.8 \mu\text{s}$ , 90 times longer than the characteristic time of the hydrodynamic problem ( $h/v \sim 2 \times 10^{-8}$  s). In other words, radiative heat transport also does not affect the plasma dynamics. To conclude this section: dissipative processes in the problem of the stability of the He-H interface are negligible, and Euler equations are a legitimate description.

The third topic on our list is the equation of state. Hydrodynamic equations are structurally stable with respect to minor differences in the equation of state, so these differences cannot change the general similarity between the astrophysical system and its laboratory counterpart. What matters is the presence of phase changes. In this case, there are none during the postshock evolution of interest. Therefore, we conclude that the laboratory experiment correctly simulates all aspects of the hydrodynamics of a real supernova at



**Fig. 10.3.** Experiment and supernova profiles from simulations. (a) He–H interface in SN1987A at 2000 s. (b) CHBr–Foam interface in experiment at 21 ns

the stage of the shock propagation through the progenitor star. As to the relation of scales, 21 ns in the laboratory setting corresponds to 2000 s in the supernova, and  $20 \mu\text{m}$  corresponds to  $10^{10}$  cm.

Our fourth topic is initial conditions. Figure 10.3 shows an example of the profiles in the supernova and the laboratory experiment, according to computer simulations. If these were identical, and if the laboratory experiment were spherical, then the experiment would evolve identically to the star. In reality any laboratory experiment can reproduce only a limited time-segment of a real process. One can create conditions in which the scaled spatial distribution of all essential parameters in the experimental package are, at some moment in time, similar to those of the astrophysical system (at least in the vicinity of the region of interest). Then, the similarity arguments above show that the two systems will evolve in a similar fashion. It is reasonable to use a planar experiment, like this one, to simulate a limited portion of a spherical explosion, but then the experiment will be a good model of the star only until the divergence of the star becomes significant. A bigger limitation is that the initial similarity can be attained only in a limited region of space.

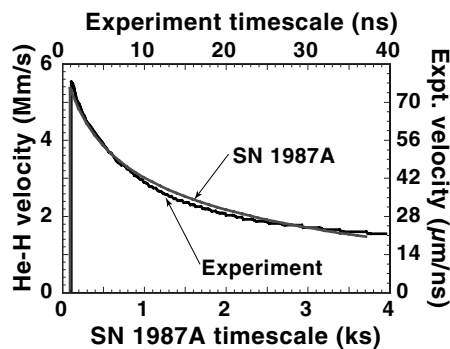
As a result, the temporal evolution will be similar only during some limited time, which is set by the time of propagation of hydrodynamic perturbations from the edges of the zone where the initial similarity exists.

The specific design of an experiment involves the specification of the structure of the target and the parameters of the laser (or other) drive, within the limits of available target-fabrication and laser-system technologies. This typically involves conceptual analysis followed by 1D hydrodynamic simulations to establish a viable approach. For example, in the experiment just described one drives a steady shock with the laser pulse and sizes the target so that the rarefaction from the front surface, which develops after the laser pulse ends, overtakes the shock to form a blast wave before the shock reaches the region of interest (see Fig. 4.23). 2D or 3D simulations can then evaluate the effects of finite experiment size and the amount of instability growth one can anticipate. One of the challenges is to obtain sufficient instability growth within the limited time-interval during which the two systems are similar.

### 10.2.3 Experiments on Interface Instabilities in Type II Supernovae

In one of the first attempts to conduct experiments that were a well-scaled study of an astrophysical process (as opposed to measuring a property of astrophysical matter), a team of researchers conducted a series of experiments aimed at the problem of hydrodynamic instabilities at the H–He interface in the years around the turn of the 21st century. The point of these experiments was to observe the dynamics of this process in a well-scaled environment. All these experiments apply an initial ablation pressure either through direct laser irradiation or through x-ray irradiation. They all produce a blast wave by launching a shock that is later overtaken by a rarefaction (see Sect. 4.3). The subsequent structure in the target then determines which aspects of unstable dynamics are being explored. Figure 10.4 (adapted from Robey et al. 2001) compares the time dependence of the velocity at the interface in such an experiment with the velocity at the He–H interface in SN1987A, as determined by a simulation of the explosion.

The first such experiments (Remington et al. 1997; Kane et al. 1997, 1999a, 2000) used the Nova laser to examine the RT growth from a single-mode initial perturbation at a planar interface. The instability grew until the distance from the valleys to the peaks in the observed modulations (known as the bubble-to-spike distance) became equal to the initial wavelength of 200  $\mu\text{m}$ . This is very nonlinear (see Sect. 5.7). Simulations, using the astrophysical code PROMETHEUS (Fryxell et al.) and the laboratory code CALE (Barton, Tipton), reproduced this result, but the details of the structures did not strongly resemble what seemed to be present in the rather poor data obtained in this first attempt. Related experiments were also undertaken during the same period by a French group (Benuzzi-Mounaix et al.).

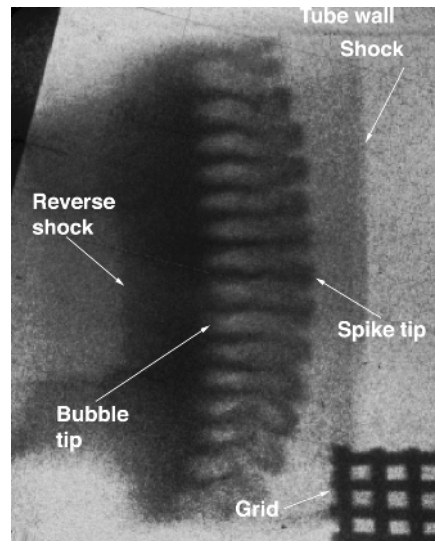


**Fig. 10.4.** Interface velocity in SN1987A and an experiment, from computer simulations

Subsequent experiments (Drake 2004, Robey, Kane et al., 2001, Robey, Miles et al., 2003, Robey, Zhou et al., 2003, Miles 2004, Miles, Blue et al., 2004, Miles, Braun, et al. 2004, Miles, Edwards, et al. 2004) improved the quality of the data. The scaling parameters used in Tables 10.1 and 10.2 were taken from these experiments. Figure 10.3 shows the spatial profiles of SN1987A and these experiments, based on simulations. One can see that these profiles are similar, or in other words that the functions  $f$  and  $g$  are similar. (Figure 10.4 showed that the interface trajectory is similar. One could view this as showing the similarity of the generalized time-dependence for the strong-shock case or as implying the similarity of the function  $h$ , a scalar here.) One concludes from the tables and from these figures that the experiment is a well-scaled (though not perfectly scaled) model of a segment of the exploding star. This will remain true until the interface in the experiment discovers that it is not spherical, when the interface is affected by disturbances from the edges.

Figure 10.5 shows data from one such experiment. The image is an x-ray radiograph, taken at 18 ns in the experiment. Darker regions in the image show smaller x-ray intensity, produced primarily by absorption in the Br dopant that was included in a tracer layer within the denser material. In this case, a single mode with an initial wavelength of  $50\ \mu\text{m}$  and an initial peak-to-valley amplitude of  $5\ \mu\text{m}$  has grown to a very large amplitude. The tips of the spikes in the image have been broadened by the Kelvin–Helmholtz instability and will soon begin to interact (Miles, Blue et al. 2004). The experiments just mentioned set the stage for experiments that proceeded to take up the challenge developed above – the study of how the instability would develop in well-scaled experiments that employed more-complex, three-dimensional, and ultimately realistic initial conditions (Robey, Miles, et al. 2003, Drake, Leibrandt et al. 2004, Kuranz, Drake et al. 2004, Kuranz, Drake et al. 2004).

Related experiments have explored the effect of coupling between interfaces, discussed in Sect. 10.2.1, on the RT instability. These are worth men-



**Fig. 10.5.** Radiographic data from a supernova simulation experiment, at 18 ns

tioning because they became the explicit focus of an extensive code validation study. The experiment to examine this (Kane, et al., 2001) produced data that were shown in Fig. 1.4. One can see the Cu spikes, extending to the right, and the modulations in the second interface, made visible by the tracer strip in the plastic below the interface. Detailed simulations of this experiment were carried out as part of a validation study (Calder et al.) for the astrophysical code FLASH, which included adaptive grids and other advanced features. A detailed comparison of the simulation results with the experimental images supported the same conclusion as in the previous case. The simulations reproduced the qualitative features of the data very well. Quantitatively, several details were not accurately reproduced, including the exact spike length, the height of the structuring in the interface, and the behavior at the edges of the system. In the specific case of the FLASH simulations, the length of the Cu spikes was found to change with the number of levels of refinement in the simulation but did not appear to be converging toward the experimental value.

### Homework 10.2

Design a diverging experiment to address the coupling of two structured, unstable interfaces that are affected by a blast wave. Beyond the basic requirements for hydrodynamic scaling, identify other specific parameters that are important to the dynamics. (Hint: review blast-wave propagation and shock stability as part of your work.)

Thus, well-scaled experiments can address significant issues in astrophysical systems that are purely hydrodynamic. The above examples have detailed one case in which initial experiments were useful for code validation and in which eventual experiments will address physical questions that cannot be addressed in simulations of the complete astrophysical dynamics. In addition, the process of doing such experiments initiated a productive interplay of astrophysical data, laboratory experiment, and computer simulation whose ultimate outcome, as of this writing, remains to be seen.

### 10.3 A Second Example: Cloud-Crushing Interactions

The previous section discussed a case in which producing a system that was well scaled in a global sense produced useful results. We also saw in Sect. 10.1 that producing the correct density structure and very strong shocks might be sufficient to obtain good scaling in many hydrodynamic systems. However, even when this is possible, it may not be sufficient to obtain results that are genuinely useful, especially when one hopes to go beyond code validation and address complex questions that simulations cannot answer. To obtain genuinely useful results, one often must consider additional parameters that assure that an experiment has the scales in time and space required to access the interesting dynamics. We consider one such case here.

This case involves the interaction of shock waves or blast waves with clumps of material. This is common in astrophysics. Shock waves, generally produced by supernovae, propagate through an interstellar medium (or molecular clouds) that is inherently very clumpy. One would like to observe the resulting destruction of the clumps in laboratory experiments and to develop and test the ability to accurately simulate this destruction. In the process, one would like to identify whatever regimes exist, such as regimes in which the dynamics of a given clump may be affected by the presence of other clumps. One would also hope to identify whether any processes develop in a scaled laboratory system that cannot be produced in a computer simulation, such as the onset of turbulence.

In the case we will consider, the experiment and the astrophysical system are both hydrodynamic and the shock wave that induces the destruction of the clump is definitely a strong shock. As a result, any system in which a strong shock encounters a dense clump of some specific shape will be hydrodynamically well scaled to any other such system, subject to the other limitations discussed in Sect. 10.1. However, one can identify two other parameters that characterize the interaction of a blast wave and a clump in a given experiment. Suppose the blast wave propagates at a given velocity  $u_s$  through a medium having a density  $\rho_m$  and interacts with a clump having a characteristic density  $\rho_{cl}$  and radius  $r_{cl}$  (and presumed to be spherical). The most-important scaling parameter describing this interaction is the cloud crushing time, defined as  $t_{cc} = (r_{cl}/u_s)\sqrt{\rho_{cl}/\rho_m}$ . The corresponding



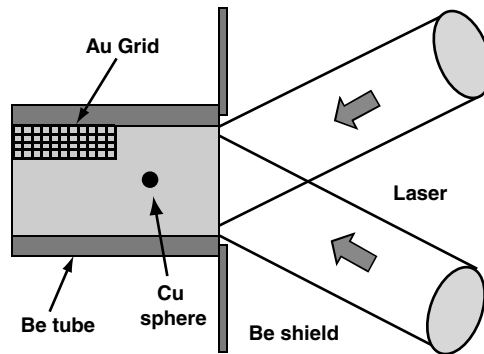


Fig. 10.6. Schematic of experiment on interaction of blast wave with clump

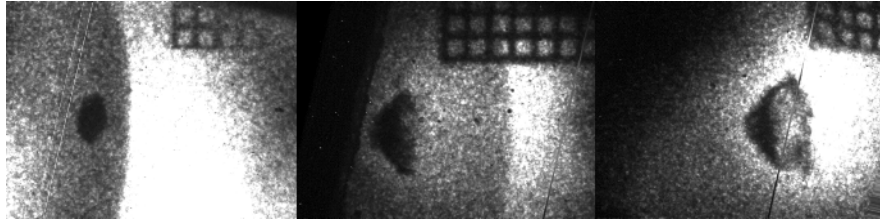
dimensionless parameter in the experiment is the ratio of the duration of the experiment (as a well-scaled system) to  $t_{cc}$ . This must be large enough to see the dynamics of interest. The second scaling parameter that describes this interaction is the ratio of the width of the blast wave between its half-pressure points to  $r_{cl}$ . To model a specific astrophysical system in detail, one would also have to match this parameter to the astrophysical value.

### Homework 10.3

Determine why  $t_{cc}$  as just defined is the relevant timescale for the crushing of the cloud.

Figure 10.6 is a sketch of the geometry used in experiments to address the destruction of a single clump by a planar blast wave (Klein et al.; Robey et al. 2002). A number of laser beams irradiated a layer of plastic material, driving a shock into it. After the laser pulses ended, the rarefaction of the front surface overtook this shock, creating a planar blast wave just as described in the previous section. This blast wave eventually encountered a dense (Cu) sphere, whose evolution was observed for several cloud-crushing times. In a fundamental sense, this experiment was not as well scaled as the one described in Sect. 10.2, because the Cu sphere was liquefied but not vaporized by the shock and thus had an equation of state rather different from that of an astrophysical cloud. Later, improved experiments used lower-density materials that could be vaporized by the shock. As a result, this specific experiment is of the type in which the experiment is not completely well scaled but certain key dimensionless parameters are well scaled, so that it can be instructive regarding the dynamics and perhaps useful for code validation. This initial experiment is a relevant model of the incompressible fluid dynamics of cloud destruction.

These experiments observed the evolution of the sphere for several cloud-crushing times. The interaction of the sphere with the postshock flowing



**Fig. 10.7.** Data from an experiment on interaction of blast wave with clump. These three frames show what has become of the spherical obstacle after the passage of the shock, seen in the first two frames but beyond the region observed in the third frame. One can see the crushing of the sphere followed by the development of vortex rings. The squares in the reference grids are  $63\ \mu\text{m}$  on a side. From Robey et al. (2002)

plasma produces vortex rings (see Sect. 5.8). This is illustrated in Figure 10.7. These are circular spirals of fluid that initially have vorticity on their surfaces. The vortex rings are subject to bending instabilities that produce three-dimensional structure by modulating the rings in the azimuthal direction (see Widnall and Sullivan, 1973, 1974). The development and properties of these structures can be examined in the data and in astrophysical simulations of a similar system. Thus one can use these experiments to test the ability of astrophysical codes to simulate this type of shock–cloud interaction.

#### Homework 10.4

Suppose that an astrophysical blast wave of interest is produced by a supernova explosion that is a known distance  $R$  from a clump of some radius  $r_{\text{cl}}$ . Determine the properties of an experimental blast wave and the duration of the experiment that would be required to model the shock–clump interaction in this system.

### 10.4 Scaling in Radiation Hydrodynamic Systems

Radiation hydrodynamic systems are challenging for both theory and experiment. We saw in Sect. 7.4 how the addition of radiation to shock waves greatly complicates their behavior, even without considering the details of actual radiation emission and transport. We mentioned in Chap. 7 a number of examples of astrophysical systems that are radiation hydrodynamic systems. In comparison to hydrodynamic systems, radiation hydrodynamics introduces new difficulties in scaling, in simulation code development, and in experiments. At minimum, such experiments ought to be able to provide benchmarks for the implementation of radiation hydrodynamics in astrophysical codes. Beyond that, one might hope to identify specific processes that

matter in astrophysics, that could be produced in the lab, and that were difficult or impossible to simulate with computers. In this section we discuss the issues associated with scaling. While complete scaling is possible in the abstract and perhaps in special cases, it seems more likely that actual experiments will scale the important dimensionless parameters well but will not manage to successfully scale all aspects of the radiating system. In this section we discuss two types of system for which laboratory experiments seem feasible – radiative shocks and radiative jets.

We will consider here only the radiative-flux regime, leaving the scaling of radiation-dominated experiments to be developed when such experiments exist. (Our discussion parallels that of Ryutov et al. 2001.) In the radiative-flux regime, the first two Euler equations (2.1 and 2.2) remain unchanged, but a radiative-heating term must be added to the third Euler equation (2.3), which becomes

$$\frac{\partial p}{\partial t} + \mathbf{u} \cdot \nabla p = -\gamma p \nabla \cdot \mathbf{u} - (\gamma - 1) \nabla \cdot \mathbf{F}_R, \quad (10.7)$$

in which as usual  $\mathbf{F}_R$  is the radiation flux.

In the abstract, one may be able to express  $\nabla \cdot \mathbf{F}_R$  as a power law function of density, pressure, and scale length in some astrophysical system, so that

$$\nabla \cdot \mathbf{F}_{R1} = A \rho^{\alpha_1} p^{\alpha_2} r^{\alpha_3}. \quad (10.8)$$

Note that this formulation requires that the radiation transport be local. This implies that the system of interest is either optically thin so that radiation is a pure loss term or optically thick so that a diffusion model applies. For an optically thin system,  $\alpha_3$  would be zero, while for an optically thick system it might not be. This case will serve our purposes here, although even more general cases are possible. In principle  $\nabla \cdot \mathbf{F}_R$  might be a somewhat more complicated function of vectorial position  $\mathbf{r}$  and might also include a dependence on velocity.

If we now perform the variable transformation described in Sect. 10.1, we obtain

$$\frac{\partial \tilde{p}}{\partial \tilde{t}} + \tilde{\mathbf{u}} \cdot \nabla \tilde{p} = -\gamma \tilde{p} \nabla \cdot \tilde{\mathbf{u}} - (\gamma - 1) \left[ A \rho^{*(\alpha_1+1/2)} p^{*(\alpha_2-3/2)} L^{*\alpha_3} \right] \tilde{\rho}^{\alpha_1} \tilde{p}^{\alpha_2} \tilde{r}^{\alpha_3}. \quad (10.9)$$

This equation will be invariant between any two systems, such as an astrophysical system and a laboratory system, if the coefficients  $\alpha_1, \alpha_2$ , and  $\alpha_3$  and the quantity in square brackets are the same in both systems. In practice the values of  $\rho^*$  and  $L^*$  (if applicable) will be fairly tightly constrained in an experiment, so the primary adjustment that could be used to obtain a well-scaled experiment would be in increasing  $p^*$  to make the quantity in square brackets equal in the experiment to its value in the astrophysical system and to adjust the composition of the experimental materials to make the coefficients equal.

As one might expect, one loses some freedom in specifying the parameters by comparison with the purely hydrodynamic case. But all that matters to have a well-scaled experiment is the net cooling rate and its dependences on pressure and density. The microscopic mechanisms are not important. This is significant because the radiation cooling in optically thin astrophysical systems is nearly always due to line radiation while the cooling in high-energy-density experiments is more often dominated by thermal radiation. Unfortunately, achieving such a perfectly scaled experiment is a very tall order, requiring at minimum a very sophisticated design effort. In addition, for any given case it is unlikely to be practical. This at least is the view from the early 21st century.

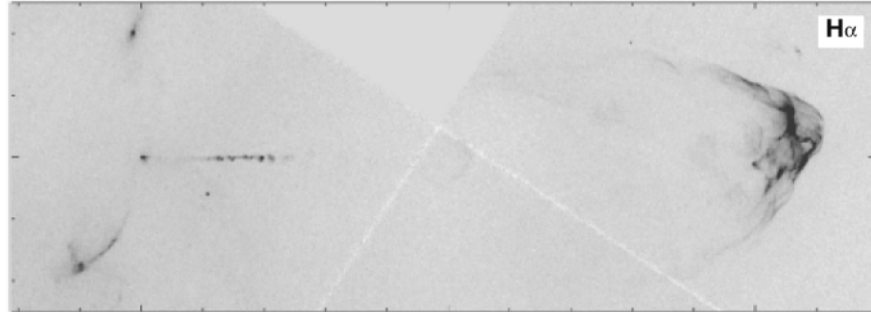
Indeed, we have seen in Chap. 7 that radiation transport is often non-local. In addition, even in cases with local radiation transport it may not prove feasible to obtain the same, power-law scaling for the radiation in both systems. In such cases, one cannot produce an experiment whose evolution will be identical, in a scaled sense, to a specific astrophysical system. Even so, the behavior of the astrophysical system may depend primarily on certain dimensionless parameters that reflect the processes which control its dynamics. Then one may be able to observe phenomena in the laboratory with the same values of these essential dimensionless parameters. In the case of radiative shocks, we saw in Chap. 7 that the controlling parameters are the optical depth of the upstream and downstream regions. In the following, we consider the case of radiative jets.

## 10.5 Radiative Astrophysical Jets: Context and Scaling

Many astrophysical jets are purely hydrodynamic, and so the discussion of Sect. 10.3 would apply to simulation experiments aimed at them. Other astrophysical jets are inherently magnetized or involve strong magnetic fields. One might do some experiments relevant to these using relativistic high-energy-density systems, but it is very unlikely that one could do so working in the nonrelativistic regime that has been our primary focus. Our present interest, in the context of radiation hydrodynamics, is in radiative jets. We proceed to discuss these here.

### 10.5.1 The Context for Jets in Astrophysics

Galactic and extragalactic jets present some of the most visually intriguing images encountered in astrophysics. One class of such objects are the stellar jets known as Herbig–Haro (HH) objects (see Reipurth and Bally 2001), thought to be collimated bipolar outflows emerging from accretion disks during the star formation process. Figure 10.8 shows an image of one such jet, HH 34. The jet shown emanates, at velocities of  $\sim 300$  km/s, from the pole of a protostar near the bottom of the image. The protostar itself is hidden;



**Fig. 10.8.** Image of the HH 34 jet. This image shows the hydrogen  $H\alpha$  emission. The protostar is near the left in the center. From Riepurth et al. (2002)

one sees reflections through the dust surrounding it. Like other HH jets (see Hartigan et al. 2000), HH34 includes multiple bow shocks, suggesting that the bipolar outflow has been episodic. It is typical that the fastest moving material (at  $\sim 200$  km/s) occurs on the axis, with slower yet higher luminosity material concentrated at the edges of the jet (see Hartigan et al. 1993). This suggests entrainment of ambient material due to the Kelvin–Holmholtz (KH) instability along the edges of the jet, leading to a greater shock excitation but slower velocities at the edges. The HH jets have typical scales of  $10^{17}$  cm, velocities of a few hundred km/s and densities  $n_{\text{jet}}$  of  $10^2$  to  $10^3$   $\text{cm}^{-3}$ . In terms of density contrast, this corresponds to  $\eta = n_{\text{jet}}/n_{\text{ambient}} \gg 1$ , where  $n_{\text{ambient}}$  is the ambient number density. The bow shocks in such jets are often radiative shocks. The internal shocks may or may not be.

Whereas HH jets are thought to be emitted during the formation phase of a star, another category of jet is formed toward the end of the evolution cycle. A star of a certain mass can pass into the asymptotic giant branch (AGB) phase and then to the planetary nebula or proto-planetary nebula (PPN) phase on its way to becoming a white dwarf. During the AGB-to-PPN transition, it appears that bipolar jets can again be emitted, one example being He 3-1475 (see Borkowski, Blondin, and Harrington). The central source for this system is a star at a distance of  $\sim 2$  kpc, which is in the midst of making the transition from a dust-enshrouded AGB star to a PPN. The star is surrounded by a torus of molecular material expanding at 12 km/s. The most spectacular features of He 3-1475 are the optical jets, and three pairs of symmetric knots, moving in the direction perpendicular to the molecular torus. The knots are located symmetrically with respect to the central star and are moving radially away at velocities of 500 km/s. Closer to the star, the jetlike outflows have velocities as high as 850 km/s. Radiative shocks moving at velocities of  $\sim 100$  km/s are thought to be the excitation mechanism for the observed emissions.

### Homework 10.5

Magnetized jets must have a ratio of plasma pressure to magnetic field pressure (usually called  $\beta$  in plasma physics) no larger than about 1. For a low- $Z$  plasma with a density of  $0.1 \text{ g/cm}^3$  and a temperature of 10 eV, determine how large a magnetic field would be required to satisfy this constraint. How does this compare with the magnetic field of order 1 MGauss that is typically produced in laser-plasma interactions and that might be produced by very clever field-compression experiments?

The basic features of a high Mach number jet, present for both radiative and purely hydrodynamic jets, are illustrated schematically in Figure 10.9 (Hartigan 1989). A source is assumed to exist that creates a beam or jet of material (labeled 1) streaming into the ambient medium (labeled 2). This launches a forward or bow shock into the ambient medium, moving at speed  $v_{BS}$ . The presence of the ambient medium causes the jet material to slow down, creating a Mach disk in the jet. Within this description, beam or jet refers to the collimated material streaming from the source to the Mach disk. Shear along the sides of the jet triggers the Kelvin-Helmholtz (KH) instability, which generates vortices and eddies that churn up mixing along the contact discontinuity. The region of shocked jet material between the contact discontinuity and jet (beam) is referred to as the cocoon. The KH vortices in the cocoon can launch shocks into the jet (beam), which act as a heating mechanism for radiative emissions from within the jet.

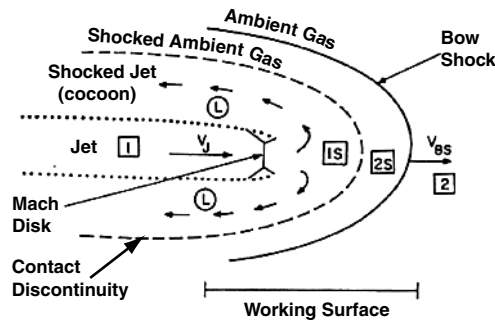
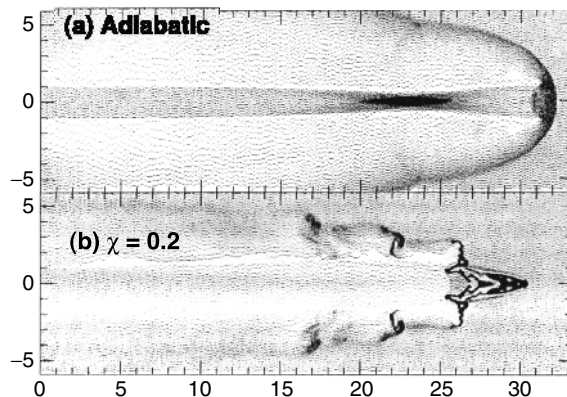


Fig. 10.9. Schematic of the structure of an astrophysical jet, from Hartigan (1989)

### 10.5.2 Scaling from Radiative Astrophysical Jets to the Laboratory

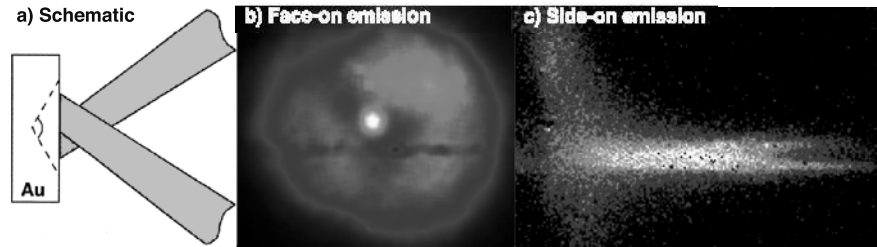
There are three dimensionless parameters that characterize the properties of a jet and the degree to which radiation is important to its dynamics. The *internal Mach number* (the ratio of the flow velocity to the sound speed within



**Fig. 10.10.** Impact of radiative cooling on jet structure, for Mach 20 jets whose density equals the initial ambient density. From Blondin et al. (1990)

the jet),  $M_{\text{int}}$ , characterizes the amount of kinetic energy that can potentially be converted to thermal energy through shocks. The *cooling parameter*,  $\chi_j$ , is defined by  $\chi_j = L_{\text{cool}}/R_{\text{jet}}$ , where  $R_{\text{jet}}$  is the jet radius and  $L_{\text{cool}}$  corresponds to the length behind the Mach disk beyond which the jet has cooled to some low value. This characterizes the relative scale on which radiation can alter the properties of the matter in the jet. The *density parameter* is  $\eta$  as defined above. This affects the amount of pressure in the shocked ambient medium, which interacts both with the head of the jet in the Mach disk region and with the jet along its length *via* the pressure in the cocoon.

The effects of radiative cooling on astrophysical jets can be very large (see Blondin et al. and Stone and Norman 1994). Figure 10.10 shows results of simulations that assume a very high Mach number jet, having  $M_{\text{int}} = 20$ , and an equal density with the ambient medium, so  $\eta = 1$ . The plasma is assumed to be optically thin, so that photons emitted by hot regions of the jet leave the system. The effects of radiation are included by means of a time-independent cooling function  $\Lambda(T)$ , assuming equilibrium conditions (see Sect. 6.2.2 and Fig. 6.5). Here  $\chi_j$  is calculated for cooling to a temperature of  $8 \times 10^3$  °K. As the magnitude of radiative cooling is increased from a purely adiabatic jet,  $\chi_j \gg 1$  (Fig. 10.10a), to a strongly cooled jet,  $\chi_j = 0.2$  (Fig. 10.10b), the jet morphology changes significantly. Radiative cooling removes heat from the system, lowering the internal pressure of the jet. The working surface and jet contract until pressure equilibrium is reestablished, making the radiatively cooled jets denser and more compact. In a more detailed description, radiation should be treated as a nonequilibrium, time-dependent process, and if the medium is not optically thin, full transport (nonlocally redepositing the photon energy) may be needed.



**Fig. 10.11.** A radiative jet experiment. Schematic (a) and images of self-emission ((b) axial and (c) side-on) from an experiment producing a radiatively collapsing Au jet. Adapted from Farley et al.

### 10.5.3 Radiative Jet Experiments

High-energy-density experiments offer the means to create high-Mach-number jets and to diagnose their subsequent dynamics and evolution (see Stone et al. 2000). For example, the effect on jet dynamics of variations in  $\eta$ ,  $M_{\text{int}}$ , and  $\chi_j$  can in principle be directly observed in laboratory experiments. Here we discuss as an example the first experiment to produce a radiative jet, done on the Nova laser at LLNL (see Farley et al. and Stone et al. 2000). A schematic of the experimental arrangement is shown in Fig. 10.11a. A gold disk had a 800  $\mu\text{m}$  diameter cone of  $120^\circ$  full opening angle machined into it. Five high-irradiance, 100 ps laser beams from the Nova laser irradiated the inside surface of this cone were symmetric (in azimuth), with an average irradiance  $I_L \approx 3 \times 10^{15} \text{ W/cm}^2$ . The high-speed, ablated Au plasma thus created expands in a direction normal to the local surface. The radial velocity component causes this plasma to implode and stagnate on the axis of the cone. The axial velocity component brings the imploded plasma out into view as a high-speed jet. Radiation from the hot, stagnated plasma causes the jet to shrink in size and increase in density.

This experiment was diagnosed with x-ray imaging. One gated x-ray framing camera looked directly face-on at the cone. This is illustrated in Fig. 10.11b with the face-on x-ray image in emission at  $t = 0.25 \text{ ns}$  relative to the peak of the Gaussian laser drive. The small bright spot near the center of the image is the imploded Au plasma that has stagnated on the axis, and that is moving at  $\sim 750 \text{ km/s}$  out of the page, directly toward the recording x-ray camera. Views of this same jet from the side in soft x-ray emission at 1.1 ns (Fig. 10.11c) show that the radiative emissions later in time are on the surface of the Au jet. The reason for the forked nature of the emissions in the side-on image is that the regions of the imploding plasma that stagnate first on the axis (at  $\sim 0.5 \text{ ns}$ ) radiatively cool at first, and appear dark (cool) later in time. By 1.1 ns, this leading tip region stops emitting in the soft x-ray band for which the instrument is sensitive, because the electron temperature  $T_e$  has dropped dramatically. A side-on radiograph found the densest part of the jet to be along the cone axis (Farley et al.). Simulations of this experiment



illustrate the importance of radiation. As the plasma collides on the axis, it heats up to over  $\sim 1$  keV and has a high ionization state,  $Z \sim 40$ , but a low density,  $\rho \sim 40$  mg/cm<sup>3</sup>. In this state, the hot Au plasma cools itself rapidly by radiative emissions, since the plasma is initially optically thin to the keV photons. The temperature was measured by Thomson scattering at 0.6 ns, at which time  $T_e$  had already dropped to 250 eV. The radiative cooling leaves a very compact, highly collimated jet moving axially away from the cone at  $\sim 750$  km/s. In simulations that do not include radiative cooling, the jet is an order of magnitude too broad, since the pressure of the stagnated Au plasma is high, which would stop the implosion (see Mizuta et al.).

A continuation of this experiment was done on the Gekko-12 laser at the University of Osaka in Japan (see Shigemori et al.). These experiments produced jets from cones of Au, Fe, Al, and CH. The results show a clear correlation: the higher the  $Z$ , the shorter the cooling time and the narrower the jet. The experiment at Gekko generating a Au jet reproduced very closely the results from the original experiment, showing that the physics being investigated is reproducible and not facility dependent. In a complementary experiment, the Magpie  $Z$ -pinch at Imperial College in London, England, was used to produce radiatively cooled, high Mach number jets (see Lebedev et al.). These radiative jet experiments observed similar trends to those just described.

These two experiments (on Nova and on Gekko) are a good example of a first attempt to produce a radiation hydrodynamic system that is relevant to astrophysics. Even as experiments that scale important dimensionless parameters, they are incomplete, because they do not include an ambient medium (thus,  $\eta \sim \infty$ ). In addition, the formation mechanism does not manage to produce an emerging source of material in a consistent initial state. Rather, the first part of the jet to form (that nearest the target) is the trailing portion of the jet that emerges. Each part of this jet has a unique history of energy input and cooling dynamics. So if one were seeking a well-scaled experiment that was a direct analog of any astrophysical jet, these first experiments would not achieve this goal. However, the experimenters did manage to vary the radiative cooling parameter  $\chi_j$  over the range of 0.7 to 40 and to vary the internal Mach numbers from 2 to 50. Given the paucity of radiation hydrodynamic experimental data, these experiments are in fact of real value for the validation of astrophysical codes. But further invention is needed to develop radiative jet experiments that are well scaled even with regard to the essential dimensionless parameters.

### Homework 10.6

An approach that has been used to form hydrodynamic jets is to create an adiabatic rarefaction by allowing a shock wave to emerge from a material into an evacuated tube and then to emerge from this tube into an “ambient medium”, at a lower density. Using the simple scaling results from this book, develop a design for a similar experiment to produce a radiative jet.

## 11 Relativistic High-Energy-Density Systems

In this chapter we address the low-density and high-temperature regime of high-energy-density physics identified in Chap. 1. While phenomena produced in this regime often connect with those discussed in the previous chapters, there are real differences in the underlying physics. A high-energy-density, thermal, relativistic plasma would have a minimum temperature of 511 keV and a density exceeding  $10^{18} \text{ cm}^{-3}$ . At the turn of the century, such plasmas did not exist in the laboratory. Producing them can be taken as a challenge for the early 21st century. However, plasmas did exist at this density with a mean electron energy exceeding 511 keV. Some such plasmas were made relativistic by the electron oscillations caused by intense lasers. We will define a *relativistic laser beam* as one producing a mean electron kinetic energy exceeding 511 keV. Some of these laser-irradiated plasmas produce beams of electrons with characteristic energies of many MeV. In addition, other, denser plasmas existed with a mean electron energy exceeding 511 keV because of the presence of a highly relativistic electron beam in a cold background plasma.

To place these systems in context, we return to the definition of high energy density as corresponding to a pressure exceeding 1 Mbar, or an energy density exceeding  $10^{12} \text{ ergs/cm}^3$ . Table 11.1 is based on Table 1.1 in the National Research Council report (Davidson). From the discussion of Chap. 8, it is clear that achieving these conditions is not so difficult. Here we consider three specific relativistic examples.

For lasers, the challenge is to get the kinetic energy of the oscillating electrons up to 511 keV. This requires  $I_{18}\lambda_\mu^2 = 1.35$ , where  $I_{18}$  is the laser irradiance in units of  $10^{18} \text{ W/cm}^2$  and  $\lambda_\mu$  is the laser-light wavelength in  $\mu\text{m}$ . This is not now difficult. It requires a laser of  $10^{10}$  to  $10^{12} \text{ W}$ , assuming the focal spot to be 1 to 10  $\mu\text{m}$ . This is less than 1 J in 1 ps or 10 mJ in 10 fs. The energy density of the electrons within such a focal spot remains a small fraction of the energy density of the laser beam.

Electron beams at the turn of the century could produce 50 GeV electrons in a 5  $\mu\text{m}$  spot, with bunches of 5 ps duration at a repetition rate of 100 Hz. The bunches contained 150 J each and thus contained  $2 \times 10^{10}$  electrons. The bunches were long and narrow, being more than 1 mm long. Their volume was  $\sim 10^{-8} \text{ cm}^3$ , so the density of these electrons is  $2 \times 10^{18} \text{ cm}^{-3}$ . When such

**Table 11.1.** Quantities corresponding to  $10^{12}$  erg/cm<sup>3</sup>

Pressure	1 Mbar = 0.1 TPascal
Irradiance of laser or relativistic particle beam	$3 \times 10^{15}$ W/cm <sup>2</sup>
Blackbody radiation temperature	400 eV
Electric field strength	$1.5 \times 10^{11}$ V/m
Magnetic field strength	5 MGauss
Ablation pressure by (1 $\mu$ m wavelength) laser at	$4 \times 10^{12}$ W/cm <sup>2</sup>
Ablation pressure by thermal radiation at	75 eV
Particle density for 511 keV mean kinetic energy	$10^{18}$ cm <sup>-3</sup>

a beam passes through a solid with an electron density of  $2 \times 10^{23}$  cm<sup>-3</sup>, the resulting average electron energy is  $\sim 500$  keV. These beams do not deposit their energy very readily, so studies with them primarily involve ways to affect the beam. This includes the important area of wakefield acceleration, discussed in Sect. 11.7.2.

Ion beams at the turn of the century (specifically the Relativistic Heavy Ion Collider) could cause ion bunches to collide at a 50 MHz repetition rate. The ions had an energy of 100 GeV per amu, or 20 TeV for Au ions; the bunches were of  $\sim 500$  ps duration,  $\sim 200$   $\mu$ m diameter, and  $\sim 3$  kJ energy. Each bunch of such ions has an energy density of about  $3 \times 10^{12}$  ergs/cm<sup>3</sup> shared among about  $10^9$  ions. Here again, the ion beams do not deposit their energy very readily, so studies with them primarily involve ways to affect the beam. In addition, as we mentioned in Chap. 8, beams of nonrelativistic heavy ions can be used to heat high- $Z$  matter into the high-energy-density regime.

Most of this chapter is devoted to the behavior of matter in the presence of electromagnetic fields strong enough to produce relativistic electron motions. The devices that produce relativistic laser beams are called *ultrafast lasers*, for reasons that will become clear. The emphasis on ultrafast lasers reflects both their broad availability and their potential to produce extremely high electric and magnetic fields. We also discuss relativistic effects that can be produced using high-energy electron beams in the area of particle acceleration. One can expect that more such applications will develop as the 21st century proceeds.

As in the previous chapter, this chapter includes many more references to journal articles than most of the book does. This reflects the relative newness of work in this area. However, once again the present chapter is not written as a review. Rather, it is intended as an introduction to the physics and the issues.

## 11.1 Development of Ultrafast Lasers

Since about 1980, the challenge of producing high-power lasers has become distinct from that of producing high-energy lasers. While the laser systems described earlier can heat cubic millimeters of material to million-degree temperatures, they cannot produce relativistic electrons or distributions of ions with billion-degree temperatures. This requires much more intense laser light. One cannot produce such light by directly amplifying a laser pulse; the amplifying glass would be damaged. The invention by Gerard Mourou of *chirped pulse amplification* (CPA) has allowed laser systems to escape this limitation (Mourou and Umstadter). All intense lasers amplify a laser beam whose spatial area is much larger than the ultimate focused laser spot. CPA goes further, also doing this in the dimension of time. In CPA, one amplifies a laser pulse whose time duration is much longer than the ultimate duration of the pulse reaching the laser spot. This is done as follows.

Despite the notion that lasers are coherent, single-frequency devices, any laser pulse in fact has a finite bandwidth. For a laser pulse that is Gaussian in time, with a frequency bandwidth  $\Delta\omega$  and time profile  $\exp[-(t/\tau)^2]$ , one can show by Fourier transforming the laser pulse that  $\Delta\omega\tau \sim 1$ . Thus, very short laser pulses may have a significant bandwidth. It is now possible to produce laser pulses of order one cycle in duration ( $\sim 1$  fs for visible light); such pulses have a very broad bandwidth. There are a number of methods for producing such pulses at low energy and low irradiance. The contribution of CPA is to provide a way to stretch these pulses in time, allowing them to be amplified at low irradiance before they are recompressed to high irradiance, after which they are focused to enormous irradiance.

Figure 11.1 illustrates a simple compressor design that can stretch a pulse in time. The first grating disperses the incoming, broadband, collimated laser pulse in angle, so that the angle of reflection of each frequency is distinct. This is illustrated in the figure by showing a pair of rays one labeled red and one labeled blue. The grating diffracts the longer wavelength, red rays through a

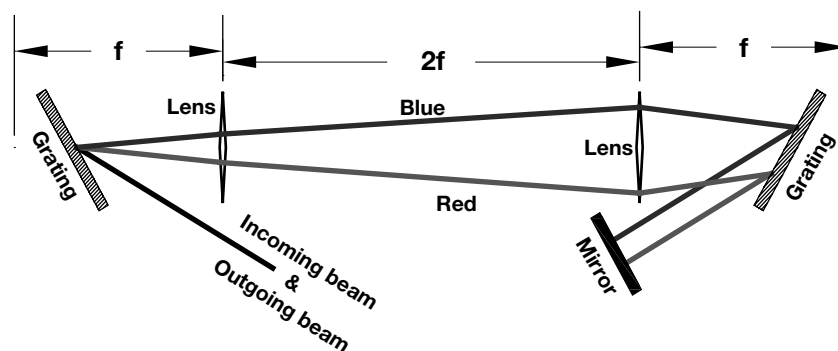


Fig. 11.1. Schematic of pulse stretcher. Credit: Enam Chowdhury

larger angle. A lens pair is used to cause the angle of incidence on a second grating to equal the angle of reflection from the first grating. The result is that a collimated laser beam emerges from the second grating, with different frequencies offset in space. A mirror reflects the light back upon itself, so that each frequency retraces its path and one obtains an outgoing beam of the same size as the initial beam. Geometrically nothing has changed, but temporally each frequency has traveled a different distance, and the redder frequencies have traveled less distance. One has dispersed the beam in time, producing a *chirped laser pulse* whose frequency varies linearly with distance along the pulse, with redder frequencies at the front.

### Homework 11.1

Design a pulse stretcher. Suppose you have a laser beam with an 800 nm central wavelength and a bandwidth of 20 nm (corresponding to a 50 fs laser pulse). Use two identical gratings, recalling that for the first diffracted order the scattered wavelength  $\lambda$  is given by  $\lambda = d(\sin \alpha + \sin \beta)$ , where  $d$  is the line spacing on the grating and  $\alpha$  and  $\beta$  are angles of incidence and reflection relative to the grating normal. Use two identical lenses, recalling that the object distance,  $o$ , image distance,  $i$ , and focal length  $f$  are related by  $o^{-1} + i^{-1} = f^{-1}$ . Note that the initial grating must be less than one focal length from the lens to obtain stretching.

At this writing some ultrafast lasers have been dedicated systems with a very short laser pulse, an excellent quality laser beam, and comparatively little laser energy. Other systems have been aimed at delivering more laser energy to the targets, which would be necessary for example for inertial fusion using fast ignition (Sect. 8.2.4). These higher-energy systems are often adaptations of a high-energy laser to the task of amplifying short laser pulses. However, none of these lasers yet produces enough energy to create an isolated, thermal, relativistic plasma.

## 11.2 Single-Electron Motion in Intense Electromagnetic Fields

Many of the fascinating phenomena that ultrafast lasers can produce are a consequence of the relativistic motion of the electrons. To develop some insight into what this makes possible, we consider first the motion of isolated electrons in the fields of such lasers. To do so, we take the electron velocity to be  $\mathbf{v}$ , the electron momentum to be  $\mathbf{p}_e$ , the vector and scalar potentials to be  $\mathbf{A}$  and  $\Phi$ , and we work as in Chap. 8 in the Coulomb gauge. In Gaussian cgs units, the electromagnetic wave equation is not changed for high velocities, although one must transform the fields properly between inertial frames

of reference. To work complicated problems in relativity, four-vector notation becomes very convenient, but we will not invest time in this here. The electromagnetic wave equation is

$$\left(\frac{\partial^2}{dt^2} - c^2 \nabla^2\right) \mathbf{A} = 4\pi c \mathbf{J}_t, \quad (11.1)$$

in which as before  $\mathbf{J}_t$  is the transverse current density. This equation is the same for relativistic and nonrelativistic systems. It is helpful to take note of some aspects of the implied fields, useful in what follows. In a plane-wave decomposition, each spectral component has a distinct  $\mathbf{k}$ ,  $\mathbf{E}$ , and  $\mathbf{B}$ , and these are all orthogonal. We let  $\mathbf{k}$  define the parallel ( $z$ ) direction. Since  $\mathbf{E} = (-1/c)(d\mathbf{A}/dt)$ , so long as the electric potential  $\Phi = 0$ , and  $\mathbf{B} = \nabla \times \mathbf{A}$ , one can see that (for any spectral component)  $\mathbf{E}$  and  $\mathbf{B}$  are out of phase with  $\mathbf{A}$  and in phase with each other. The fact that  $E$  and  $B$  are in phase and vanish simultaneously may be counterintuitive if you have not considered it previously. We allow  $\mathbf{A}$  and thus  $\mathbf{E}$  to define the  $x$  direction and  $\mathbf{B}$  to define the  $y$  direction. We define the *pump strength* of the electromagnetic wave as

$$\mathbf{a}_o = \frac{e\mathbf{A}}{m_e c^2}, \quad (11.2)$$

for reasons that will become clear shortly. In practical units,

$$a_o = \sqrt{\frac{I_L \lambda_\mu^2}{1.37 \times 10^{18} \text{W } \mu\text{m}^2/\text{cm}^2}}. \quad (11.3)$$

We now consider the motion of an electron in a single plane wave (which may for now include arbitrary spectral components). The equation of motion for an electron is not changed in its fundamental form: the time rate of change of momentum equals the force. But the electron momentum,  $p_e$ , is now relativistic. Given our definitions, the  $y$  component of the electron momentum is constant in time. The equation of motion for the  $x$  and  $z$  components of the electron momentum, labeled  $p_x$  and  $p_z$ , respectively, are

$$\frac{dp_x}{dt} = -\frac{e}{c} (cE_x - v_z B_y), \quad (11.4)$$

and

$$\frac{dp_z}{dt} = -\frac{e}{c} (v_x B_y), \quad (11.5)$$

in which  $\mathbf{p}_e = \gamma_r m_e \mathbf{v}$  and in these Gaussian cgs units  $E_x = B_y$ . To avoid complications involving products of real quantities in complex notation, we here assume that  $E_x$  and  $B_y$  are real quantities, although (as the particle experiences them) they may vary arbitrarily in time and space. We allow the momenta and velocities to be complex.

If one thinks of a given frequency component of the electric field as seen by the particle, one can see that the detailed motion of the electron has an indefinite number of harmonic components. The electric-field term in (11.4) creates a second-harmonic response in the  $z$  motion (11.5), which in turn creates a third-harmonic response in the  $x$  motion through the magnetic-field term, which then creates a fourth-harmonic response in the  $z$  motion, and so on.

Equations (11.4) and (11.5) and the related conventions enable us to derive a generally useful relation between  $p_z$  and  $p_x$ , as follows. Defining the total particle energy as  $\mathcal{E}_e$ , so that with rest mass  $m_e$  and rest energy  $\mathcal{E}_o = m_e c^2$ , one has

$$\mathcal{E}_e = \sqrt{\mathcal{E}_o^2 + p^2 c^2}. \quad (11.6)$$

The rate of change of the particle energy is due entirely to the work done by the electric field, so

$$\frac{d\mathcal{E}_e}{dt} = -ev_x E_x, \quad (11.7)$$

from which by comparison with (11.5) we can see that

$$\mathcal{E}_e - cp_z = \text{const} \equiv \alpha. \quad (11.8)$$

Here  $\alpha$  is defined for convenience. It depends on the state of the particle when the field begins. If the particle is at rest at that time, then  $\alpha = \mathcal{E}_o$ . In general, from (11.6) and (11.8), one finds

$$\mathcal{E}_o^2 = \mathcal{E}_e^2 - p^2 c^2 = (\mathcal{E}_e - p_z c)(\mathcal{E}_e + p_z c) - p_x^2 c^2 = \alpha(\alpha + 2p_z c) - p_x^2 c^2, \quad (11.9)$$

from which

$$p_z = \frac{1}{2\alpha c} (\mathcal{E}_o^2 - \alpha^2 + p_x^2 c^2). \quad (11.10)$$

If the particle is initially at rest so that  $\alpha = \mathcal{E}_o = m_e c^2$ , then we obtain the well known result

$$p_z = \frac{p_x^2}{2m_e c}. \quad (11.11)$$

This equation has some interesting things to tell us. First of all, it says that motions in both  $x$  and in  $z$  are always part of the response of an electron to a wave. Second, recalling that the momenta are complex in this representation, the motion in  $z$  includes a steady drift and an oscillation at twice the frequency of the oscillation in  $x$ . This combination creates a path that looks like a figure eight, vertically elongated when  $p_x$  is small compared to  $m_e c$ . Third, isolated electrons cannot be permanently accelerated by light waves. When an electron initially at rest is overtaken by an electromagnetic wave packet, the electron oscillates in  $x$  and drifts in  $z$ , but as the wave packet passes both these motions cease. The electron ends up displaced but once again stationary. In the absence of collective effects, (isolated) electrons must be created within the light wave, for example by ionization, to end up with

significant net energy once the wave has passed. Fourth, in the limit of very large fields,  $|p_z| \rightarrow \gamma_r m_e c$ , in which case  $|p_x|$  is smaller, being  $\sqrt{2\gamma_r} m_e c$ . As a result,  $v_x$  decreases, becoming  $v_x = p_x / (m_e \gamma_r) \rightarrow c\sqrt{2/\gamma_r}$ . Finally, the angle of the electron relative to the  $z$ -axis,  $\theta$ , is given by

$$\tan \theta = \sqrt{\frac{2}{\gamma_r - 1}} \quad (11.12)$$

This has sensible limits, going to  $90^\circ$  as  $\gamma_r$  reaches 1 and approaching 0 degrees as  $\gamma_r$  becomes very large.

### Homework 11.2

Assuming that the electron motion is due to a plane wave with a single frequency and that the electron movement is small compared to the wavelength of the light wave, solve the above equations to find the electron trajectory. Determine how it changes as the electron velocity increases (while remaining  $\ll c$ ).

Thinking strictly in terms of harmonic motion leaves out some aspects of the electron motion, because as the field of the laser increases the electron motion along  $z$  soon becomes significant on the scale of the laser wavelength. In more detail, the electric field experienced by an electron within a ( $z$ -directed) single-frequency plane wave in vacuum is

$$E_x(\mathbf{x}_p, t) = \hat{E} \cos [kz_p(t) - ct + \phi_o], \quad (11.13)$$

in which

$$z_p(t) = z_o + \int_{t_o}^t v_z(t') dt'. \quad (11.14)$$

Here  $\phi_o$  gives the phase of the electric field, the  $z$ -position of the particle is  $z_p$ , and  $z_p$  at time  $t_o$  is  $z_o$ . One can see that the particle will experience this plane wave as a simple harmonic field only if  $z_p$  is a linear function of  $t$ . Yet we have already seen that this is not the case. Thus, an electron experiences a light wave as a simple harmonic field only in the limit as the motion of the electron in  $z$  vanishes. In the other limit, as  $z_p \rightarrow ct$ , the electron will experience a nearly constant field. In this limit the maximum energy the electron can extract from a light wave of finite spot size  $d$  is  $eE_L d$ , where the electric field of the focused light wave is  $E_L$ .

It is evident that an electron may be introduced into the wave with any phase and thus may experience any field from zero to the maximum when it is born. If the electron is born at rest when the electric field is zero, then the electron returns to rest at the end of each cycle and ends up at rest when the wave has passed. If the electron is produced at rest when the oscillating field is at its maximum, then the electron ends up with the maximum possible energy. When ionization creates the electrons, they may in principle have



any phase with respect to the wave. But if the ionization is produced by the wave itself, then the electron will tend to be produced when the electric field is maximum and at near zero velocity. In this case most of the electrons will gain a significant net velocity from their interaction with the wave.

We can explore the electron motion somewhat further as follows. We write the Lorentz factor, which we label as  $\gamma_r$ , as

$$\gamma_r = \left( 1 / \sqrt{1 - \frac{v^2}{c^2}} \right), \quad (11.15)$$

or

$$\gamma_r = \sqrt{1 + \frac{p_e^2}{m_e^2 c^2}}. \quad (11.16)$$

### Homework 11.3

Prove that these definitions (11.15) and (11.16) are equivalent.

Now we consider the solutions to (11.4) and (11.5) as the velocity of the electron increases. In the event that  $v_z$  is negligible, one has from (11.2) and (11.4)

$$\gamma_r v_x / c = a_o, \quad (11.17)$$

implying

$$v_x / c = \frac{a_o}{1 + a_o}, \quad (11.18)$$

and also

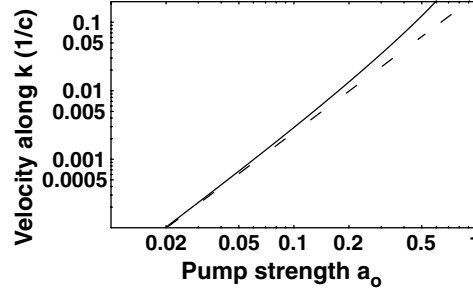
$$\gamma_r = \sqrt{1 + a_o^2} \quad (11.19)$$

in this regime. For small  $a_o$ , this is identical to the result we obtained in Chap. 8 for the oscillating velocity of the electron in a light wave. As  $a_o$  increases,  $v_x/c$  cannot exceed one, as should be the case. Equation (11.19) will fail to be accurate as  $v_z/c$  becomes significant (meaning 0.1 for most purposes). To explore this, we can use (11.18) in (11.5), finding

$$\frac{d}{dt} \left( \frac{v_z}{c} \sqrt{1 + a_o^2} \right) = - \frac{a_o}{1 + a_o} \frac{da_o}{dt}. \quad (11.20)$$

In the small  $a_o$  limit, one evidently has  $v_z/c = a_o^2/2$ . One can solve (11.20), using a computational mathematics program, to see when  $v_z/c$  approaches 0.1. The additional assumption needed is a specification of a value of  $v_z$  (e.g., zero) at some specific phase in the wave (e.g.,  $\pi/4$ ). Figure 11.2 plots the maximum value of  $\mathbb{R}[v_z]$  in the small  $a_o$  approximation and also plots  $\mathbb{R}[a_o^2]/4$ , against the magnitude of  $a_o$ . One sees that the above solutions reach their limits when  $a_o$  becomes a few tenths.

Once  $v_z$  exceeds about 0.1, the solution to (11.4) and (11.5) for the velocity and the trajectory becomes much more complex for two reasons. First,



**Fig. 11.2.** Electron oscillating velocity in direction of  $\mathbf{k}$  as  $a_o$  increases. The dashed curve shows the small- $a_o$  limit

one must deal with both terms on the left-hand side of (11.4), making their solution a nonlinear mess. Second, one must consider how the variations in  $v_z$  affect the phase of the electron in the wave and change its behavior. To see the essential feature that determines the qualitative behavior, we now pay attention to the phase of the electron in the wave.

The outcome of the interaction at very large  $a_o$  is that the electron moves along  $\mathbf{k}$  at nearly the speed of light. One can see from (11.4) that in this case the force in the  $x$ -direction will be greatly reduced. To see how this comes about, we can represent the pump strength as  $a_o \sin \phi$ , explicitly writing the phase of the wave as  $\phi$ . For convenience, we treat all the physical quantities as explicitly real for the present discussion. Now

$$\phi = (\mathbf{k} \cdot \mathbf{x} - \omega t) = (kz - \omega t) \quad (11.21)$$

in which, without loss of generality for our present purposes, we assume the phase to be zero when  $z$  and  $t$  are zero. We also take  $\mathbf{k}$  to be in the  $z$  direction, as assumed above, and chose the signs so the wave propagates in the forward direction. This is traditional but not necessary. In the discussion above, we assumed that  $z = 0$  throughout. This is reasonable if the electron is essentially stationary in  $z$ . However, as  $v_z$  increases, the electron now moves and as a result it no longer experiences a purely sinusoidal field. The position  $z_p$  of an electron is given by (11.14) with  $z_o = t_o = 0$ . The phase experienced by the electron,  $\phi_e$ , is

$$\phi_e = \left( k \int_0^t v_z(t') dt' - \omega t \right) = \omega \left( \frac{1}{c} \int_0^t v_z(t') dt' - t \right). \quad (11.22)$$

Recalling that  $B_y = E_x$ , (11.5) becomes

$$\frac{d}{dt}(\gamma_r v_z) = -\omega \left( 1 - \frac{v_z}{c} \right) (v_x a_o \cos \phi_e). \quad (11.23)$$

One can see that whether the electron is accelerated or decelerated in  $z$  depends upon the sign of  $v_x$  and  $\cos \phi_e$ . But now consider the impact of

$v_z$  on the duration of the acceleration. One can see from (11.22) that  $\phi_e$  decreases as time increases. The question now is what time  $\Delta t$  it takes for  $\phi_e$  to change by a certain amount, equal to  $-\Delta\phi_e$ . Suppose the average velocity during this period is  $\bar{v}_z/c$ . Then  $-\Delta\phi_e = \omega\Delta t(\bar{v}_z/c - 1)$ , so

$$\Delta t = \frac{\Delta\phi_e}{\omega} \frac{1}{1 - \bar{v}_z/c}. \quad (11.24)$$

The important point is that periods when  $\bar{v}_z$  is larger, as will occur if the electron is being accelerated along  $\mathbf{k}$ , last longer than periods when  $\bar{v}_z$  becomes negative. The electron rides the wave when going forward but quickly moves through it when going backward. The result is that the electron is accelerated longer when moving forward, acquiring a high average forward velocity.

The electron eventually will stop ever moving backward and instead will move forward at nearly the speed of light, with the changes occurring primarily in  $\gamma_r$  rather than in  $v_z$ . To take an approximate look at this regime, let  $v_z/c = 1 - \delta$ , where  $\delta$  is assumed to be small. Also suppose that the total speed equals  $c$ , so that  $v_x/c = \sqrt{2\delta}$  to the lowest order. Then after defining  $\eta = \omega t$  for convenience, (11.23) becomes, to the lowest order in  $\delta$ ,

$$\frac{d\delta}{d\eta} = 4\delta^3 a_o \cos\left(\pi + \int_{\eta_o}^{\eta} \delta[\eta'] d\eta'\right), \quad (11.25)$$

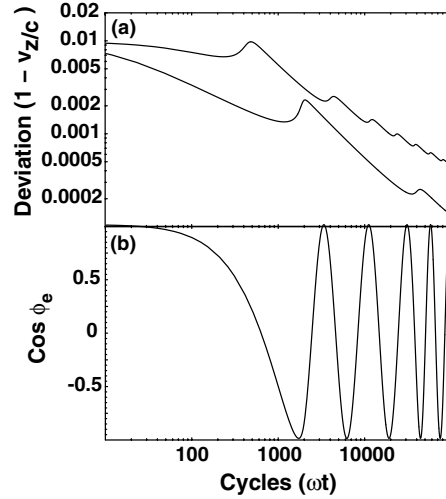
in which we have added  $\pi$  to the phase to initialize the electrons moving forward.

One can integrate this equation to see how  $\delta$  behaves as time ( $\eta$ ) increases. Figure 11.3 shows the value of  $\delta$  and the value of the cosine in (11.25), taking  $\delta = 0.01$  when  $\eta = 0$ . One can see that the brief periods of deceleration (which decrease  $v_z$  and thus increase  $\delta$ ) are not sufficient to greatly increase  $\delta$  and (allowing for the logarithmic abscissa) that the duration of a cycle increases as  $\delta$  decreases.

#### Homework 11.4

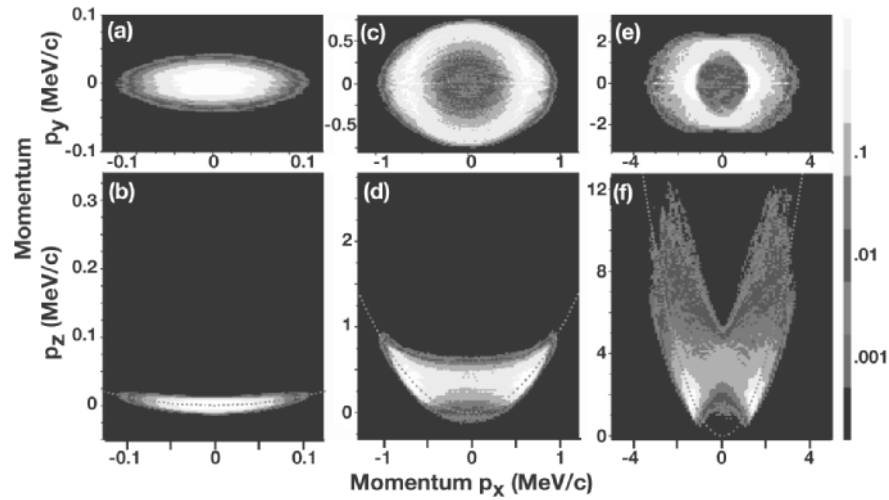
Solve (11.25) for a range of values of the initial phase (i.e., change  $\pi$  to various other values, for fixed  $a_o = 100$  and  $\delta(0) = 0.01$ . Comment on the variations in the behavior.

The simple analysis above is not self-consistent, and it ignores the range of initial conditions produced when an electron is introduced to the light wave with an arbitrary phase. Enam Chowdhury, in his Ph.D. thesis at the University of Delaware (2004), did a numerical treatment of a related problem. He used a tunneling-ionization model to inject electrons into fields corresponding to a model of a focused laser beam. It is helpful to review some of his results here.

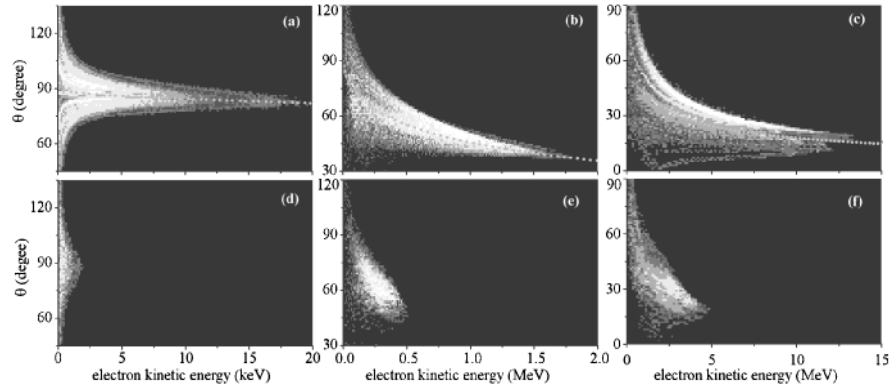


**Fig. 11.3.** Electron acceleration. (a) Deviation from speed of light for  $a_o = 10$  (upper curve) and  $a_o = 100$  (lower). (b) Phase  $\phi_e$  of electron in wave for  $a_o = 10$

Figure 11.4 shows the distribution of final momentum states that result from ionization of various ions in intense fields produced by a focused laser. Even though the initial laser beam is polarized with the electric field in the  $x$  direction, focusing the beam introduces finite electric fields in the  $y$  and



**Fig. 11.4.** Final state momentum plots for the ionization of atoms in a focus having an  $f$ -number of 2.5. The dotted lines represent the relation given in (11.11). (a) and (b) are for  $\text{Ne}^{7+}$  at  $10^{17} \text{ W/cm}^2$ , (c) and (d) are for  $\text{Ar}^{8+}$  at  $10^{19} \text{ W/cm}^2$ , and (e) and (f) are for  $\text{Ar}^{15+}$  at  $10^{20} \text{ W/cm}^2$ . Credit: Enam Chowdhury



**Fig. 11.5.** The electron energy spectrum for the same physical conditions as in Fig. 11.4, with the left, center, and right columns corresponding to  $10^{17}$  W/cm<sup>2</sup>,  $10^{19}$  W/cm<sup>2</sup>, and  $10^{20}$  W/cm<sup>2</sup>, respectively. The top panels show the angle of deviation from the  $z$ -axis, in the  $x$ - $z$  plane. The bottom panels show the angle of deviation from the  $z$ -axis, in the  $y$ - $z$  plane. Here again the dotted lines represent the relation given in (11.11). Credit: Enam Chowdhury

$z$  directions. This produces the finite momentum in  $y$  seen in the top set of panels in this figure. The relation between  $p_z$  and  $p_x$  is dominated by the behavior corresponding to an electron at rest, shown as a dashed line in the bottom panels. This is because most electrons are ionized near the peak of the laser field and are nearly at rest when they are produced. Such electrons form the most intense peaks in Fig. 11.4f, with momenta along the mean electric field of about  $\pm 1.5$  MeV/ $c$ . The other ionization events produce the range of electron momenta shown.

Figure 11.5 shows the angle of the electrons relative to the  $z$ -axis. The top panels show the distribution in the  $x$ - $z$  plane. This tends to follow the relation for an electron initially at rest at the maximum field (11.11), shown as a dashed line. The bottom panels show the distribution in the  $y$ - $z$  plane. These electrons are influenced by the  $y$ -component of the electric field produced by focusing.

Thus, one can produce a beam of energetic electrons, having a distribution of energies like those seen in Figs. 11.4 and 11.5, from ionization produced during the laser pulse. This technique can be used to obtain a brief, energetic electron beam that can be used as a probe or to drive other processes. In contrast, the production of intense, directed beams of electrons for accelerators tends to require the interaction of a high-energy-density source with a plasma. We discuss this at more length in Sect. 11.7.1 after a general introduction to relativistic laser-plasma interactions.

### 11.3 Initiating Relativistic Laser–Plasma Interactions

One cannot abruptly initiate a laser pulse at an irradiance high enough to produce strongly relativistic effects. The very best one can do is to produce a laser pulse with a Gaussian shape in time, with a pulse width whose characteristic time to reach  $1/e$  from its maximum is  $\sim 1/\Delta\omega$ , where  $\Delta\omega$  is the bandwidth. The problem this creates is that any target experiences all irradiances from zero to the maximum as the laser pulse arrives. The laser and optical system must have three properties in order to obtain the cleanest-possible laser–solid interactions. First, the laser spot size must be close to the diffraction limit, so that the experimental results will not be confused by structure within the laser spot. Second, most of the laser energy must be present in this spot, to avoid large signals from lower-intensity interactions outside it. Third, the pulse shape in time must be close to Gaussian, without significant structure early in time. Such early structure is known as a prepulse.

These issues have led laser systems devoted to the pure study of interaction phenomena at high fields to produce shorter and shorter laser pulses, to minimize the interactions at lower intensities. Such lasers also need to use deformable mirrors as part of their focusing system, to produce nearly a diffraction-limited spot. These developments lead in their limit to systems described as  $\lambda^3$  lasers ( $\lambda$  is the wavelength of the light), whose goal is to obtain a laser pulse of one cycle in the duration of its maximum irradiance (and thus of length  $\lambda$  in space), focused to an area of approximately  $\lambda^2$ . For light at a central wavelength of 800 nm, the duration of one cycle is about 3 fs. In contrast, lasers devoted to delivering large energies to a target, for example to attempt fast ignition (see Chap. 9), end up using much longer pulses, of order picoseconds. Some of the consequences of this are discussed below.

In discussing these consequences, it will prove useful to refer to three laser systems of the early 1990's. The Vulcan laser, at the Rutherford Appleton Laboratory in Britain, produced at that time relatively high energy in a long ( $\sim 2$  ps) pulse, by doing chirped pulse amplification within a laser system capable of large output energies (see Danson et al.). The TITANIA laser in Britain (see Chambers et al.) produced a spot that was five times the diffraction limit yet still contained substantial internal structure in addition to a prepulse. The laser facility of that same era at the Center for Ultrafast Optical Sciences (CUOS) at the University of Michigan produced a diffraction-limited spot. It could put high irradiance on target without ever producing a plasma layer whose thickness exceeded a skin depth. (The *skin depth* is the penetration distance of the evanescent laser pulse into a medium past a sharp, reflecting interface. It is the inverse of the imaginary value of the wavenumber in the region where the light cannot penetrate. For a sharp, plasma interface at a very high density, the skin depth is  $c/\omega_{pe}$ .)

To produce a very thin, plasma layer imposes three requirements on the laser system. First, the irradiance of the laser light that independently makes its way through the laser optics to the target and that arrives nanoseconds before the main pulse, must be kept below  $10^8$  W/cm<sup>2</sup>, to avoid producing vapor or plasma in front of the target. (see Combis et al., Lindley et al., and Sauerbrey et al.). Such “prepulses” are often produced by amplified spontaneous emission (ASE) from laser amplifiers, or by leakage through Pockels cells, and can easily be several tens of nanoseconds in extent. ASE prepulses have proven useful for the purpose of maximizing the production of hot electrons (see Kmetec et al.) or x-rays (see Rousse et al.), but they are harmful if one’s goal is to obtain very clear evidence regarding laser interactions with solid matter.

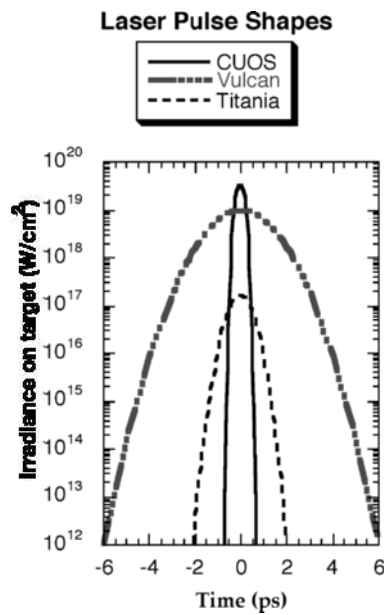
Second, the irradiance of any “pedestal” on the main laser pulse itself must be kept below  $10^{12}$  W/cm<sup>2</sup> to avoid plasma production that is too early. Such pedestals are typically a few ps to a few hundred ps in duration. They are often caused by effects within the laser system that alter the frequency spectrum of the laser light, such as group velocity dispersion in the laser glass or gain-bandwidth narrowing. Early short-pulse glass lasers typically had a pedestal of order  $10^{-3}$  times the maximum irradiance. Later, with the advent of Ti:sapphire oscillators, this was reduced to less than  $10^{-5}$ . The laser at CUOS in 1992 had a measured pedestal of  $2 \times 10^{-6}$  times the maximum irradiance, next improved to  $10^{-8}$  times the maximum infrared irradiance (see Nantel et al.). Doubling the frequency of this pulse kept the pedestal below  $10^8$  W/cm<sup>2</sup>, even as  $I_L$  approached  $10^{20}$  W/cm<sup>2</sup>. In contrast, the highest-energy, ultrafast, glass lasers of that era had pedestals above  $10^{12}$  W/cm<sup>2</sup> for many ps.

The third requirement on the laser system is that the main laser pulse must rise steeply enough at low irradiance. Once  $I_L$  exceeds about  $10^{12}$  W/cm<sup>2</sup>, plasma forms at the surface of the solid and begins to expand. We give  $I_L$  because the onset of preplasma depends primarily on the power delivered. (This does not rule out small effects that depend on wavelength and/or absorption, which we ignore here.) The rate of critical-surface expansion scales as the sound speed, which is proportional to  $T_e^{1/2}$ , where  $T_e$  is the electron temperature. It reaches of order 100 nm/ps at  $I_L \sim 10^{15}$  W/cm<sup>2</sup>. Once  $I_L \lambda_\mu^2$  exceeds  $10^{15}$  W  $\mu\text{m}^2/\text{cm}^2$ , the ponderomotive pressure of the reflecting light wave becomes large enough to stop the plasma expansion (see Liu and Umstadter). At higher intensities, the ponderomotive pressure compresses the plasma and pushes the critical density layer back toward the solid material. A sufficiently intense pulse can first steepen an existing preplasma, so that the local density scale length,  $L/\lambda$ , becomes quite small, and can then push the critical surface inward, decreasing  $D$ . We distinguish between  $L$ , the local scale length of the density profile at the critical density, and  $D$ , the distance from critical density to solid density.  $L$  and  $D$  are simply connected for a free expansion but may be quite different in plasmas that have expanded

and then been compressed. In the end, the only way to retain smooth, thin, planar plasma is to limit the initial plasma expansion so as to keep both  $L$  and  $D$  small at all times.

The CUOS laser of the early 1990's accomplished this by doubling the frequency of a very clean, Gaussian laser pulse of 400 fs FWHM. The doubling efficiency was saturated at the highest intensity, so that the FWHM of the converted pulse was also about 400 fs. However, at lower intensity on the rising edge, the intensity of the second harmonic, proportional to the square of the first-harmonic intensity, rose quite steeply. The second-harmonic intensity, at 527 nm wavelength, increased from  $10^{12}$  to  $10^{19}$  W/cm<sup>2</sup> (16 e-foldings) in 500 fs. Confirmation that this laser system produced negligible preplasma was provided by studies of the high-density plasma using x-ray spectroscopy (see Jiang et al. 1995). The increase in  $I_L$  from  $10^{12}$  W/cm<sup>2</sup> to  $10^{15}$  W/cm<sup>2</sup> took only 180 fs, which allowed the plasma to expand only about 10 nm. This is less than a skin depth for solid density (17 nm at  $10^{23}$  cm<sup>-3</sup>). After that, the light pressure prevented the plasma layer from expanding further.

Figure 11.6 compares the nominal profiles of the pulses of our three reference laser systems. A laser system with a clean, Gaussian pulse of 1 ps FWHM, having a peak  $I_L$  of  $1.6 \times 10^{17}$  W/cm<sup>2</sup>, requires 700 fs to increase in



**Fig. 11.6.** Reference laser pulses of lasers in the early 1990's. A schematic CUOS laser pulse is compared to clean, Gaussian pulses having FWHM of 1 ps and 2.5 ps. The two longer pulses are the best pulses that might have been produced by the TITANIA and Vulcan laser systems of that era, respectively. The longer pulses permit much more time for plasma formation



$I_L$  from  $10^{12}$  W/cm<sup>2</sup> to  $10^{15}$  W/cm<sup>2</sup>. This would allow a plasma expansion above 50 nm. This would have been achieved by the TITANIA laser system if it had a clean pulse and no pedestal or prepulses. A laser system with a clean, Gaussian pulse of 2.5 ps FWHM, having a peak  $I_L$  of  $10^{19}$  W/cm<sup>2</sup>, requires 1500 fs to increase in  $I_L$  from  $10^{12}$  W/cm<sup>2</sup> to  $10^{15}$  W/cm<sup>2</sup>. This would allow a plasma expansion above 100 nm. This would have been achieved by the Vulcan laser system if it had no prepulse. The resulting parameters are compared in Table 11.2.

**Table 11.2.** Comparison of plasma size and excursion distances in planar experiments

Experiment	$I\lambda_\mu^2$ (W $\mu\text{m}^2/\text{cm}^2$ )	Plasma size $D/\lambda$	Excursion distance $x_{\text{os}}/D$
Vulcan (1993)	$3 \times 10^{15}$	0.2	0.04
Titania (1998)	$1 \times 10^{16}$	0.2	0.08
CUOS (1999)	$7 \times 10^{18}$	0.02	7

One issue that determines how thin a plasma layer is thin enough to study laser–solid interactions is whether the interaction dynamics involves the solid-density matter or only involves the lower-density plasma near its surface. A demanding measure of this, and one which determines whether *Brunel electrons*, discussed further below, can participate in the absorption dynamics, is the ratio  $x_{\text{os}}/D$ . Here  $x_{\text{os}} = a_0 c / (\gamma_r \omega) = (\lambda/2\pi)(a_0/\gamma_r)$  is the excursion distance of the electrons in the electric field of the laser. We compare this parameter in Table 11.2 for the three reference cases.

## 11.4 Absorption Mechanisms

The absorption mechanisms change as  $I_L$  increases in ultrafast laser–target interactions. At low enough irradiance ( $<10^{14}$  W/cm<sup>2</sup>), the laser cannot heat the dense plasma very much in a short pulse, and collisional absorption of the evanescent laser electric field dominates the interaction. Strong laser absorption is observed at normal incidence and in *s-polarization* (with the laser electric field parallel to the surface of the target) is consistent with that expected from collisions. The absorption is enhanced in *p-polarization* (with the laser electric field parallel to the target normal), for modest scale lengths  $L$ , when the evanescent electric field of the laser light penetrating to critical density is resonantly enhanced, leading to *resonance absorption* (see Kruer).

As  $I_L$  increases above  $10^{15}$  W/cm<sup>2</sup>, the plasma becomes hot enough that collisional effects become small. Then the absorption becomes more complicated, introducing a number of mechanisms that we will not explore here.

These include *sheath inverse bremsstrahlung*, the high-frequency skin effect, and the *anomalous skin effect*, all of which are discussed in a common context by Rozmus et al. and by Gibbon and Forster. The simple relation between electron temperature and laser irradiance that we derived in Chap. 8 now breaks down, as the outward convection of hot material is no longer the important energy loss channel for the electrons. Even so,  $T_e$  scales as some power of  $I_L$  that is not too far from  $1/2$ . In this regime, resonance absorption is still strong but the energy transfer mechanism may not be collisional. The noncollisional absorption might be due to breaking of the laser-driven electron plasma waves, but there is also evidence in some simulations of repeated “Langmuir collapse” as in Gibbon (1994). Experimentally, there is clear evidence of resonance absorption in this regime as described in Chaker et al. and Meyerhofer et al.

As  $I_L$  increases above  $10^{18}$  W/cm<sup>2</sup>, the physical excursions of the electrons can become comparable to the thickness of the plasma layer at the surface of a solid. In this case, the excursions of the electrons into the vacuum, accelerated by the laser light wave, can lead to enhanced absorption when they return and enter the solid. This effect, often called the *Brunel effect* as it was first identified theoretically by Brunel (1987) and later replicated in theory and simulation by others including Bonnaud et al. and Kato et al., can significantly enhance the absorption for p-polarized light. Resonance absorption gives way to vacuum excursions as  $I_L$  increases and the scalelength decreases, but the transition is complex (see Gibbon and Bell, 1992). As discussed above with reference to Table 11.2, the condition for significant absorption by vacuum excursions is  $x_{os}/D > 1$ . (Here  $x_{os}/D$  is the relativistic generalization of the parameter given by Brunel,  $v_{os}/\omega$ , where  $v_{os}$  is the nonrelativistic, electron oscillation velocity in the laser electric field.)

It is worth noting that much earlier experiments produced plasmas that were recompressed, by the ponderomotive force, to have density profiles that were locally very steep. Both some experiments using CO<sub>2</sub> lasers (see Bach et al. and Fedosejevs et al.) and more recent ones using 1  $\mu$ m lasers (see Norreys et al.) produced plasmas that, probably, satisfied  $x_{os}/L > 1$  at the critical surface. However, in the process of recompression these initially much-thicker plasma layers became rippled. This precluded clean diagnosis of the interaction processes.

In this same irradiance regime the anomalous skin effect begins to become important for normal incidence, if the plasma layer is sufficiently thin. The electrons are heated enough that the normal skin depth becomes smaller than the electron excursion length and the electron mean free path, leading to increased penetration of the laser electric field into the solid. Ruhl and Mulser used Vlasov simulations to study laser–solid interactions in this irradiance regime. They explain the connection between the Brunel effect and the anomalous skin effect as the natural limits of the force on the electrons through its dependence on the angle of incidence.

The Brunel effect eventually saturates in consequence of  $\mathbf{v} \times \mathbf{B}$  forces (see Brunel 1988). As  $I_L$  increases into the strongly relativistic regime, further new effects are predicted to arise. The  $\mathbf{v} \times \mathbf{B}$  motion of the relativistic electrons in the laser field begins to produce a significant electron velocity directed into the target (see Pukhov and Meyer-ter-Vehn). This can lead to hole boring and collisionless shocks, discussed in Sect. 11.8.

Despite the variation in specific absorption mechanisms, all the absorption processes involve transferring laser energy to electrons. When laser light is absorbed at a solid surface, this produces a distribution of electrons, often approximately Maxwellian in shape. One can write an energy flux balance equation that qualitatively describes the heating, as

$$f_{\text{abs}} I_L = n_c k_B T_e v_{\text{eff}}, \quad (11.26)$$

in which  $f_{\text{abs}}$  is the fraction of the laser energy that is absorbed,  $n_c$  is the critical density, and  $v_{\text{eff}}$  is the effective velocity at which energy flows away from the absorption region. In the slow (ns), large-plasma regime of Chap. 8,

$$f_{\text{abs}} \sim 1 \text{ and } v_{\text{eff}} \sim c_s \propto T_e^{1/2} \text{ so } T_e \propto I_L^{2/3}. \quad (11.27)$$

As the temperature increases and the collisional absorption becomes smaller, one enters a regime in which  $f_{\text{abs}} \propto \nu_{\text{ei}}/\omega_o$  but  $v_{\text{eff}}$  is still of order the sound speed. Then, since  $\nu_{\text{ei}} \propto T_e^{-3/2}$ , one has

$$T_e \propto I_L^{1/3} \text{ in the weakly collisional regime.} \quad (11.28)$$

Beyond these regimes the excursions of the electrons begin to matter and one finds that  $f_{\text{abs}}$  depends more weakly on  $T_e$ , bringing the increase of  $T_e$  with  $I_L$  up closer to

$$T_e \propto I_L^{1/2} \text{ in the excursion regime.} \quad (11.29)$$

As the laser irradiance increases into the strongly relativistic regime, the electrons are observed to have a characteristic energy of order the kinetic energy of the electron oscillations in the laser light. This is a natural result – any process that manages to deflect an electron, changing its velocity in the direction of the laser electric field, will have the effect of converting some fraction of this energy into randomized motion. So the quiver energy must set the energy scale once the absorption loses its temperature dependence and the electrons leave at  $c$ . In the relativistic regime, the characteristic energy of these electrons,  $\mathcal{E}_{\text{hot}}$ , is

$$\mathcal{E}_{\text{hot}} = \left( \sqrt{1 + a_o^2} - 1 \right) m_e c^2. \quad (11.30)$$

For simple estimates, one often describes the “temperature” of these electrons using the relation  $k_B T_h = \mathcal{E}_{\text{hot}}$ . Once  $a_o$  is very large, one has

$$k_B T_h \sim a_o m_e c^2. \quad (11.31)$$

## 11.5 Harmonic Generation

As  $I_L$  increases above  $10^{17}$  W/cm<sup>2</sup>, the interaction of the laser with the overdense target can lead to the emission of many harmonics of the laser light. This phenomenon has utility both as a diagnostic of the interaction mechanisms and as a potential source of coherent, short pulses of soft x-rays for other applications. Early theory on harmonic production from solids (see Bezzerides et al. and Grebogi et al.) was motivated by the observation of many harmonics from long-pulse experiments using CO<sub>2</sub> lasers (see Burnett et al. and Carman et al.). The important mechanism in this regime is the oscillation of the critical surface in the electric field of the laser light.

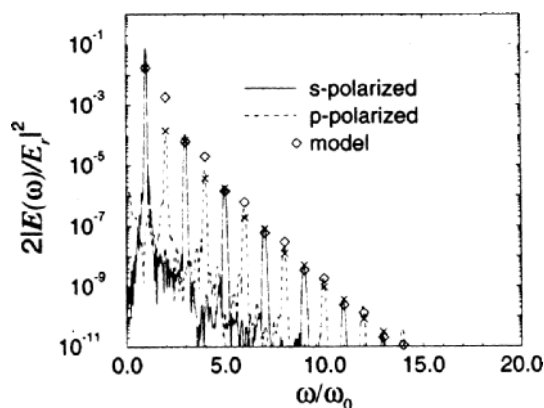
If the interaction geometry remains simple and planar, then the pump laser light will be specularly reflected and any harmonics will be emitted within the cone angle of the reflected pump light. It is well established that this requires a plasma layer that is much less than one wavelength in extent. Thus, the angular distribution of harmonic light is an indicator of how planar the surface is. In experiments on the TITANIA laser system, a transition from specular to diffuse harmonic emission was observed when the maximum intensity of the laser pulse (of 1 ps FWHM for these experiments) exceeded  $1.6 \times 10^{17}$  W/cm<sup>2</sup>. The 1 ps FWHM laser pulse shown in Fig. 11.6 corresponds to this maximum intensity. One should note that, with such a pulse, the amount of preplasma produced by the leading edge of the laser pulse increases significantly as  $I_L$  increases. In experiments on the Vulcan laser system, whose long laser pulse (illustrated in Fig. 11.6) produced much more preplasma, diffuse harmonic emission was always observed (see Norreys et al.). In experiments at Toronto (see Zhao), there was a transition from specular to diffuse harmonic emission when a prepulse was introduced 1.5 ns before the main pulse. A consistent interpretation of these data, and of much other data with lower intensities and longer pulse lengths not discussed here, is as follows. Given sufficient preplasma production, the critical surface where the laser light reflects becomes rippled. This is discussed, for example, by Wilks et al. (1992). Possible mechanisms include corrugation through electron clustering, irregular lateral motion of the ions, and Rayleigh–Taylor instabilities. This causes the laser–plasma interaction to occur over a wide range of angles, causing the scattering and reflection of the laser light to become diffuse. Once diffuse scattering sets in, the angular structure of the signal becomes worthless as a diagnostic of the interaction dynamics.

Theoretical work in the 1990s, nicely reviewed by Gibbon (1997), concluded that efficient harmonic production is possible from the interactions of sufficiently intense laser pulses with the overdense plasma at the surface of a solid. Wilks et al. (1993) observed odd harmonics in both 1D and 2D PIC simulations and observed (weaker) even harmonics in the 2D case. Relativistic oscillations of the critical surface introduce harmonic structure into the light reflected from it. One can think of this first in terms of the driven currents that emit the reflected radiation. At all angles of incidence, the

ponderomotive force drives density oscillations with even harmonic content, which beat with velocity oscillations at the pump frequency  $\omega$  to produce a current source with odd harmonic content. (We ignore here the case of circularly polarized light.)

For oblique incidence, the harmonic content also depends on the polarization. In  $s$  polarization, a current source also develops with only even harmonic content. In  $p$  polarization, the density oscillations have both odd and even harmonic content. Those with odd harmonic content beat with velocity oscillations at  $\omega$  to produce a current source with even harmonic content. In both  $s$  and  $p$  polarizations, the even harmonics are expected to be very weak near normal incidence and to increase as the angle of incidence increases.

An alternative way to think through the harmonic generation process is to treat the reflection from the critical surface as a reflection from an oscillating mirror, as suggested by Bulanov et al. Lichters et al. have done this to produce a cold-plasma model which accounts for the harmonic production observed in their simulations. They note that a correct treatment of the retarded source terms introduces substantial anharmonic content to this oscillation and that including this effect gives quantitative agreement between the model and the PIC simulations in some cases. (Highly resolved calculations for oblique incidence become possible by boosting the frame of a 1D PIC calculation to the Lorentz frame in which the electromagnetic wave appears to be normally incident.) They provide a quantitative model involving few assumptions that can be used to calculate the predicted spectrum of the harmonics. They have difficulty with the magnitude of the lowest even harmonics in oblique incidence with  $s$  polarization, as illustrated in Fig. 11.7. They suggest that the sharpness of the plasma–solid boundary significantly affects the production of these harmonics.



**Fig. 11.7.** The spectrum of harmonic emission from simulations by Lichters et al., along with predictions from their model (diamonds). The crosses are from an ad hoc model intended to mimic the effects of surface plasma

Another source of harmonic emission is also discussed by Gibbon (1996). He argues that the particles which undergo vacuum excursions produce a very anharmonic current source, repeating at the pump frequency, as they reenter the solid and are strongly decelerated. Gibbon points out that for oblique incidence at fairly large angles, such as  $60^\circ$ , vacuum excursions will become the dominant source of harmonic emission.

## 11.6 Relativistic Self-Focusing and Induced Transparency

We now ask what effect the plasma has on the laser light. To do so we return to the wave equation for the laser light, (11.1), and evaluate  $J_t = -en_e v_x = -en_e c a_o / \gamma_r$ . Converting  $A$  to  $a_o$ , we obtain

$$\left( \frac{\partial^2}{\partial t^2} + \frac{\omega_{pe}^2}{\gamma_r} - c^2 \nabla^2 \right) a_o = 0, \quad (11.32)$$

in which  $\omega_{pe}$  is the plasma frequency corresponding to the electron density  $n_e$ . Upon first glance, this may appear to be a simple equation, in which the plasma frequency is reduced by a factor of  $\gamma_r$ . However, since  $\gamma_r$  depends on  $a_o$  through  $v$ , it also varies in time on the same timescale as  $a_o$  does. One must work out the effects of this when seeking a detailed solution, but one can formulate the solution in terms of an appropriately averaged  $\gamma_r$ , which we designate as  $\langle \gamma_r \rangle$ . One obtains a modified dispersion relation,

$$\omega^2 - \frac{\omega_{pe}^2}{\langle \gamma_r \rangle} - c^2 k^2 = 0, \quad (11.33)$$

from which the phase velocity is seen to be

$$\frac{\omega}{k} = c \frac{1}{\sqrt{1 - \frac{\omega_{pe}^2}{\langle \gamma_r \rangle \omega^2}}} = c \frac{1}{\sqrt{1 - \frac{n_e}{\langle \gamma_r \rangle n_c}}}. \quad (11.34)$$

Two conclusions follow from these equations. First, the wave travels slower in regions of higher  $a_o$ . Since this corresponds to the center of a focused laser beam, the phase fronts become curved and the laser beam tends to self-focus, a phenomenon known as *relativistic self-focusing*. This effect is opposed by diffraction of the laser beam, but above some threshold laser-beam power the beam does self-focus. This threshold laser-beam power,  $P_{sf}$ , is

$$P_{sf} = 17.4 (n_c/n_e) \text{ GW}, \quad (11.35)$$

in which  $n_c$  is the critical electron density as defined in Sect. 8.1. We saw that lasers producing relativistic high-energy-density conditions typically exceed

this power. If such a laser-beam interacts with a sufficient volume of plasma, it will self-focus.

The second conclusion is that a relativistic laser beam can penetrate to a higher density than a nonrelativistic one. Taking (11.11) at face value, the density of reflection, where the phase velocity goes to zero, is  $n_e = \langle \gamma_r \rangle n_c$ . Thus, the laser can penetrate to higher density by a factor of  $\langle \gamma_r \rangle$ . In detail the effect is more complicated because of the fluctuations in  $\gamma_r$ . Despite this, the net effect is that a relativistic laser beam can penetrate to a higher density than a nonrelativistic beam. This phenomenon is known as *induced transparency*.

## 11.7 Particle Acceleration

Much of the interest in relativistic laser–plasma interactions revolves around the acceleration of particles by such lasers. It is hoped that beams of particles might be produced for fast ignition. Beams of protons from such lasers may prove useful for proton radiography. Alternatively, beams of electrons may produce  $\gamma$ -rays for radiography. By interacting intense lasers or particle beams with plasmas, researchers are working at this writing to produce advanced particle accelerators. This is possible because one can produce much larger electric fields in plasmas than one can between electrodes in vacuum. For these reasons, it is sensible to discuss the mechanisms of particle acceleration here.

### 11.7.1 Acceleration Within Plasmas

The potential applications of high-energy-density plasmas and beams to electron accelerators nearly all involve the behavior of wakes in plasmas. The mechanism responsible for generating wakes is perhaps best understood by taking some time to throw rocks in a lake. The rock itself pushes some water outward, generating a first circular outgoing wave. But the rock also displaces water, in response to which the remaining water rushes in toward the center and then rebounds outward, with the consequence that a second circular outgoing wave is created. When one proceeds from throwing rocks to watching boats, one can see that these two waves correspond to the bow wave and to the trailing wake. The wakes propagate at an angle to the path of the boat, depending on their phase speed relative to the boat. If one gets onto the water and is able to match speeds with the wake, perhaps on a windsurfer or in a kayak, one can ride the wake, gaining speed and extracting energy from it. At this point one will understand the *wakefield accelerator*, save for a few (i.e. many) technical details. The seminal paper on the application of wakes to acceleration in plasmas was written by Tajima and Dawson in 1979.

When one works inside a plasma, the key to producing electron acceleration is to generate some kind of wake. (In some limits, the wake begins to

look more like an extended plasma wave). The fundamental requirement is to create a local source of pressure that moves through the plasma, creating a wake that moves near the speed of light and whose electric field is large. Suppose for a moment that we repeat the experiment with the rock, except by creating a local and brief source of pressure, of some size  $\lambda_p$ , within a uniform cold plasma. (All realistic plasmas are cold when compared to motions at  $c$ .) The source is brief in the sense that it moves the electrons but causes little displacement of the ions. The local disturbance of the plasma creates a spherical wake, potentially including structure analogous to the two waves in the water, for the same reasons. The wake propagates because the electron density oscillates in response to a displacement, and because the displacement is local. We know from Sect. 2.4 that the electrons will oscillate in response to a charge separation at the electron plasma frequency,  $\omega_{pe}$ . Since the perturbation moves a distance  $\lambda_p$  in one plasma cycle, the velocity of this wake is

$$v_p = \omega_{pe}/k_p, \quad (11.36)$$

where  $k_p = 2\pi/\lambda_p$ . If it happens that  $v_p \approx c$  and that the wake is strong enough to trap some electrons and carry them along, then our imaginary spherical wake would accelerate these electrons to relativistic velocities.

The simplest application of this idea to real systems is to create a packet of photons or electrons of very-high energy density (and thus high pressure), whose characteristic length is close to  $\pi c/\omega_{pe}$ , as this will most effectively excite a plasma wake for which  $v_p \approx c$ . As this high-pressure packet traverses the plasma, it creates a strong plasma wake. The particles in such a wake can be trapped by it and substantially accelerated. The intense field produced immediately behind the high-pressure packet can be particularly effective at producing a beam of accelerated particles having a narrow energy spread and low divergence.

More complicated applications of the wakefield idea abound, because they require less-demanding experimental hardware. A laser beam above the threshold for relativistic self-focusing will tend to focus to produce a small region with very intense fields. When one or more extended laser beams resonantly drives an extended, intense plasma wave with  $k \sim \omega_{pe}/c$  the resulting electric field will tend to self-modulate on the scale of  $\omega_{pe}/c$  through the action of an instability (exactly how depends on details). This will drive in turn a series of wakes in the plasma and can accelerate particles. When two laser beams are used to create such a plasma wave by beating, in a process identical to the beating involved in stimulated Raman scattering (Chap. 8), the accelerator is known as a *beat-wave accelerator*. In the beat-wave accelerator and some of the other approaches, one cannot expect the plasma wake to trap cold particles and instead must inject particles that are then accelerated to higher energy. Overall, there is a veritable zoo of possibilities with an alphabet-soup of acronyms describing various different approaches to the goal of creating plasma wakes to accelerate electrons. We will leave the exploration of this zoo to those who develop a specialized interest.



To determine how worthwhile it may be to pursue all of this, one would like to know the possible electron energy gain. We evaluate this next, following Tajima and Dawson. The maximum energy gain will occur when the wake becomes so large that it breaks. This occurs when the electron excursion distance during a plasma period,  $x_{\text{os}}$ , reaches  $k_{\text{p}}x_{\text{os}} \approx 1$ . We observe this oscillation in the lab frame, where

$$x_{\text{os}} = eE_{\text{w}}/(m_e\omega_{\text{pe}}^2). \quad (11.37)$$

Here  $E_{\text{w}}$  is the electric field in the wake, related to the potential  $\Phi$  by  $|E_{\text{w}}| \approx k_{\text{p}}\Phi$ . This gives

$$e\Phi = m_e\omega_{\text{pe}}^2/k_{\text{p}}^2 \approx m_e c^2 \quad (11.38)$$

in the lab frame.

The energy gain of the electron is most simply evaluated in the moving frame of the plasma wake, where the electron oscillates in a stationary potential well. Referring to the volume of Landau and Lifshitz on the theory of fields for the relativistic transformations, we find the depth of the potential well in this moving frame,  $\Phi^{(\text{w})}$ , in terms of the lab-frame quantities, as

$$e\Phi^{(\text{w})} = \gamma_r e\Phi \approx \gamma_r m_e c^2, \quad (11.39)$$

in which

$$\gamma_r = 1/\sqrt{1 - v_{\text{p}}^2/c^2}. \quad (11.40)$$

Thus  $\gamma_r$  is the relativistic factor corresponding to the motion of the plasma wake. We also take  $\beta \equiv v_{\text{p}}/c$ . The electron has its maximum energy in the lab frame just when it has the maximum energy in the frame of the wake, where the energy is  $\gamma_r m_e c^2$ . To find the corresponding laboratory frame energy,  $\mathcal{E}_{\text{lab}}$ , we Lorentz transform the energy back into the lab frame, finding

$$\mathcal{E}_{\text{lab}} = \gamma_r[\gamma_r m_e c^2 + \beta c(\gamma_r \beta m c)] = \gamma_r^2 m_e c^2(1 + \beta^2) \approx 2\gamma_r^2 m_e c^2. \quad (11.41)$$

It is helpful to know the relativistic factor  $\gamma_{\text{max}}$  of the accelerated electron, seen in the lab frame, that has this much energy. This is evidently

$$\gamma_{\text{max}} = 2\gamma_r^2. \quad (11.42)$$

For the specific case of using a light pulse to create the pressure, the optimum light pulse will have a group velocity,  $v_{\text{g}}$ , that equals  $v_{\text{p}}$ . As we discussed in Chap. 8,

$$v_{\text{g}} = c/\sqrt{1 - \frac{\omega_{\text{pe}}^2}{\omega^2}}, \quad (11.43)$$

from which one can infer that

$$\gamma_{\text{max}} = 2\omega^2/\omega_{\text{pe}}^2. \quad (11.44)$$

Thus, for example if one uses laser light of wavelength  $1\ \mu\text{m}$  to drive a wake in a plasma of density  $10^{17}\ \text{cm}^{-3}$ , one finds  $\gamma_{\text{max}} = 2 \times 10^8$  and  $\mathcal{E}_{\text{lab}} = 10\ \text{GeV}$ . There are reasons this is not trivial, though. The required laser pulse duration of  $\pi/\omega_{\text{pe}}$  is 180 fs. The length over which a uniform plasma must be provided so that this acceleration can occur,  $\ell_a$  is

$$\ell_a \approx \mathcal{E}_{\text{lab}}/|eE_w| = 2c\omega^2/\omega_{\text{pe}}^3, \quad (11.45)$$

which would be 34 cm. Sustaining a uniform plasma and a focused laser pulse over this distance would be a severe challenge. But if one raises the plasma density (i.e., increases  $\omega_{\text{pe}}$ ) to make the distance shorter, the laser pulse length soon approaches one laser cycle. Clever technique will be needed to achieve energies of tens of GeV or more using lasers. At this writing, this is an active area of research.

### Homework 11.5

Find the time required to accelerate the electron to  $\sim 30\ \text{GeV}$  in the example just given.

### Homework 11.6

Suppose one has a laser beam that can be focused to  $10^{20}\ \text{W}/\text{cm}^2$  in a  $10\ \mu\text{m}$  diameter spot. Would one obtain higher-energy electrons from tunnel ionization (as in Sect. 11.2) or from using the laser for wakefield acceleration?

Among other alternatives, one can use an electron bunch to create the pressure, in which case the plasma wake will accelerate some members of this bunch to higher energy. At this writing, energy increases above 1 GeV have been obtained on a  $\sim 30\ \text{GeV}$  electron beam. In this application, the optimum plasma density is determined by the length of the electron bunch,  $\ell_b$ , which should be

$$\ell_b = \pi c/\omega_{\text{pe}} \quad (11.46)$$

for the reasons discussed above. To keep the electron bunch focused as it propagates requires a balance between beam divergence and self-focusing in the plasma. This sets a relation between the plasma density and the beam size at the plasma entrance,  $r_{\text{bo}}$ ,

$$\omega_{\text{pe}}^2 = 2\gamma_b c^2 \epsilon_b^2 / r_{\text{bo}}^4, \quad (11.47)$$

in which the relativistic factor for the bunch is  $\gamma_b$  and the emittance of the bunch (related to divergence, and having units of distance times angle) is  $\epsilon_b$ . Thus, the electron bunch must be carefully shaped for optimum acceleration. It remains true in this case that  $\gamma_{\text{max}} = 2\gamma_r^2$  and that  $eE_w \approx m_e c \omega_{\text{pe}}$ , so the acceleration length is

$$\ell_a \approx \gamma_{\max} c / \omega_{\text{pe}} = 2\gamma_r^2 c / \omega_{\text{pe}}. \quad (11.48)$$

One can see that keeping  $\ell_a$  from growing while increasing  $\gamma_{\max}$  requires increasing  $\omega_{\text{pe}}$ . This in turn requires shortening the bunch and making it smaller in diameter. At this writing, this is also an active area of research.

### 11.7.2 Acceleration by Surface Potentials on Solid Targets

The wakefield and beat-wave processes just discussed might perhaps be developed into next-generation particle accelerators. Accelerators place demanding constraints on the dispersion of the accelerated particles in energy and in angle. Some other applications, such as proton radiography, may not be so demanding in these respects. Beams of protons (and other ions) are typically observed when relativistic laser beams strike solid targets. One way that they arise, which might be called *electrostatic ion acceleration* or *sheath acceleration*, is easy to understand. The laser interacts with the electrons, and it removes many of them from the target. The result is that the target becomes positively charged, expelling ions from its surface. Because the surface is planar, the ions are expelled along the target normal. Because the laser produces plasma on the front surface of the target and produces electrons that easily penetrate a target that is not too thick, it is not uncommon to see beams of ions that emerge from both surfaces.

We can analyze this type of acceleration by building on our analysis of the self-similar rarefaction in Sect. 4.2 and following the discussion of Mora. In Sect. 4.2 we used a Boltzmann distribution to describe the electrons, finding

$$e\Phi = k_B T_e \ln(n_e/n_{e0}), \quad (11.49)$$

in which the potential is  $\Phi$  and the initial electron density at the target surface is  $n_{e0}$ . This equation is valid so long as electron collisions are fast enough to sustain a Maxwellian distribution, and this remains true even in ultrafast experiments. So we will assume here that the expansion is isothermal, at least while the most energetic ions are produced before the expanded plasma cools significantly.

We also previously derived the self-similar solution for the ions, finding

$$\rho = \rho_0 \exp\left(-\frac{x}{c_s t} - 1\right), \quad (11.50)$$

in which  $c_s$  is as usual the sound speed. In an ultrafast experiment, the ions have little time to heat so  $c_s = \sqrt{Z k_B T_e / m_i}$ . The derivation of this equation assumed quasineutrality of the plasma, so  $\rho = n_e A m_p / Z$ . We considered in Chap. 2 the possibility that such an expansion might extend to an electron density at which interactions with background gas were common, providing a maximum potential energy and thus a maximum ion energy of order  $20 k_B T_e$ .

One can easily show that the electric field,  $E_{ss}$ , corresponding to the self-similar solution, is constant and is given by

$$eE_{ss} = k_B T_e / (c_s t). \quad (11.51)$$

However, in an ultrafast experiment the self-similar model breaks down because the assumption of quasineutrality becomes invalid. Charge separation can always occur on the scale of a Debye length,  $\lambda_D$ . Once  $\lambda_D$  exceeds the local scale length of the ion density profile,  $c_s t$ , the self-similar model breaks down. Then the electrons are able to stay ahead of the ions, which creates an expansion front and pulls the ions along. This roughly occurs when

$$\lambda_D = \lambda_{D0} \sqrt{n_{e0}/n_e} = \lambda_{D0} \exp[(1 + c_s t)/2], \quad (11.52)$$

in which the Debye length at  $n_{e0}$  is  $\lambda_{D0}$ . From the self-similar rarefaction, this occurs at  $x = (c_s t)(2 \ln(\omega_{pi} t) - 1)$ , where  $\omega_{pi}$  is the ion plasma frequency at  $n_{e0}$ , equal to  $\sqrt{4\pi n_{e0} Z e^2 / m_i}$ . For fully ionized, solid-density Be, one has  $\omega_{pi} \sim 3 \times 10^{14}$  rad/s, so  $\omega_{pi} t \sim 100$  in a few hundred fs. In this case the electric field at the ion front,  $E_{front}$ , is estimated to be  $2E_{ss}$ , due to the surplus of electrons at larger  $x$ . Simulations of the plasma expansion have shown that to excellent accuracy

$$eE_{front} = \frac{\omega_{pi}}{c_s} \frac{2k_B T_e}{\sqrt{5.44 + \omega_{pi}^2 t^2}}. \quad (11.53)$$

One can integrate (11.53) (which is  $m_i dv_i/dt$ ) to find the ion velocity  $v_i$  and the maximum ion energy,  $\mathcal{E}_{max}$ . These are

$$v_i = c_s [\ln(2\omega_{pi}^2 t^2) - 1], \quad (11.54)$$

and

$$\mathcal{E}_{max} = 2Zk_B T_e \left[ \ln(\omega_{pi} t \sqrt{2/2.72}) \right]^2. \quad (11.55)$$

For  $\omega_{pi} t \sim 100$ , this gives  $\mathcal{E}_{max} \sim 40Zk_B T_e$ . In the strongly relativistic regime, where  $k_B T_e \sim a_o m_e c^2$ , this implies  $\mathcal{E}_{max} \sim 20Za_o$  MeV. This implies that ion accelerations to many MeV are plausible by sheath acceleration even for modest values of  $a_o$ . This process is most likely to be important on the front side of solid targets and on the rear side of solid targets that are thick enough that the explosion processes discussed below are not dominant.

### 11.7.3 Acceleration by Coulomb Explosions

A third important acceleration mechanism, also responsible for high-energy ion production and even for ion beams, is known as a *Coulomb explosion*. Coulomb explosions, created by the irradiation of molecules with photons or particle beams, have been used in chemistry since at least the 1970's. We will

see shortly why it took longer to produce them using lasers. Our analysis draws in part on Zweiback.

First consider a spherical distribution of ions at constant density  $n_i$ , from which the electrons have been magically removed. One can integrate Gauss' law to find the electric field at location of any ion at radius  $r$  inside the cluster, in cgs units, as

$$E = 4\pi n_i Z e r, \quad (11.56)$$

then solve  $\mathbf{E} = -\nabla\Phi$  to find the potential  $\Phi$  as

$$\begin{aligned} \Phi &= 2\pi n_i Z e \left[ \left( \frac{8\pi}{3} \right) r_o^2 + (r_o^2 - r^2) \right] \text{ for } r < r_o \text{ and} \\ \Phi &= 2\pi n_i Z e \left( \frac{8\pi}{3} \right) \frac{r_o^3}{r} \text{ for } r > r_o, \end{aligned} \quad (11.57)$$

in which as usual  $n_i$  is the ion density,  $Z$  is the ionic charge, and  $e$  is the charge of an electron. In addition, the initial radius of the sphere is  $r_o$  and we have chosen  $\Phi$  to be zero as  $r \rightarrow \infty$ .

The dynamics of such an ion cloud is simple. All the ions are accelerated outward, with the outermost ions being accelerated most greatly. As  $r$  increases for each ion, the acceleration decreases. So no interior ion overtakes any outer ion. As a result, each ion at radius  $r$  ends up converting the electrostatic potential energy created by the charge at radii  $< r$  into kinetic energy. The maximum ion energy  $\mathcal{E}_{\max}$  is obtained when an ion at the outer edge of the sphere has been accelerated until the remaining potential is negligible. Then the maximum ion energy is

$$\mathcal{E}_{\max} = \frac{8\pi}{3} 2\pi n_i Z^2 e^2 r_o^2. \quad (11.58)$$

The derivation of the corresponding, normalized, ion energy distribution function  $f(\mathcal{E}_i) = (1/N)dN/d\mathcal{E}_i$  is left as homework. It is

$$\begin{aligned} f(\mathcal{E}_i) &= \frac{3}{2} \frac{\sqrt{\mathcal{E}_i}}{\mathcal{E}_{\max}^{3/2}} \text{ for } \mathcal{E}_i < \mathcal{E}_{\max} \text{ and} \\ &= 0 \text{ for } \mathcal{E}_i > \mathcal{E}_{\max}. \end{aligned} \quad (11.59)$$

For  $r_o = 10 \mu\text{m}$ , which is a plausible laser-beam spot,  $\mathcal{E}_{\max}$  evaluates to  $0.76n_i/(10^{18} \text{ cm}^{-3}) \text{ MeV}$  for  $Z = 1$ . Thus, MeV energies might plausibly be obtained from gasses and much larger energies could be obtained from (thin) solids, if in fact such explosions can be produced. We now consider some aspects of this.

**Homework 11.7**

Solve for the potential of a spherical cloud of ions having uniform density, and for the energy distribution function of the ions produced by a Coulomb explosion of this cloud.

To create a Coulomb explosion, the laser must strip the electrons from the ions and expel them from the cloud. The laser will have to turn on very quickly and reach sufficient irradiance. Two timescales matter. These are the explosion time itself and the sonic expansion of the cloud. Let us consider these in reverse order. The sound speed here again is  $c_s = \sqrt{Zk_B T_e/m_i}$ , so the sonic disassembly time is

$$\tau_{\text{sonic}} = r_o/c_s = \sqrt{r_o^2 m_i / (Z k_B T_e)}. \quad (11.60)$$

If we crudely estimate  $k_B T_e = a_o m_e c^2$  from (11.31), then we find

$$\tau_{\text{sonic}} = 140 \sqrt{\left(\frac{A r_\mu^2}{Z a_o}\right)} \text{ fs}, \quad (11.61)$$

in which  $r_\mu$  is  $r_o$  in  $\mu\text{m}$ .

The explosion time can be found by solving the equation of motion for an ion at the outer surface of the cloud. Because the charge within the cloud is fixed and remains within the radius of the ion, one can relate the ion velocity to the potential energy. One has, for the ion velocity  $v_i$ ,

$$\frac{1}{2} m_i v_i^2 = \mathcal{E}_{\text{max}} \left(1 - \frac{r_o}{r}\right), \quad (11.62)$$

which can be written as

$$\left(\frac{v_i}{v_{\text{max}}}\right)^2 = 1 - \frac{r_o}{r}. \quad (11.63)$$

This allows the equation of motion,

$$\frac{d}{dt} \left(\frac{v_i}{v_{\text{max}}}\right) = \frac{ZeE}{v_{\text{max}}} = \frac{\mathcal{E}_{\text{max}}}{m_i r_o v_{\text{max}}} \left(\frac{r_o}{r}\right)^2, \quad (11.64)$$

to be written as

$$\frac{d}{dt} \left(\frac{v_i}{v_{\text{max}}}\right) = \frac{v_{\text{max}}}{2r_o} \left(1 - \frac{v_i^2}{v_{\text{max}}^2}\right)^2. \quad (11.65)$$

By integrating this equation to find when the ion energy reaches half of  $\mathcal{E}_{\text{max}}$ , we can obtain a reasonable value for the duration of the explosion,  $\tau_{\text{exp}}$ , which is

$$\tau_{\text{exp}} = 3 \text{ ps} \frac{\sqrt{A}}{Z \sqrt{n_{18}}}, \quad (11.66)$$

in which  $n_{18}$  is the ion density in units of  $10^{18}$  ions per  $\text{cm}^3$ . Note that this turns out to be independent of the radius of the cloud. At the densities associated with solids or with clusters in gasses, of order  $10^{23}$  ions per  $\text{cm}^3$ , this time is of order 10 fs.

The third requirement is that the laser must be able to expel the electrons. The electrons are affected by the ponderomotive force produced by the laser light, discussed in Sect. 8.1. The nonrelativistic expression for this force can be written as the gradient of a ponderomotive potential, defined by  $U = m_e \langle v_{\text{os}}^2 \rangle / 2$ , where the brackets denote an average that introduces a factor of 2. One finds that this potential contributes to the total energy of the electrons and thus can balance other sources of potential energy, such as the Coulomb potential associated with a charged cloud. The relativistically correct ponderomotive potential can be written

$$U = m_e c^2 \left( \sqrt{1 + a_o^2} - 1 \right). \quad (11.67)$$

Setting this equal to the maximum Coulomb potential, which is  $\mathcal{E}_{\text{max}}$ , we find

$$a_o > 1.2 \sqrt{n_{18}} \sqrt{2 + 1.5 n_{18}} \quad (11.68)$$

to have enough ponderomotive potential to completely remove the electrons. If instead the ponderomotive potential is only large enough to remove some of the electrons, the cloud of ions will still explode. However, it will not produce the maximum possible ion energies.

In summary, to obtain a Coulomb explosion the laser pulse must rise quickly enough to avoid thermal expansion. This time decreases as the laser irradiance increases or the laser spot (or the target cloud, if it is smaller) shrinks. The laser pulse must also rise quickly enough to avoid a premature explosion before the electrons are expelled and must be of long enough duration for the explosion to occur. This duration decreases as density increases, reaching femtoseconds for solid densities. Finally, the laser pulse must be powerful enough to expel the electrons, and the required irradiance increases roughly as the square of the ion density.

## 11.8 Hole Drilling and Collisionless Shocks

When the laser irradiance onto a high-density target plasma (having  $n_e > n_c$ ) becomes large enough, the ponderomotive pressure no longer merely keeps the density profile steep, but actually drills a hole into the plasma, pushing the ions ahead of it. Under circumstances we will discuss, this can drive a *collisionless shock* into the plasma. The shock reflects plasma ions, but often will not be relativistic. As a result the reflected ions are given roughly twice the velocity of the shock (in the laboratory frame). This is another mechanism, in addition to sheath acceleration and Coulomb explosions, that can produce energetic ions in ultrafast laser–plasma interactions.

The phenomenon of *hole drilling* by relativistic lasers was first discussed by Wilks et al. (1992). One can understand the basic behavior by considering the momentum exchange between the laser beam and the plasma ions. The electrons, of course, carry negligible momentum because of their small mass. The momentum of each photon is  $\hbar k$ , so that the total incident momentum flux is  $\hbar k \times I_L/(\hbar\omega) = I_L/c$  and the total momentum flux delivered to the plasma by the laser is  $(1 + \eta)I_L/c$ , where the fraction of the laser power reflected is  $\eta$ . This must be balanced by the momentum flux of the ions. If the ions that have already been swept up move into the target at  $v_i$ , then the ion flux being swept up is  $n_i v_i$  and the momentum flux being delivered to the newly-swept-up ions is  $m_i n_i v_i^2$ . Thus,

$$\left(\frac{v_i}{c}\right)^2 = \frac{I_L(1 + \eta)}{n_i m_i c^3}, \quad (11.69)$$

or if we formulate this in terms of  $a_o$ , using also (8.13),

$$\left(\frac{v_i}{c}\right)^2 = \frac{a_o^2(1 + \eta)}{m_i c^3} \frac{n_c}{n_i} \frac{1.37 \times 10^{18}}{1.1 \times 10^{21}} = 2.7 \times 10^{-4} a_o^2 (1 + \eta) \frac{n_c}{n_{eo}} \frac{Z}{A}, \quad (11.70)$$

where the electron density in the target is  $n_{eo}$  and the critical density for the laser, from (8.13) is  $n_c$ . This equation applies to the regime of non-relativistic ion velocity once  $a_o$  is large enough that other pressures are negligible. It implies that a picosecond-timescale laser having  $a_o \sim 30$  can drill a hole into a critical-density target that is 50  $\mu\text{m}$  deep. This may prove important in the context of fast ignition (see Chap. 9).

We can compare this ion velocity to the sound speed as follows. For large  $a_o$ , the laser-heated electron temperature  $T_h$  is given by  $k_B T_h \sim a_o m_e c^2$ , as discussed above. Thus, the sound speed is

$$\left(\frac{c_s}{c}\right)^2 = \frac{1}{c^2} \frac{Z k_B T_h}{m_i} = a_o \frac{m_e}{m_p} \frac{Z}{A}. \quad (11.71)$$

Thus, the ions penetrate the plasma with a Mach number given by

$$M^2 = \left(\frac{v_i}{c_s}\right)^2 = 0.5 a_o (1 + \eta) \frac{n_c}{n_{eo}}. \quad (11.72)$$

This in turn gives a threshold value of  $a_o$  to make this Mach number greater than 1, as

$$a_o \geq \left(\frac{2}{1 + \eta}\right) \frac{n_{eo}}{n_c}. \quad (11.73)$$

Thus, lasers will drive supersonic ion fluxes into critical-density plasmas beginning at  $a_o \sim 1$  and into solid-density plasmas beginning at  $a_o \sim 100$ .

In the context of hydrodynamics, we would conclude that a shock must form once  $M > 1$ . However, collisions are far too weak to produce a shock in such a target, so if one is to form it must be by means of collisionless



dynamics. This does occur, as follows (for a further description, see the two papers by Forslund and Shonck and the one by Silva et al.). Interpenetrating plasmas are unstable to an instability known as the *two-stream* instability once the interpenetration velocity exceeds the thermal velocity of one of the species. This instability can resonate between any two species in the plasma, so that there is an ion–ion and an ion–electron two-stream instability. Both these instabilities have a growth rate of some fraction of the ion–plasma frequency, given by  $\omega_{\text{pi}}^2 = 4\pi n_i Z^2 e^2 / m_i$  (in cgs units). This means that they grow to a large amplitude rapidly on the timescale on which the ions move. They are so-called electrostatic instabilities, producing longitudinal electric fields but no magnetic fields.

These instabilities lead to a shock of sorts, as follows. The electrostatic waves become quite large and trap electrons locally where the ions begin to interpenetrate. This takes a time of order  $5 \times 2\pi / \omega_{\text{pi}}$ , as has been confirmed by PIC simulations. The trapped population of electrons prevents further electrons from crossing the interaction zone. As a result, the ions that do cross this zone establish a positive potential and are reflected. The system reaches a steady state. In the shock frame, one would say that the ions coming in from upstream are in large part elastically reflected once they cross the trapped-electron region, while the downstream ions are stationary, being held in place by the ram pressure associated with the reflected ions on one side and the momentum flux delivered by the laser on the other side. In the laboratory frame, one would say that the laser pushes ions into the target, that at their leading edge a group of trapped electrons is produced, and that the resulting positive potential barrier strikes the ions beyond it and sends them ahead of the shock at twice the shock velocity. The density of the ions downstream of the shock, which is effectively compressed from both sides, ends up somewhat larger than the density of the ions ahead of the shock.

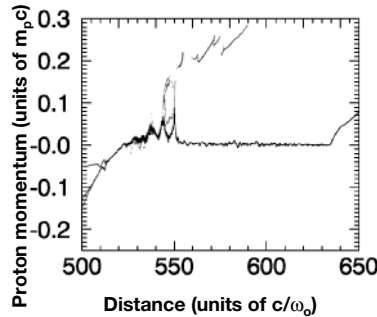
Figure 11.8 shows the results of a simulation of such a shock. Like most displays of PIC results, the dots on this plot correspond to the location of a given particle in  $z$  velocity and in  $z$  position. The initial location of the target was from 500 to 640 distance units. One can see sheath-accelerated particles at the two ends of the target. The shock is at about 550 distance units and is followed by structure that we will not concern ourselves with here. The shock-reflected ions are the highest-energy particles, seen streaming ahead of the shock with momenta above  $0.15m_p c$ .

Since few ions join the ions behind the shock, the “piston” velocity at which the laser pulse pushes the ions ahead of it equals the shock velocity. One thus has a reflected-ion velocity,  $v_{\text{ref}}$ , given by

$$\left(\frac{v_{\text{ref}}}{c}\right) = 0.03a_0 \sqrt{\frac{n_c}{n_{\text{eo}}} \frac{Z}{A}}, \quad (11.74)$$

which we can also write as

$$\left(\frac{v_{\text{ref}}}{c}\right) = \frac{0.06}{(1 + \eta)} \sqrt{\frac{n_{\text{eo}}}{n_c} \frac{Z}{A}} M^2. \quad (11.75)$$



**Fig. 11.8.** Particle velocities from a PIC simulation of a collisionless shock. A laser pulse having  $a_o = 16$  has been incident on a plasma initially having  $n_e = 10n_c$  for a time of  $1024/\omega_o$  from Silva et al.

Even for  $M$  not much larger than 1, this can easily reach a few tenths of the speed of light. Under the right conditions, with a solid-density target and a very large  $a_o$ , even a shock with  $M \sim 2$  might drive relativistic reflected ions.

### Homework 11.8

Derive the relativistic version of this theory and find the relativistically correct revision to (11.75).

It would seem from the above equations that relativistic ions would be straightforward to produce by sufficiently increasing  $M$ . This, however, does not work in practice because the ability of the fluctuations produced by the two-stream instability to trap electrons is limited, and as a result ions above a certain energy (in the shock frame) cannot be reflected. The limiting Mach number, known as a *critical Mach number*, is  $M \sim 3$ . Above this Mach number fluctuations are still driven and the interpenetrating ion beams are affected, but there is no longer a shock. Faster ions cannot then be driven by shocks, but they can be driven in the “laser piston” regime, discussed in the next section.

The mechanism of laser-driven collisionless shocks is most effective and most important in comparatively thin targets. This better enables the laser to keep the electrons heated throughout the target so that the shock remains below the critical Mach number and in addition allows the ions to move through the target in time to be further accelerated by the sheath on its rear surface.

A different type of “shock” is produced during any realistic Coulomb explosion of more than a small, isolated cluster. This develops as follows (see Kaplan et al.). The density profile of the ions is never entirely uniform, and in addition the electron expulsion may not be complete at the outer edges of the initial ion cloud. This has the consequence that the force on the ions reaches a maximum within the cloud, and that some ions within the

cloud are accelerated more strongly than those on its outer edge. As a result, some ions overtake others. If the explosion truly involves cold ions and no electrons, then the Mach number of this interpenetration will be too large to drive the type of shock described above. (Such shocks might develop if the ions were warm enough or some laser-heated electrons were present, to sufficiently increase the sound speed.) But even in the absence of instabilities or other ion interactions, one ends up with ions of three velocities overlapping in a certain region. These are the slow ions and two groups of faster ions. The two groups of fast ions originate either just inside or just outside the initial radius of the fastest (most strongly accelerated) ions. This leads the ion density to increase in the region where the overlap occurs. In addition, there is a large concentration of ions per unit velocity at the two ends of this region, where in each case two of these groups merge. The resulting structure is often referred to as a *shock shell*.

## 11.9 Other Phenomena

We have already seen that a host of phenomena can be produced when a laser beam irradiates a target at a relativistic irradiance. We discuss several more here.

As the laser irradiance increases beyond the levels that drive collisionless shocks, (11.70) would predict from momentum balance that the laser can produce relativistic ions. In detail, however, the laser pulses push on the electrons and they in turn create an electric field that accelerates the ions. When the electrons are able to accelerate the ions to relativistic velocities during the laser pulse, the system has entered the *laser piston* regime, discussed by Esirkepov et al. The requirements for this regime can be calculated in simple limits as follows.

We suppose that the laser beam pushes all of the electrons completely out of the initial target, which is a thin layer of thickness  $d$ . For simplicity suppose that  $d$  is much less than the width of the laser spot, which implies that the electric field is independent of exactly how far the electrons have been pushed. The electric field is produced by both the electron layer and the ion layer, each of which has the same charge. From Gauss' law one can show that the electric field experienced by the ions closest to the electrons is then

$$E_{\parallel} = 16\pi^2 en_{e0}d, \quad (11.76)$$

which is in cgs units. In SI units the coefficient that is  $16\pi^2$  would be  $4\pi$ . The relativistically correct ion energy is

$$\mathcal{E}_i = \sqrt{m_i^2 c^4 + p_i^2 / (2m_i)} = \sqrt{m_i^2 c^4 + (eE_{\parallel}ct)^2}, \quad (11.77)$$

in which the rightmost expression is specific to ions that acquire energy while moving through constant electric field at speed  $c$  (which is clearly an

approximation for early times). The approximate time at which the kinetic energy of the ions reaches their rest mass, thus making them relativistic, can be found from this to be

$$t_{\text{rel}} = \frac{m_i c^2}{e E_{\parallel}} = \frac{m_i c^2}{16\pi^2 e^2 n_{\text{eo}} d} = \frac{Z(c/d)}{4\pi\omega_{\text{pi}}^2}. \quad (11.78)$$

If we hope to avoid complications not present in this model, this acceleration should happen within a laser cycle, so that  $\omega_o t_{\text{rel}} \sim 2\pi$ . We would like to know how intense a laser is required to accomplish this. We find this by recognizing that the electric field in the laser,  $E_L$ , must be larger than  $E_{\parallel}$  and using (11.78) to solve for  $d$ . We find this implies that the target thickness should be

$$d \sim \frac{Zc\omega_o}{8\pi^2\omega_{\text{pi}}^2} = \frac{c}{8\pi^2\omega_o} \frac{\omega_o^2}{\omega_{\text{pe}}^2} \frac{m_i}{m_e} = \frac{c}{8\pi^2\omega_o} \frac{n_c}{n_{\text{eo}}} \frac{m_i}{m_e}, \quad (11.79)$$

which will typically be some fraction of a laser wavelength. Using this result in (11.76),  $E_L \geq E_{\parallel}$ , and  $a_o = eE_L/(m_e\omega_o c)$ , we find

$$a_o = \frac{1}{2\pi} \frac{m_i}{m_e}, \quad (11.80)$$

which for protons is an  $I_L \lambda_{\mu}^2$  of  $\sim 1.2 \times 10^{23}$  W  $\mu\text{m}^2/\text{cm}^2$ .

Another phenomenon that results from laser-solid dynamics is *magnetic-field generation*. The beams of electrons produced by the laser carry substantial currents. As a result, they drive substantial magnetic fields. This is easier than it may seem. For example, suppose one wants to produce a 1 GGauss field. From (8.69), supposing that one drives an electron current by sending a beam of electrons, at  $c$ , through a circle of diameter 10  $\mu\text{m}$ . The density of this beam must be only  $\sim 10^{15}$   $\text{cm}^{-3}$ . One can achieve this by accelerating one electron in  $10^8$  to  $\sim c$ . Measuring the field, however, is a severe challenge. At his writing, it is believed that GGauss fields have been produced.

Currents can be driven either into the target or along the surface. Ruhl and Mulser identified the production of a surface magnetic field in their calculations, arising from the ponderomotive force, which generates charge separation and surface currents for oblique incidence. These in turn produce a dc magnetic field, which was found in the simulations to vary strongly with irradiance. This field, which can approach 100 MG, causes dramatic variations in the absorption with  $I_L$  and angle of incidence (see their figure 6.) Wilks et al. (1992) also observed a dc magnetic field in (fully relativistic and electromagnetic) 2D PIC simulations for normal incidence, at intensities larger than those considered by Ruhl and Mulser. They attribute it to electron heating at the light-plasma interface. In addition, other magnetic-field generation mechanisms come into play at relativistic intensities (see Bychenkov and Tikhonchuk and Sudan).

A third phenomenon of potential interest in astrophysical contexts is *positron production*. It requires more than  $10^{28}$  W/ $\text{cm}^2$  to produce electron-positron pairs directly from vacuum. However, when the laser light interacts

with solid targets, it becomes possible to produce pairs at much lower irradiance. This is discussed, for example, by Liang, Wilks, and Tabak. This first requires the production of energetic electrons, with the characteristic energy  $E_{\text{hot}}$  from (11.20). The threshold electron energy for pair production by interaction with a nucleus is  $2mc^2$ , and the cross-section scales as the square of the nuclear charge,  $Z_{\text{nuc}}^2$ . The threshold irradiance for pair production in steady state is about  $10^{20}$  W/cm<sup>2</sup>. For a dynamic, ultrafast interaction, Liang et al. find a comparable threshold value. At this writing, positron production has been observed. It remains to be seen whether one can produce a plasma whose energy density is dominated by the pairs and whether a laser–target system can become a practical positron source.

Relativistic laser beams can produce *nuclear reactions* within a target. The first step is the production of electrons having energies of many MeV. For example, when a beam having an irradiance above  $10^{20}$  W/cm<sup>2</sup> and a total energy of 50 J, with a substantial prepulse, was used by Cowan et al. to irradiate a Au target, a distribution of electrons was produced with energies up to 100 MeV. These electrons in turn produced bremsstrahlung photons with energies of tens of MeV by interaction with the Au nuclei. These photons exceeded the energy threshold of  $\sim 10$  MeV for photonuclear reactions in the Au and in Cu located near the target. This caused transmutation of these elements.

We mentioned above the use of electron bunches to drive wakefield acceleration in a plasma. Other effects can also be explored using high-energy electron beams (see Joshi et al.). One can use a tailored plasma to focus, defocus, or steer the beam. One can cause the beam to undulate in the transverse direction in the plasma. On the one hand, this happens naturally if the parameters are such that the beam repeatedly self-focuses and due to the natural tilt of the beam with respect to its propagation direction. On the other hand, one could imagine more active steps to create waves in the plasma that undulate the beams. This would produce tunable, forward-directed radiation with potential applications.

Beyond the above, the collisions involved in high-energy (as distinct from high-energy-density) physics create temporary, relativistic environments of an extremely high energy density. While much of the behavior might be best described in terms of particle physics, one does encounter concepts such as the “quark–gluon plasma” that have much in common with various plasma systems. In addition, some of these systems may have applications to astrophysics (see Chen). This potentially includes cosmic acceleration experiments, the spectroscopy of heavy elements, experiments related to event horizons, and experiments with the dynamics of relativistic jets.

The area of relativistic high-energy-density experiments has been developing very rapidly, in recent years, at the time of this writing. One can look forward to exciting further ideas and discoveries in the coming years.

## 12 Appendix A: Constants, Acronyms, and Standard Variables

**Table 12.1.** Table of Constants

Name	Symbol	cgs value	useful alternative	SI value
Bohr radius	$a_o$	$5.29 \times 10^{-9}$ cm	0.529 Å	$5.29 \times 10^{-11}$ m
Speed of light	$c$	$3 \times 10^{10}$ cm/s		$3 \times 10^8$ m/s
Electronic charge	$e$	$4.8 \times 10^{-10}$ statcoul		$1.6 \times 10^{-19}$ Coul
Planck constant	$h$	$6.63 \times 10^{-27}$ erg-sec		$6.63 \times 10^{-34}$ J-sec
Boltzmann constant	$k_B$	$1.6 \times 10^{-12}$ erg/eV	$1.6 \times 10^{-16}$ J/keV	$1.38 \times 10^{-23}$ J/K
Electron rest mass	$m_e$	$9.11 \times 10^{-28}$ g		$9.11 \times 10^{-31}$ kg
Proton mass	$m_p$	$1.67 \times 10^{-24}$ g	$1836m_e$	$1.67 \times 10^{-27}$ kg
Stefan-Boltzmann	$\sigma$	$5.67 \times 10^{-5}$ ergs/( $\text{cm}^2 \text{ s deg}^4$ )	$1.03 \times 10^5$ W/( $\text{cm}^2 \text{ eV}^4$ )	$5.67 \times 10^{-8}$ W $\text{m}^{-2} \text{ K}^{-4}$

**Table 12.2.** Table of Acronyms

Acronym	Represents
ASE	Amplified spontaneous emission
CPA	Chirped pulse amplification
DPP	Distributed phase plate
DPR	Distributed polarization rotator
ICF	Inertial confinement fusion
LTE	Local thermodynamic equilibrium
NLTE	Nonlocal thermodynamic equilibrium
RT	Rayleigh Taylor
KH	Kelvin Helmholtz
RM	Richtmyer Meshkov
RPP	Random phase plate
SBS	Stimulated Brillouin scattering
SN	Supernova
SRS	Stimulated Raman scattering
SSD	Smoothing by spectral dispersion
exp[]	Equivalent to $e^{\square}$

**Table 12.3.** Table of standard variables

Name	Symbol
Atomic weight (average)	$A$
Vector potential	$\mathbf{A}$
Atwood number	$A_n$
Area of capsule	$A_c$
Area of laser spots	$A_L$
Area of walls of hohlraum	$A_w$
Magnetic field	$\mathbf{B}$
Thermal intensity	$B(T)$
Thermal spectral intensity	$B_\nu(T)$
Isentropic sound speed	$c_s$
Specific heat at const vol	$c_V$
Small vortex diameter	$d$
Element of area	$d\mathbf{A}$
Critical to solid density distance	$D$
Electron charge	$e$
Electric field	$\mathbf{E}$
Spectral kinetic energy	$E(k)$
Thermal energy density	$E_{BB}$
Energy released by fusion	$E_{fus}$
Hydrogen ionization energy	$E_H$
Electric field of laser beam	$E_L$
Total radiation energy density	$E_R$
Energy in Marshak wave	$E_w$
Spectral radiation energy density	$E_\nu$
Energy difference between ionization states $j$ and $k$	$E_{jk}$
Electron total energy	$\mathcal{E}_e$
Electron rest mass energy	$\mathcal{E}_o$
Ion total energy	$\mathcal{E}_i$
Thermal flux	$F_{BB}$
Electron free energy	$F_e$
Electromagnetic force density	$\mathbf{F}_{EM}$
Lorentz force	$\mathbf{F}_L$
Radiative energy flux	$\mathbf{F}_R$
Total radiation flux	$F_R$
Photon flux	$\mathbf{F}_\gamma$
Spectral radiation flux	$F_\nu$
Eddington factor	$f_\nu = p_\nu/E_\nu$
Distribution function	$f(v)$
“Gravitational” acceleration	$g$
Laser irradiance	$I_L$
Total intensity	$I_R$
Laser irradiance in units of $10^{xx}$ W/cm <sup>2</sup>	$I_{xx}$
Spectral intensity	$I_\nu$

**Table 12.3.** (continued)

Name	Symbol
Current density	$\mathbf{J}$
Richardson number	$J_r$
Total mean intensity	$J_R$
Transverse current density	$\mathbf{J}_t$
Mean spectral intensity	$J_\nu$
Riemann invariants	$J_+$ or $J_-$
Wave number	$k$
Wave vector	$\mathbf{k}$
Coefficients in Maxwell's equations	$k_1, k_2, k_3$
Scale length of a profile	$L$
Eddy diameter	$\ell$
Compton mean free path	$\ell_C$
Mach number	$M$
Upstream Mach number	$M_u$
Internal Mach number	$M_{\text{int}}$
Fusion fuel mass	$m_f$
Mass ablation rate	$\dot{m}$
Shock normal	$\mathbf{n}$
Electron density	$n_e$
Ion density	$n_i$
Critical density	$n_c$
Scalar fluid pressure	$p$
Total scalar pressure	$\tilde{p}$
Electron momentum	$\mathbf{p}_e$
Fermi degenerate pressure	$p_F$
Scalar radiation pressure	$p_R$
General pressure tensor	$\underline{\mathbf{P}}$
Ablation pressure	$P_{\text{abl}}$
Power threshold for relativistic self-focusing	$P_{\text{sf}}$
Turbulent energy dissipation	$P_t$
Radiation spectral pressure tensor	$\underline{\mathbf{P}}_\nu$
Thermal Heat flux	$\mathbf{Q}$
Radiation Strength parameter	$Q$
Spitzer–Harm heat flux	$Q_{\text{SH}}$
Free-streaming heat flux	$Q_{\text{FS}}$
Internal energy	$R$
Gas constant $p/(\rho T)$	$R$
Ion sphere radius	$R_o$
Radiation strength parameter	$R_r$
Poynting flux	$\mathbf{S}$
Specific entropy	$s$
Specific entropy of electrons	$s_e$
Source of quantity $Q$	$S_Q$
Spectral source function	$S_\nu$
Time	$t$



**Table 12.3.** (continued)

Name	Symbol
Temperature	$T$
Immediate post-shock temperature	$T_2$ or $T_s$
Fermi-degenerate temperature	$T_d$
Electron temperature	$T_e$
Effective temperature	$T_{\text{eff}}$
Ion temperature	$T_i$
Temperature corresponding to a radiation flux	$T_{\text{min}}, T_{\text{eff}}$
Energetic electron temperature	$T_{\text{hot}}$
Precursor temperature	$T_p$
Radiation temperature	$T_R$
Immediate postshock plasma (mainly electron) temperature	$T_s$
Hohlraum wall temperature	$T_w$
Fluid velocity	$\mathbf{u}$
Zeroth-order fluid velocity	$U$
Characteristic velocity for scaling arguments	$U$
First-order components of fluid velocity	$\mathbf{u}_1 = (u, v, w)$
Kolmogorov velocity scale	$u_k$
Particle velocity	$\mathbf{v}$
Velocity difference between frames of reference	$\mathbf{v}$
Phase velocity	$v_p$
Oscillating velocity of electron in light wave	$\mathbf{v}_{\text{os}}$
Electron thermal velocity	$v_{\text{th}}$
Rocket velocity (or capsule velocity)	$V$
Exhaust velocity	$V_{\text{ex}}$
Vertical component of velocity	$w$
Vortex rotational velocity	$w$
Eddy rotational velocity	$w_e$
Marshak wave scaling variable	$W$
Mag. Energy den.	$W_B$
Electric energy density	$W_E$
Space	$x$
Marshak wave penetration depth	$x_M$
Fusion yield	$Y$
Ionic charge (average)	$Z$
Albedo	$\alpha$
Various angles	$\alpha$
Fraction of incoming photons ionized	$\alpha_i$
Various angles	$\beta$
Relativistic velocity ( $v/c$ )	$\beta$
Various angles	$\chi$
Coeff of thermal diffusivity	$\chi$
Electron momentum	$\chi_e$

**Table 12.3.** (continued)

Name	Symbol
Jet cooling parameter	$\chi_j$
Rosseland-mean opacity	$\chi_R$
Spectral total opacity	$\chi_\nu$
Specific internal energy	$\epsilon$
Total specific internal energy density	$\tilde{\epsilon}$
Downstream emissivity	$\epsilon_d$
Specific internal electron energy	$\epsilon_e$
Fermi energy	$\epsilon_F$
Specific internal ion energy	$\epsilon_{ii}$
Specific kinetic ion energy	$\epsilon_{ik}$
Upstream emissivity	$\epsilon_u$
Efficiency of ideal rocket	$\epsilon_R$
Various angles	$\phi$
Fusion burn fraction	$\phi$
Phase of a wave	$\phi$
Phase experienced by an electron	$\phi_e$
Scalar electric potential	$\Phi$
Polytropic index	$\gamma$
Relativistic $\gamma$	$\gamma_r$
Instability growth rate	$\gamma_o$
Strong coupling parameter	$\Gamma$
Flux of quantity $Q$	$\Gamma_Q$
various angles and fractions	$\eta$
x-ray conversion efficiency	$\eta$
Kolmogorov length scale	$\eta_k$
Spectral emissivity	$\eta_\nu$
Spectral scattering emissivity	$\eta_{\nu sc}$
Spectral thermal emissivity	$\eta_{\nu th}$
Absorption opacity	$\kappa$
Total coefficient of heat conduction	$\tilde{\kappa}$
Opacity of thin layer using cooling function	$\kappa_{astro}$
Thermal bremsstrahlung absorption coefficient	$\kappa_b$
EM wave absorption coefficient	$\kappa_{EM}$
Specific Planck mean opacity	$\kappa_m$
Planck mean opacity	$\kappa_P$
Radiative coefficient of heat conduction	$\kappa_{rad}$
Thermal coefficient of heat conduction	$\kappa_{th}$
Spectral absorption opacity	$\kappa_\nu$
Wavelength of a wave	$\lambda$
Vortex characteristic scale	$\lambda$
Taylor microscale	$\lambda_T$
Debye length	$\lambda_D$
Electron Debye length	$\lambda_{De}$

**Table 12.3.** (continued)

Name	Symbol
Ion Debye length	$\lambda_{Di}$
Mean free path	$\lambda_{mfp}$
Wavelength in microns	$\lambda_\mu$
Astrophysical cooling function	$\Lambda$
Chemical potential	$\mu$
Classical chemical potential	$\mu_c$
Vortex characteristic scale	$\lambda$
Atomic mass per charge ( $A/Z$ )	$\mu_e$
Electron–ion collision rate	$\nu_{ei}$
Optically thin cooling rate	$\nu$
Kinematic viscosity	$\nu$
Radiation frequency	$\nu$
Extinction rate	$\nu_e$
Cooling rate normalization for thin layer	$\nu_{rad}^*$
Cooling rate more general	$\nu_1$
Cooling rate using cooling function	$\nu_{astro}$
Radiation cooling rate for thin layer	$\nu_{rad}$
Kinematic photon viscosity	$\nu_{rad}$
Scaling variable $n_e/T_e^{3/2}$	$\theta$
Degeneracy parameter	$\Theta$
Mass density	$\rho$
Charge density	$\rho_c$
Density of Q	$\rho_Q$
Scattering opacity	$\sigma_s$
Spectral scattering opacity	$\sigma_\nu$
Viscosity stress tensor	$\underline{\sigma}_\nu$
Kolmogorov time scale	$\tau_k$
Optical depth at frequency $\nu$	$\tau_\nu$
Optical depth	$\tau$
Wave frequency	$\omega$
Laser light frequency	$\omega_o$
Normalized frequency	$\omega_n$
Electron plasma frequency	$\omega_{pe}$
Ion plasma frequency	$\omega_{pi}$
Scattered light frequency	$\omega_s$
Irradiance conversion by hohlraum	$\xi$
General similarity variable	$\xi$
Gravitational potential	$\Psi$

## 13 Appendix B: Sample *Mathematica* Code

This notebook provides an example of a computational math derivation of the basic shock relations.

- Begin by loading packages we may want for plotting

```
<< Graphics`MultipleListPlot`
<< Graphics`Graphics`
<< NumericalMath`ListIntegrate`
$TextStyle =
{FontWeight -> "Bold", FontFamily -> "Helvetica", FontSize -> 12}

{FontWeight -> Bold, FontFamily -> Helvetica, FontSize -> 12}
```

- First suppose  $\gamma$  does not change. Write the balance equations as quantities equal to zero

$$\begin{aligned} \text{eq1} &= \rho_1 u_1 - \rho_2 u_2 \\ \text{eq2} &= \rho_1 u_1^2 + p_1 - \rho_2 u_2^2 - p_2 \\ \text{eq3} &= \left( p_1 u_1 + \rho_1 \epsilon_1 u_1 + \frac{\rho_1 u_1^3}{2} \right) - \left( p_2 u_2 + \rho_2 \epsilon_2 u_2 + \frac{\rho_2 u_2^3}{2} \right) \end{aligned}$$

- Analysis: 3 equations.  
First want relations between pressure and density.  
Use EOS to eliminate  $\epsilon$ .  
Then use first two equations to eliminate  $u_1$  and  $u_2$ .
- The next step is the first example of a pattern replacement. This is a key technique for doing algebra in *Mathematica* without being forced to define dummy variables. It saves a lot of confusion.

$$\text{eq3a} = \text{eq3} /. \left\{ \epsilon_1 \rightarrow \frac{p_1}{\rho_1 (\gamma - 1)}, \epsilon_2 \rightarrow \frac{p_2}{\rho_2 (\gamma - 1)} \right\}$$

```
cond1 = Solve[{eq1 == 0}, u2]
(* Here we solve an equation to create a condition giving the
   value of a variable. It is useful to use a systematic
   notation for this. Then we substitute the results *)
eq4 = eq2 /. cond1[[1]]
eq5 = (eq3a /. cond1[[1]])
```

- The next two steps are why this is a pain to do on paper. We solve for  $u_1$  and then substitute in the energy equation, and finally solve for  $p_2$ . If you try it on paper, the algebra is a lot easier using  $1/\rho$  as a variable.

```
cond2 = Solve[eq4 == 0, u1]
eq6 = eq5 /. cond2
```

```
cond3 = Solve[eq6 == 0, p2]
```

```
cond3[[2]]
```

```
prat = Simplify[(p2 /. cond3[[2]]) / p1]
(* creating a normalized ratio of p2/p1 *)
```

```
prativ = Simplify[prat /. {ρ1 → 1 / V1, ρ2 → 1 / V2}]
```

- The above two results are the standard expressions for the pressure ratio. Next find the density ratio.

```
cond4 = Solve[eq6 == 0, ρ2]
```

```
rhorat = Simplify[(ρ2 /. cond4[[1]]) / ρ1]
(* rhorat is the ratio ρ2/ρ1 *)
```

```
eq7 = ρ2 / ρ1 - rhorat
```

- Here we created an equation for the density ratio, eq7, which is equal to zero. That lets us proceed to find other solutions, for example involving Mach number. This follows:

```
cond5 = Solve[eq4 == 0, p2]
```

```
cond6 = Solve[eq7 == 0 /. cond5[[1]], ρ2]
```

```
cond7 = Simplify[cond6 /. p1 → ρ1 cs2 / γ]
(* using the standard definition of sound speed *)
```

```
cond8 = Simplify[(cond7[[2]]) /. u1 → Mu cs]
(* Mu is the upstream Mach number *)
```

```

rhorat2 = (ρ2 /. cond8[[1]]) / ρ1
(* get the standard expression for the density ratio *)

```

```

prat2 = Simplify[prat /. ρ2 → ρ1 rhorat2]
(* get the standard expression for the pressure ratio *)

```

```

Trat = Collect[Simplify[prat2 / rhorat2], Mu]
(* develop an expression for the temperature ratio *)

```

```

test = Simplify[Trat /. Mu → 1] (* check sensibility *)

```

```

p2a = Simplify[p1 (prat2 /. Mu2 → (ρ1 us2 / (γ p1)))]

```

```

p2b =  $\frac{\rho 2}{A \text{ mp}}$  kB (Z + 1) T /. ρ2 → ρ1 rhorat2
p2b = p2b /. Mu2 → (ρ1 us2 / (γ p1))

```

```

cond9 = Solve[p2b - p2a == 0, T]

```

```

T2 = Collect[T /. cond9[[1]], us]

```

```

Ten = T2 /. kB -> 1

```

```

Tenig = Ten /. γ → 5 / 3

```

```

(* With p1 = 0, this is the "standard" result *)

```

**Notebook to do Fermi distributions and related calculations**

- Begin by loading useful packages

```
<< Graphics`MultipleListPlot`
<< Graphics`Graphics`
<< NumericalMath`ListIntegrate`
$TextStyle =
{FontWeight -> "Bold", FontFamily -> "Helvetica", FontSize -> 12}
```

- Sections that follow
  1. Fermi degenerate distributions
  2. Classical vs Degenerate density and pressure calculations

- 1. First section produces fermi degenerate plot, using the expression in the text for the electron distribution function

```
f = 
$$\frac{\text{Exp}\left[\frac{-e\text{Fermi}}{kT}\right] + 1}{\text{Exp}\left[\frac{-e\text{Fermi} + e n}{kT}\right] + 1}$$

g = Exp[-en / kT]
f1 = Simplify[f /. {en -> α eFermi}]
g1 = Simplify[g /. {en -> α eFermi}]
f2 = Simplify[f1 /. {eFermi ->  $\frac{1}{\beta}$  kT}]
g2 = Simplify[g1 /. {eFermi ->  $\frac{1}{\beta}$  kT}]
```

```
g1 = Plot[{f2 /. β -> .01, f2 /. β -> 1, f2 /. β -> 10, g2 /. β -> 10},
{α, 0, 10}, PlotRange -> {{0, 10}, {0, 1.1}}, Frame -> True,
PlotStyle -> {{Thickness[0.008]}, {Thickness[0.008]},
{Thickness[0.008]}, {Thickness[0.015], GrayLevel[.5]}}
```

```
Export["fermi distributions.eps", g1, "EPS"]
```

```
LinearLogPlot[
{f2 /. β -> .01, f2 /. β -> 1, f2 /. β -> 5, g2 /. β -> 5},
{α, 0, 10}, PlotRange -> {{0, 10}, {.2, 1.1}}
```

■ 2. Classical vs Degenerate density and pressure calculations

■ Basic relations

```
efermi = (6 π^2)^(2/3) (h / (2 π))^2 ne^(2/3) / (2 me)
θ1 = (kB Te) / efermi (* this is θ *)
θψ = PowerExpand[θ1 /. ne -> (ψ Te^(3/2))]
coefθψ =
  θψ /. {ψ -> 1, me -> 9.11 10^-28, h -> 6.63 10^-27, kB -> 1.6 10^-12}
θF = PowerExpand[θ1 /. ne -> (Te^(3/2) (4 π (2 me kB)^(3/2) Fonehalf) / h^3)]
```

Some quantities relevant to results in the book by Lindl

$$neLindl = \frac{0.25}{2.5 \cdot 1836 \cdot 9.11 \cdot 10^{-28}}$$

$$eFermiLindl = 7.9 * \frac{neLindl}{10^{23}}$$

$$pFermiLindl = 9.9 \left( \frac{0.25}{2.5} \right)^{5/3}$$

$$\thetaLindl = \theta1 /. \{me \rightarrow 9.11 \cdot 10^{-28}, h \rightarrow 6.63 \cdot 10^{-27}, kB \rightarrow 1.6 \cdot 10^{-12}, \\ Te \rightarrow \frac{11.5}{11604} \cdot 0.001, ne \rightarrow neLindl\}$$

(\* the range of ψ of interest is 10<sup>14</sup> to 10<sup>26</sup> \*)  
 (\* so the range of θ is \*)  

$$\frac{coef\theta\psi}{(10^{14})^{2/3}} \text{ (* to *)}$$
  

$$\frac{coef\theta\psi}{(10^{26})^{2/3}}$$

(\* i.e. 0.001 to 10<sup>6</sup> \*)

This next plot shows why this is the range



```
g2 = LogLogPlot[{ni, 10 ni, 20 ni, 40 ni, 80 ni}, {ni, 1019, 1024},
  PlotRange -> {{1019, 1024}, {1. 1020, 1. 1026}}, Frame -> True]
```

```
N[1019 / 10003/2]
```

- This part calculates  $\mu/(k_B T_e)$  for  $\Theta = T/T_d$

```
Solve[ $\Theta \psi = \Theta, \psi$ ]
 $\mu_{class\Theta} = \text{Log}[\text{PowerExpand}[\frac{h^3}{2 (2 \pi m_e k_B)^{3/2}} \Theta] /. \Theta \rightarrow (\psi /. \text{Solve}[\Theta \psi = \Theta, \psi])][[1]]$ 
 $\mu_{fermi\Theta} = \text{PowerExpand}[\left( (3 \pi^2)^{2/3} \left( \frac{h}{2 \pi} \right)^2 \frac{1}{2 m_e k_B} \Theta^{2/3} \right) /. \Theta \rightarrow (\psi /. \text{Solve}[\Theta \psi = \Theta, \psi])][[1]]$ 
```

```
 $\Theta_{crit} = \text{N}[\Theta /. \text{Solve}[\mu_{class\Theta} = 0, \Theta][[1]]]$ 
```

```
(* range of  $\mu$  us *)
N[ $\mu_{class\Theta} /. \Theta \rightarrow 10^6$ ] (* to *)
N[ $\mu_{fermi\Theta} /. \Theta \rightarrow .001$ ]
```

```
 $\Theta_{class} = \Theta /. \text{Solve}[\mu_1 == \mu_{class\Theta}, \Theta][[1]]$ 
 $\Theta_{fermi} = \Theta /. \text{Solve}[\mu_1 == \mu_{fermi\Theta}, \Theta][[1]]$ 
```

- One desires to generate plots showing how the chemical potential is related to the temperature. This is difficult both because one must construct tables of results from numerical integrals and because *Mathematica* does not happily do this. The following approach works. One has to
  - set up the integral,
  - do the integral to fill a table (here using finer resolution as the potential approaches zero),
  - arrange the arrays of numbers as needed for plotting, and
  - make the plots.

```
Clear[ $\mu_1$ ]
eq100 =  $\left( (3/2) \text{HIntegrate}\left[\frac{\sqrt{x}}{1 + \text{Exp}[x - \mu_1]}, \{x, 0, 200\}\right] \right)^{-2/3}$ 
eq100 /. { $\mu_1 \rightarrow 15, \text{HIntegrate} \rightarrow \text{NIntegrate}$ }
```

$$\text{eq101} = \left( (3/2) \text{HIntegrate} \left[ \frac{\sqrt{x}}{1 + \text{Exp}[x - \mu 1]}, \{x, 0, \mu \text{end}\} \right] \right)^{-2/3}$$

$$\text{eq102} = \left( (3/2) \text{HIntegrate} \left[ \frac{\sqrt{x}}{1 + \text{Exp}[x - \mu 1]}, \{x, 0, 2 \mu 1\} \right] \right)^{-2/3}$$

```

t1 = Table[{-μ1, N[θclass], N[θfermi],
  eq101 /. {HIntegrate → NIntegrate, μend → Max[200, 2 μ1] }}
, {μ1, -21, -2, 1}];
t2 = Table[{-μ1, N[θclass], N[θfermi],
  eq102 /. {HIntegrate → NIntegrate, μend → Max[200, 2 μ1] }}
, {μ1, -1.99, -.01, .01}];
tneg = Join[t1, t2];
thetaplot = Transpose[tneg][[4]];
μplot = Transpose[tneg][[1]];
neglist = Transpose[Append[{thetaplot}, μplot]];
classplotneg = Transpose[tneg][[2]];
negclass = Transpose[Append[{classplotneg}, μplot]];
Null

```

```

g1 = LogLogListPlot[neglist, PlotRange → {{10-3, 103}, {0.1, 20}},
  PlotJoined → True,
  Frame → True, PlotStyle → {Thickness[0.005]};
g2 = LogLogListPlot[negclass,
  PlotRange → {{10-3, 103}, {0.1, 20}},
  PlotJoined → True, Frame → True, PlotStyle →
  {Thickness[0.008], GrayLevel[0.5], Dashing[{0.02, 0.02]}}];
g3 =
Show[
  g1,
  g2]

```

```

g4 = LogLogListPlot[neglist, PlotRange -> {{102, 106}, {0.1, 25}},
  PlotJoined -> True, Frame -> True]
g5 =
  LogLogListPlot[negclass, PlotRange -> {{102, 106}, {0.1, 25}},
  PlotJoined -> True, Frame -> True, PlotStyle ->
  {Thickness[0.008], GrayLevel[0.5], Dashing[{0.02, 0.02}]];
g6 =
  Show[
  g4,
  g5]

```

- The above did the work for negative chemical potential. Now do it again for positive chemical potential

```

t3 = Table[{μ1, N[θclass], N[θfermi],
  eq101 /. {HIntegrate -> NIntegrate, μend -> Max[200, 2 μ1] }}
, {μ1, .1, 20, .1}];
t4 = Table[{μ1, N[θclass], N[θfermi],
  eq101 /. {HIntegrate -> NIntegrate, μend -> Max[200, 2 μ1] }}
, {μ1, 30, 1000, 10}];
tpos = Join[t3, t4];
thetaplot = Transpose[tpos][[4]];
μplot = Transpose[tpos][[1]];
poslist = Transpose[Append[{thetaplot}, μplot]];
posclass = Transpose[Append[{Transpose[tpos][[2]]}, μplot]];
posfermi = Transpose[Append[{Transpose[tpos][[3]]}, μplot]];

```

```

g11 =
  LogLogListPlot[poslist, PlotRange -> {{10-3, 103}, {0.1, 1000}},
  PlotJoined -> True, Frame -> True,
  PlotStyle -> {Thickness[0.005]};
g12 = LogLogListPlot[posclass,
  PlotRange -> {{10-3, 103}, {0.1, 1000}},
  PlotJoined -> True, Frame -> True, PlotStyle ->
  {Thickness[0.008], GrayLevel[0.5], Dashing[{0.02, 0.02}]];
g13 = LogLogListPlot[posfermi,
  PlotRange -> {{10-3, 103}, {0.1, 1000}},
  PlotJoined -> True, Frame -> True,
  PlotStyle -> {Thickness[0.008], Dashing[{0.02, 0.02}]];
g14 = Show[g11, g12, g13]

```

Now one can exploit the plots for the two regimes, to be combined in graphics software

```
Export["muvsthetaneg.eps", g3, "EPS"]
Export["muvsthetapos.eps", g14, "EPS"]
```

■ Now do the pressure integral, for  $p/(ne \text{ kB Te})$

```
eqF32 = HIntegrate[ $\frac{x^{1.5}}{1 + \text{Exp}[x - \mu1]}$ , {x, 0, \muend}]
eqF12 = HIntegrate[ $\frac{x^{0.5}}{1 + \text{Exp}[x - \mu1]}$ , {x, 0, \muend}]
eq200 =  $\frac{2}{3} \frac{\text{eqF32}}{\text{eqF12}}$ 
```

- Can leave out  $\mu < 0$  as this is obviously  $\sim 1$ .

```
p1 = Table[
  {\mu1, eq200 /. {HIntegrate -> NIntegrate, \muend -> Max[200, 2 \mu1]}},
  {\mu1, 0.11, 99.01, .1}]
```

```
g100 = LogLogListPlot[p1, PlotRange -> {{0.1, 100}, {0.5, 40}},
  PlotJoined -> True, Frame -> True]
```

```
Export["p vs \mu fermi.eps", g100, "EPS"]
```

- From this calculation, at large  $\mu$  one has  $p/(ne \text{ kB Te}) = (2/5) \mu / (\text{kB Te})$   
But in the degenerate regime  $\mu / (\text{kB Te}) = 1/\Theta$ . So  $p = pF = (2/5) \mu \epsilon F$

■ Compare electron and ion energy contributions in classical regime.

```
jmax = IntegerPart[0.63 \sqrt{Te}]
eion =  $\sum_{i=1}^{jmax} i^2 EH$  /. EH -> 13.6
eelec =  $\frac{3}{2} 0.63 \sqrt{Te} Te$  (* Te and EH in eV here *)
```

```
gener = LogLogPlot[{eion, eelec},
  {Te, 5, 1000}, PlotPoints -> 1000, Frame -> True]
```

```
Export["energy comparison.eps", gener, "EPS"]
```

## 14 Appendix C: A List of the Homework Problems

### Homework 2.1

One approach to deriving the Euler equations is to identify the density, flux, and sources of mass, momentum, and energy and then to use (2.5). Do this for a polytropic gas and then simplify the results to obtain (2.1) through (2.3).

### Homework 2.2

Linearize the Euler equations to derive (2.7) and (2.8). Find appropriate divisors to make these equations nondimensional and discuss which terms are smaller than others. Then derive (2.9).

### Homework 2.3

Take the actual, mathematical Fourier transform of (2.9) to find (2.10).

### Homework 2.4

Substitute, for the density in (2.9), the actual, mathematical Fourier transform of the spectral density  $\tilde{\rho}(\mathbf{k}, \omega)$ . Show how the result is related to (2.10).

### Homework 2.5

Derive (2.14) from (2.1), (2.2), and (2.4).

### Homework 2.6

Generalize the above derivation to a plasma with an arbitrary number of ion species, each of which may have a distinct temperature.

### Homework 2.7

Derive (2.63).

**Homework 2.8**

Derive a replacement for (2.65), keeping an appropriate version of the drag term at the end of (2.62).

**Homework 2.9**

Find the sizes and directions of the particle orbits. Explain from fundamental laws of electromagnetics why their direction is as it is. Show pictorially why the  $\mathbf{E} \times \mathbf{B}$  drift moves particles in the same direction.

**Homework 3.1**

Inertial fusion designs typically involve the compression of DT fuel to about 1,000 times the liquid density of  $0.25 \text{ g cm}^{-3}$ . Assuming that this compression is isentropic and that the fuel remains at absolute zero, determine the energy per gram required to compress this fuel. Compare this to the energy per gram required to isentropically compress the fuel to this same density, assuming the fuel is an ideal gas whose final temperature is to be the ignition temperature of 5 keV.

**Homework 3.2**

Argue conceptually that the contribution of the denominator in (3.16) at large  $\mu/(k_B T_e)$  is a step function. Evaluate this integral numerically to determine how rapidly it becomes a step function as  $\mu/(k_B T_e)$  increases.

**Homework 3.3**

Show, in the limit as  $T_e \rightarrow 0$ , that  $n_e \epsilon_e = (3/5)n_e \epsilon_F$ .

**Homework 3.4**

Derive 3.24 and 3.26 and discuss their differences.

**Homework 3.5**

Make plots comparing  $Z_{\text{bal}}$  from (3.35) with the estimate  $20\sqrt{T_e}$  as a function of  $T_e$ , for ion densities of  $10^{19}$ ,  $10^{21}$ , and  $10^{23} \text{ cm}^{-3}$ . Discuss the results.

**Homework 3.6**

Carry out the evaluation in (3.2.9) and compare the result to  $Z_{\text{bal}}$ , for  $T_e = 1 \text{ keV}$ ,  $Z_{\text{nuc}} = 30$ , and  $n_i = 10^{21} \text{ cm}^{-3}$ .

**Homework 3.7**

Plot the ratio of  $\Delta E$  to the ionization energy versus  $Z_i$  from 1 to 80 for ion densities of  $10^{19}$ ,  $10^{21}$ ,  $10^{23}$ , and  $10^{25}$   $\text{cm}^{-3}$ . Discuss the results.

**Homework 3.8**

Derive (3.73).

**Homework 3.9**

The value of  $R$  used here ignores the internal energy in excited states (as well as the energy lost by radiation during ionization, which would properly have to be treated by more general equations). Again assuming hydrogenic ions, estimate what fraction of the internal energy is present in excited states, and how this varies with  $Z$ .

**Homework 4.1**

Add a gravitational force density and gravitational potential energy to (4.2) and (4.3) and derive the modified jump conditions.

**Homework 4.2**

Suppose that during the shock transition significant energy is lost by radiation. Write down the modified jump conditions.

**Homework 4.3**

Determine from energy arguments how to generalize (4.20) for a two-species plasma.

**Homework 4.4**

Appendix B shows a derivation of (4.10)–(4.15). For  $\gamma_1 = \gamma_2$ , derive (4.18) and (4.20). Using a computational mathematics program is suggested.

**Homework 4.5**

Derive from (4.10) and (4.12) a general expression for  $T_2$ , valid for weak and strong shocks, for  $\gamma_1 = \gamma_2$ . Express the result in physically clear parameters, so the relation among the terms is evident. Check your result by finding it as a limit of (4.19) and by finding (4.20) as a limit from it.

**Homework 4.6**

Evaluate the entropy variation of (4.24) as the Mach number approaches 1.

**Homework 4.7**

Derive (4.28)–(4.31).

**Homework 4.8**

Derive (4.34) and (4.35).

**Homework 4.9**

Derive (4.42). This requires thinking about which frame of reference one is working in, a key element in all such problems.

**Homework 4.10**

Determine the equations and derive the behavior of the simpler case in which a shock is incident on a stationary wall. Let state 0 be the state of the unshocked fluid, state 1 be that of the once-shocked fluid, and state 2 be the state of the reshocked fluid produced when the shock reflects from the wall.

**Homework 4.11**

For the simpler case in which  $p_1 = p_4 = 0$ ,  $\rho_1 = \rho_4$ , and  $\gamma_1 = \gamma_4 = \gamma$ , which is not a bad approximation for many flyer plate collisions, solve (4.44)–(4.1.50) to find the pressure and velocity of the shocked material.

**Homework 4.12**

Show that the conservation of mass in fact requires  $x \geq -c_s t$  in (4.61) and (4.62).

**Homework 4.13**

Obtain (4.74) from (4.73).

**Homework 4.14**

Sketch the  $C_+$  and  $C_-$  characteristics in a fluid flowing uniformly with velocity  $u$ .



**Homework 4.15**

Plot the minimum density and pressure in the rarefaction as a function of  $U$ . Discuss the meaning of the plots. Reasonable normalizations are recommended.

**Homework 4.16**

Show that this type of analysis produces  $\alpha = 1/2$  for cylindrical blast waves and  $\alpha = 2/3$  for planar blast waves

**Homework 4.17**

Find the coefficients  $\alpha$  for cylindrical and planar momentum-conserving snowplows.

**Homework 4.18**

Derive (4.95)–(4.97).

**Homework 4.19**

Derive (4.106)–(4.108).

**Homework 4.20**

Use a computational mathematics program to integrate these equations to find and plot the profiles, and to evaluate  $Q$ , for a cylindrical case. Apply this to find the behavior of a lightning channel produced by a deposited energy of  $10^{10}$  ergs/cm.

**Homework 4.21**

Assuming that a strong shock reaches an interface beyond which the density ( $\rho_4$ ) is 0.1 times the density of the shocked material to the left of the interface ( $\rho_1$ ), solve for the profiles of the fluid parameters in the rarefaction that results.

**Homework 4.22**

Assuming that  $\gamma_1 = \gamma_4$  (or not, if you wish), derive (4.116) from (4.44)–(4.52) by letting  $p_3$  approach  $p_1$  as the definition of the transition to a rarefaction. Hint: This one is not easy. Taking a limit will be necessary and the approach to the solution will matter.

**Homework 4.24**

An entertaining aspect of this specific problem is that it is one case where the traditional model in which shocks are driven by moving pistons does not produce correct qualitative behavior. Consider a rarefaction as it approaches a piston that is moving forward at a constant velocity. What will happen?

**Homework 4.25**

To obtain these results, one must evaluate the equations in cylindrical polar coordinates. Beginning with the first two Euler equations, carry out this evaluation.

**Homework 4.26**

Thus, a property of uniform flow is that  $u_r = -\partial u_\phi / \partial \phi$  in any cylindrical polar coordinate system. Landau and Lifshitz use a geometric argument to demonstrate this. Instead, demonstrate this using a vectorial argument. (Hint: Begin by taking dot products of unit vectors along  $r$  and  $\phi$  with an arbitrary velocity vector.)

**Homework 5.1**

Consider a system with water above oil as just described. Suppose there is a small, sinusoidal ripple on the surface. Find the vertical profile of the force density between the lower and upper boundaries of the ripple for a region of denser fluid and for a region of less-dense fluid. Discuss the comparison of the two fluids and the shape of the force density profile.

**Homework 5.2**

The final relation in (5.22) is significant for our specific application, in which one needs to integrate, across an interface, equations that contain discontinuous quantities along with derivatives of discontinuous quantities. By treating the delta function and the step function as limits of appropriate functions (see a mathematical methods book), prove this relation.

**Homework 5.3**

Find the solution for the velocity profiles and the growth rate for the RT instability for two uniform, constant density fluids that are confined by two planar surfaces each a distance  $d$  from the interface, which is accelerated at constant  $g$ .

**Homework 5.4**

The discussion above (5.43) shows that  $\tilde{n} = (n/\sqrt{kg})\sqrt{\tilde{k}}$ . This would suggest that it might make more sense to separate the meaning of the axes more cleanly by using  $\tilde{\delta} = (n/\sqrt{kg})$  and  $\tilde{k} = [(k^2\nu)/\sqrt{gk}]^{2/3}$  as the two variables. Recast this equation in terms of these new variables, solve it, and plot the real roots from  $\tilde{k} = 0$  to 2. Discuss the results and compare them to  $n = \sqrt{A_n g k}$ .

**Homework 5.5**

Derive (5.44) and (5.45) from (5.41). Comment on the nature of the terms that have been dropped.

**Homework 5.6**

Find the plane-wave solutions to (5.48) and discuss their behavior.

**Homework 5.7**

Consider an exponential density profile that decreases in the direction of the acceleration,  $g$ , as  $\rho = \rho_0 e^{-z/L}$ , and thus is the opposite of the case analyzed above. Apply the RT instability analysis to find  $n$  for this case. Discuss the results.

**Homework 5.8**

Carry out this calculation and find (5.66). Then find the limits when (a)  $k_p \rightarrow 0$  and  $k_x L \gg 1$  and (b) when  $A_n = 0$  and  $L_p = 0$ . Compare these with previous results in the chapter.

**Homework 5.9**

Work out the linear theory to find an expression for the growth rate for the case of a density gradient that extends for a finite distance between two layers of constant density.

**Homework 5.10**

By operating on (5.82) and (5.84), create two scalar differential equations that can be subtracted to eliminate terms involving  $p$ . Compare the resulting differential equation to (5.21) and discuss.

**Homework 5.11**

If we take the point of view that the modulations of interest are proportional to  $e^{\text{int}}$ , then we would insist on finding negative imaginary  $n$  in order to have growth of the modulations, as opposed to damping, in time. However, this should give us pause because the complex representation is only a mathematical convenience while the physical quantities are real. Considering the real, physical quantities, what is the significance of finding positive or negative imaginary  $n$ . (The chapter in Jackson, which introduces waves, may be of some help regarding the connection of real physical quantities and a complex representation.)

**Homework 5.12**

Suppose  $\beta$  is small enough that terms involving  $\beta$  in (5.127) can be dropped. Determine whether the two boundaries seen in Fig. 5.10 ever cross, completely eliminating the instability.

**Homework 5.13**

Analyze the shock conditions for a small-amplitude ripple and show that the change due to the ripple in the  $\hat{z}$  component, relative to that from a planar shock, is second order in the ripple amplitude [i.e., generalize (5.130)].

**Homework 5.14**

Solve (5.133) through (5.136) to find the ratio of  $\alpha$ ,  $\eta$ , and  $\chi$  to  $\beta$ . Plot the results for various values of  $\gamma$  and comment on what you observe.

**Homework 5.15**

Evaluate the small-angle limit of the equations for a shock at an oblique interface with a density decrease, and produce a plot similar to Fig. 5.19 for this case.

**Homework 5.16**

Consider the qualitative behavior of the postshock interface when there is a rarefaction but  $\chi < 0$ . Redraw Fig. 5.19 for this case. Discuss the evolution of the interface.

**Homework 5.17**

Develop (5.144) and (5.145) from the equations in Chap. 2.

**Homework 5.18**

Derive (5.147) through (5.149).

**Homework 5.19**

To be more precise about this point, one should recognize that what moves with the fluid is the vorticity passing through a surface  $S$ . Prove this by taking the time derivative of the integral of  $\boldsymbol{\omega} \cdot d\mathbf{S}$  over a surface  $S$  that moves with the fluid and may change its shape in time. Relate the result to (5.154). Hint: The key here is the evaluation of the partial derivative in time of the surface as a contour integral involving the edge of the surface.

**Homework 5.20**

Obtain (5.153) through (5.155) from the momentum equation.

**Homework 6.1**

Integrate the thermal intensity over  $2\pi$  steradians to find the total radiation power per unit area from a surface at temperature  $T$ .

**Homework 6.2**

Using the particle treatment of the radiation, derive an expression for the total radiation momentum density, and show that it equals  $\mathbf{F}_R/c^2$ .

**Homework 6.3**

Derive (6.14).

**Homework 6.4**

From the uncertainty principle, the spectral width in frequency,  $\Delta\nu$ , of an emission line is roughly the inverse of the decay time. For a typical decay time of 1 ns, find the normalized spectral width  $\Delta\nu/\nu$ , for emission lines in the visible and in the soft x-ray with a photon energy of 100 eV.

**Homework 6.5**

Derive (6.27)

**Homework 6.6**

Take moments of the radiation transfer equation to derive (6.38) and (6.40).

**Homework 6.7**

Beginning with (6.47), derive Eqs. (6.48) to (6.51).

**Homework 6.8**

Derive (6.56). Discuss the result.

**Homework 6.9**

Demonstrate this.

**Homework 6.10**

Given these relations, show that the radiative transfer equation is relativistically invariant.

**Homework 6.11**

Derive (6.73), (6.74), and (6.75). Discuss the limits on  $v/c$  for this specific description if the emission and absorption are dominated by a) continuum emission or b) line emission.

**Homework 6.12**

Rework (6.78) into the form of a conservation equation and discuss the meaning of the terms that result.

**Homework 7.1**

Carry out the calculations just described and compare the behavior of pure hydrogen as opposed to  $C_1H_1$  (used in Fig. 7.1).

**Homework 7.2**

Derive the dispersion relation for isothermal acoustic waves from the Euler equations. That is, demand constant temperature and see what happens.

**Homework 7.3**

Figure 7.4 shows the wave properties as  $\omega$  varies for fixed  $\eta$ . Consider how the wave properties vary with  $\eta$  for  $\beta = 1$  and fixed  $\omega/(\nu_e c_s^2/c^2)$ . Plot the normalized phase velocity and damping length for  $0.01 \leq \eta \leq 10$  and discuss the results.

**Homework 7.4**

We did not explore the angular variation in the contributions to (7.34). One might imagine that the largest contributions could come at grazing angles, where  $\mu$  is very small and the optical depth along a line of sight becomes large. The model used here would be less realistic if most of the emission came at grazing angles, because real systems will have layers that are not truly planar and certainly are not infinite in extent. Use a computational mathematics program to derive (7.34). Then modify the calculation in order to explore how large the contribution is from such grazing angles. Conclude whether or not the results above might be reasonable estimates for real layers.

**Homework 7.5**

It is curious that (7.39) and (7.41) do not depend on  $\beta$ , so that these waves seem not to care whether the system is fully ionized. Beginning with (7.37), derive (7.41) and discuss why there is no  $\beta$  dependence.

**Homework 7.6**

Beginning with  $\rho(\partial\epsilon/\partial t) = \nabla \cdot (\kappa_{rad}\nabla T)$ , derive (7.51).

**Homework 7.7**

Work through the constant-flux model, providing all the missing mathematical steps. Then plot the positions vs. time of the radiation wave and of a disturbance (in the radiation-heated material) moving at Mach 1 or Mach 10. Discuss the results.

**Homework 7.8**

Show that (7.74) is a solution to (7.73). Clearly annotated work with a computational mathematics program is preferred.

**Homework 7.9**

Consider a gold container shaped so that a planar approximation is reasonable, having planar walls spaced 1 mm apart in vacuum. Assume  $\rho = 20 \text{ g/cm}^3$  and treat  $c_V = 10^{12} \text{ ergs/(g eV)}$  as constant. Use other parameters from Ch. 6 as appropriate. Suppose  $100 \text{ kJ/cm}^2$  is the initial energy content of the vacuum between the walls and that the initial wall temperature is negligible. Approximate the heat front in the walls as a square wave. From zero to 10 ns, find the position of the heat front and the temperature of the surface as a function of time. Plot the ratio of the energy content of the walls to the energy content of the vacuum. Discuss the result.

**Homework 7.10**

Develop the equivalent of (7.76) for a spherically symmetric system.

**Homework 7.11**

Demonstrate this point explicitly by considering a system having a planar flow of material within a cylinder of some diameter and of finite length yet losing radiation both radially and axially, and integrating over the cylinder.

**Homework 7.12**

Derive (7.82).

**Homework 7.13**

Working with the Planck description of blackbody radiation, find and plot the fraction of photons that are ionizing as a function of temperature. You will need a computational mathematics program to generate the plot.

**Homework 7.14**

Determine whether (7.91) admits a self-similar solution, assuming a diffusive of  $F_R$

**Homework 7.15**

Solve (7.94) numerically, for several relevant values of  $n$ . Comment on the results.

**Homework 7.16**

Evaluate the net radiation flux ( $F_R - F_o$ ) for an optically thin precursor using a calculation similar to that done in (7.101) and (7.105).

**Homework 7.17**

Assuming that the upstream radiation flux at the shock is  $2\sigma T_f^4$ , the intensity is isotropic, and the absorption and emission from the upstream medium contribute negligibly to  $J_R$ , find the steady-state temperature of the upstream medium.



**Homework 7.18**

Beginning with (7.80)–(7.82), derive the final inverse compression (7.115) under the assumptions of the present section.

**Homework 7.19**

Consider a truly radiation-dominated case, so  $p$  can be neglected in (7.124) and (7.125). Solve these equations for  $p_R$  and  $\rho$ . Find the dependence of the post-shock  $T$  on the shock velocity  $v$ , and compare it to the dependence of a non-radiative shock.

**Homework 7.20**

Express  $p$  and  $p_R$  as reasonable functions of  $T$  and solve (7.124) and (7.125) to find  $T$  and  $\rho$  in the post shock state. This may be a numerical solution, for which you should make reasonable choices about the parameters and show a few cases. Provide at least one graph based on these equations as part of the analysis.

**Homework 8.1**

Derive (8.3) from Maxwell's equations.

**Homework 8.2**

Derive an equation for the conservation of charge from (8.3).

**Homework 8.3**

Using the equation of motion for the electron fluid in the fields of an electromagnetic wave in a plasma of constant density, determine the time-averaged distribution of energy among the electric field, the magnetic field, and the kinetic energy of the electrons. Discuss how this varies with density.

**Homework 8.4**

Derive (8.20).

**Homework 8.5**

Derive (8.22). Calculate the energy density of the laser light wave and show how this is related to the source term on the right-hand side.

**Homework 8.6**

Develop an energy equation for the electron fluid including a Spitzer–Harm heat flux, and show that it is a diffusion equation.

**Homework 8.7**

Determine the range of electron velocities that contribute significantly to the heat flux, by plotting the first-order contribution to the argument of the heat-flux integral (8.28).

**Homework 8.8**

Find the approximate expression for  $\epsilon_R$  to second order in the quantity  $m_a/m_o$ . Plot the corresponding rocket efficiency and the value of (8.49). Discuss the comparison.

**Homework 8.9**

By analyzing the isothermal rarefaction, derive the ratio of the energy required to sustain the rarefaction to the energy injected into the rarefaction at the heat front.

**Homework 8.10**

Evaluate the ablation pressure ( $p_1$ ) for the expansion heat front case, assuming the ablator is Be with a density  $\rho_o$  of  $1.8 \text{ g/cm}^3$ , as a function of radiation temperature from 100 eV to 300 eV. Compare the result with the value given by (8.58).

**Homework 8.11**

Assume that a hohlraum of 1 mm radius is heated for 1 ns at a temperature of 200 eV. Estimate the pressure produced at the center of the hohlraum when the plasma expanding from the gold walls reaches the axis. (Note: this is not an application of (8.67). Instead, you will need to think about the rarefaction produced during the heating pulse.)

**Homework 8.12**

While one can vary the properties of the Z-pinch load from one experiment to the next, one can modify the pulsed-power device itself on a somewhat longer timescale. Such devices are typically characterized by the number of Volt-Seconds they can produce, and operate so that  $V\tau = \text{constant}$ . First,

consider and then explain why Volt-Seconds is a reasonable way to characterize a pulsed-power device. Second, using the scaling relations developed in Sect. 8.3.1, discuss how to optimize the stagnation power for a device with  $V\tau = \text{constant}$ .

### Homework 8.13

An alternative way to think about what could be done with an imploding radiative shock is to imagine that one can drive a converging shock in an optically thin system. Assuming that such a shock reaches steady state, plot the radiation flux and its characteristic temperature against shock velocity for densities of 0.01 and 0.1 g/cm<sup>3</sup>. Comment on the comparison with the above calculation.

### Homework 8.14

Revisit the derivation at the beginning of Sect. 8.3. Consider two infinitely wide, plane parallel conductors carrying opposing currents. Find the force per unit area between them and express it in terms of the magnetic field magnitude. Discuss how the force per unit area compares to the energy density of the magnetic field.

### Homework 9.1

Plot the burn fraction versus  $\rho r$ . Discuss the impact of the assumptions made in deriving the burn fraction on this curve, and on the size of a system designed to produce a certain quantity of fusion energy.

### Homework 9.2

Carry out the evaluation just described. For deuterium at a density of 0.1 g/cm<sup>3</sup>, plot the pressure as a function of temperature for deuterium treated as bosons and for deuterium treated as a classical gas. Discuss the comparison.

### Homework 9.3

Derive the classical relation between entropy and pressure (normalized by the Fermi pressure of the electrons).

### Homework 9.4

Plot the minimum required implosion velocity, for  $\rho r = 3 \text{ g/cm}^2$ , versus final fuel density. Discuss the result.

**Homework 9.5**

Derive (9.26). Why do we need to express this result using a  $1/3$  power?

**Homework 9.6**

Suppose that one could apply a pressure  $p$  for a time  $t$ , using some energy source. With this source, we could accelerate some amount of mass per unit area,  $\rho_o \Delta r$ , to  $v_{\text{imp}} = 300$  km/s. Define a fusion capsule using the reflected pressure due to sunlight for 12 h as the pressure source. Approximate sunlight as light with a wavelength of 580 nm and an irradiance of 1 kW/m<sup>2</sup>. How long would such a capsule take to implode?

**Homework 9.7**

Derive the spectrally averaged absorption coefficient for bremsstrahlung in DT. Check your value against the value found in the NRL plasma formulary.

**Homework 9.8**

Evaluate the appropriate integral of the radiative transfer equation over solid angle to obtain  $F_R$  from a spherical volume of DT. Find the value of the characteristic distance. Compare your result to the result in (9.31), which assumes that the integral over solid angle of the distance across the fuel gives  $\pi R_h$ . Extra credit: generalize this calculation to include arbitrary optical depth and discuss the results.

**Homework 9.9**

The Lawson criterion is generally written as  $n\tau > 10^{14}$  s/cm<sup>3</sup>, with density  $n$  and confinement time  $\tau$ . Find a way to relate this to (9.15) and comment on the comparison.

**Homework 9.10**

One choice in a central hot spot design is how much to increase the pressure above the minimum value of 13.5 Gbars. Increasing the pressure decreases the size of the hot spot but increases the energy required to create this pressure. Keeping the constraints on density and  $\rho r$  found above, consider the effects of scaling the pressure in the hot spot.

**Homework 9.11**

Evaluate the amount of RT growth for the sunlight-driven fusion system of the problem 9.6.

**Homework 10.1**

Show that the Euler equations are in fact invariant under the transformations just described.

**Homework 10.2**

Design a diverging experiment to address the coupling of two structured, unstable interfaces that are affected by a blast wave. Beyond the basic requirements for hydrodynamic scaling, identify other specific parameters that are important to the dynamics. (Hint: review blast-wave propagation and shock stability as part of your work.)

**Homework 10.3**

Determine why  $t_{cc}$  as just defined is the relevant timescale for the crushing of the cloud.

**Homework 10.4**

Suppose that an astrophysical blast wave of interest is produced by a supernova explosion that is a known distance  $R$  from a clump of some radius  $r_{cl}$ . Determine the properties of an experimental blast wave and the duration of the experiment that would be required to model the shock-clump interaction in this system.

**Homework 10.5**

Magnetized jets must have a ratio of plasma pressure to magnetic field pressure (usually called  $\beta$  in plasma physics) no larger than about 1. For a low-Z plasma with a density of  $0.1 \text{ g/cm}^3$  and a temperature of 10 eV, determine how large a magnetic field would be required to satisfy this constraint. How does this compare with the magnetic field of order 1 MGauss that is typically produced in laser-plasma interactions and that might be produced by very clever field-compression experiments?

**Homework 10.6**

An approach that has been used to form hydrodynamic jets is to create an adiabatic rarefaction by allowing a shock wave to emerge from a material into an evacuated tube and then to emerge from this tube into an “ambient medium”, at a lower density. Using the simple scaling results from this book, develop a design for a similar experiment to produce a radiative jet.

**Homework 11.1**

Design a pulse stretcher. Suppose you have a laser beam with an 800 nm central wavelength and a bandwidth of 20 nm (corresponding to a 50 fs laser pulse). Use two identical gratings, recalling that for the first diffracted order the scattered wavelength  $\lambda$  is given by  $\lambda = d(\sin \alpha + \sin \beta)$ , where  $d$  is the line spacing on the grating and  $\alpha$  and  $\beta$  are angles of incidence and reflection relative to the grating normal. Use two identical lenses, recalling that the object distance,  $o$ , image distance,  $i$ , and focal length  $f$  are related by  $o^{-1} + i^{-1} = f^{-1}$ . Note that the initial grating must be less than one focal length from the lens to obtain stretching.

**Homework 11.2**

Assuming that the electron motion is due to a plane wave with a single frequency and that the electron movement is small compared to the wavelength of the light wave, solve the above equations to find the electron trajectory. Determine how it changes as the electron velocity increases (while remaining  $\ll c$ ).

**Homework 11.3**

Prove that these definitions (Eqs. 11.15 and 11.16) are equivalent.

**Homework 11.4**

Solve (11.25) for a range of values of the initial phase (i.e., change  $\pi$  to various other values, for fixed  $a_o = 100$  and  $\delta(0) = 0.01$ . Comment on the variations in the behavior.

**Homework 11.5**

Find the time required to accelerate the electron to  $\sim 30$  GeV in the example just given.

**Homework 11.6**

Suppose one has a laser beam that can be focused to  $10^{20}$  W/cm<sup>2</sup> in a  $10 \mu\text{m}$  diameter spot. Would one obtain higher-energy electrons from tunnel ionization (as in Sect. 11.2) or from using the laser for wakefield acceleration?

**Homework 11.7**

Solve for the potential of a spherical cloud of ions having uniform density, and for the energy distribution function of the ions produced by a Coulomb explosion of this cloud.

**Homework 11.8**

Derive the relativistic version of this theory and find the relativistically correct revision to (11.75).

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